Mathematics in Natural Sciences: how History of Science sheds some light on this "Marriage of Convenience"

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The context…

- Mathematics is relevant to describe Nature.
  Wigner: “The Unreasonable Effectiveness of Mathematics in Natural Sciences”…

- What “kind” of mathematics are used in natural sciences?
  - Mathematics are “multiform”…
  - Physics is a big consumer of “mathematical structures”.

- Look at history of physics to get details on this “Marriage of Convenience”.

- Main points exposed here were developed for lectures at Aix-Marseille Univ. in the Master “Philosophy and History of Fundamental Sciences”.
  ➞ Reflections by a “teacher-historian-physicist-mathematician”…

Mathematics are "multiform"

Some definitions...

Aristotle: *The science of quantities.*

➙ discrete quantities = arithmetic; continuous quantities = geometry.

Oxford Dictionary: *The abstract science of number, quantity, and space, either as abstract concepts (pure mathematics), or as applied to other disciplines such as physics and engineering (applied mathematics).*

Cambridge Dictionary: *The study of numbers, shapes, and space using reason and usually a special system of symbols and rules for organizing them.*

Mathematics are "multiform"

Three important components to consider...

1. The study of **numbers and their relations**, *i.e.* concrete quantities...
   ➙ useful for enumerations and measurements.

2. The study of **structures, shapes, and patterns**, *i.e.* abstract relations...
   ➙ useful for classification and structuring.

3. The study of **logic**...
   ➙ useful for reasoning.

We will look at the first and then the second component...

Study of numbers and their relations

Dynamical Systems in Natural Sciences

Dynamical systems are archetypal of the use of this first component of mathematics…

- A dynamical system describes the **deterministic evolution of quantities** (state) **along a parameter** (*e.g.* time).

- Discrete parameter: the evolution is given by a **recurrence relation**.

- Continuous parameter: the evolution is given by a **differential equation**.
  - This appears after the notion of derivation was formalized.
  - Key change in history of science (no time to elaborate).

- Dynamical systems are everywhere in natural sciences where something evolves in times and can be enumerated or measured (numerically).
  - They are also used in social sciences (econometrics for instance)…
  - Generalizations: stochastic processes (random variables)…

- They can give rise to complicated mathematics…
  ➞ Look at books on "Mathematical Biology" (for instance).

Dynamical Systems: example 1
The Fibonacci sequence (1202)

- Dynamical system describing the growth of an idealized rabbit population.
- Discrete time (in unit of two months).
- State of the system at time n: integer a_n (number of pairs of rabbits).
- Evolution: $a_{n+2} = a_{n+1} + a_n$.
- From the initial state $a_0 = 0$ and $a_1 = 1$, one gets the sequence
  0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, …
- This sequence is related to the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$.
- This sequence appears already in Indian mathematics as a combinatorial problem: Pingala (200 BC), Virahanka (c. 700 AD)…

Dynamical Systems: example 2

Newton's laws of motion (1687)

- This dynamical system motivated Newton to define differential calculus.
- Dynamical system describing the motion of an idealized single point of mass m.
- Continuous (absolute) time.
- State of the system at time t: position $(x(t), y(t), z(t))$ in an inertial frame.
- Evolution: it depends on the applied force $(F_x(t), F_y(t), F_z(t))$ and obeys the system of second order differential equations
$$m\ddot{x}(t) = F_x(t), \qquad m\ddot{y}(t) = F_y(t), \qquad m\ddot{z}(t) = F_z(t).$$
- Vector notation: $\vec{r} = (x, y, z)$ and $\vec{F} = (F_x, F_y, F_z)$:
$$m\vec{a} = \vec{F} \qquad \text{with } \vec{a} = \ddot{\vec{r}} \text{ (acceleration)}$$
(Here we introduce a structure going beyond "measured quantities"...)
- There is a unique solution for a given initial condition $(\vec{r}(0), \dot{\vec{r}}(0))$.
- Choosing (or determining) $\vec{F}$ permits to describe a lots of mechanical systems.
➙ gravitation, friction, electrostatic...

Dynamical Systems: example 3

Sea shells patterns

(See Meinhardt, *The Algorithmic Beauty of Sea Shells*, 2009)

- Patterns of sea shells depend on the species.
- Related to the growth of the shell.
- Can be reproduced using a **unique dynamical system**...

Dynamical Systems: example 3 (cont'd)

Sea shells patterns

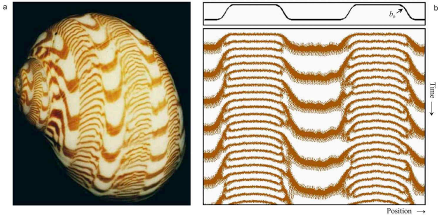
Dynamical system in terms of activator and inhibitor of pigmentation…

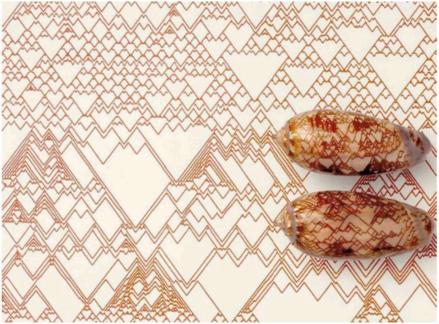
$$\begin{cases} \dfrac{\partial a}{\partial t} = s\left(\dfrac{a^2}{b} + b_a\right) - r_a a + D_a \dfrac{\partial^2 a}{\partial x^2} \\ \dfrac{\partial b}{\partial t} = s a^2 - r_b b + D_b \dfrac{\partial^2 b}{\partial x^2} + b_b \end{cases}$$

- x is the position along the edge of the shell in formation;
- $a(x, t)$ is the concentration of the activator;
- $b(x, t)$ is the concentration of the inhibitor;
- $s\frac{a^2}{b}$ describes the production of the activator (non-linear autocatalysis);
- $-r_a a$ is the destruction rate of the activator;
- $D_a \frac{\partial^2 a}{\partial x^2}$ describes the propagation of the activator by diffusion;
- b_a is the basic production of activator;
- similar terms for b…

Dynamical Systems: example 3 (cont'd)

Sea shells patterns





- The modeling ignores the details of the chemistry and biology underlying the pigmentation of the shell.
- They are phenomenological models. ➞ **free parameters**.
- Changing the values of the parameters and the initial states permits to reproduce various patterns.
- This shows the **adaptability of dynamical systems in natural sciences**.
- This leads to the idea of **"effectiveness" of Mathematics in natural sciences**.

Study of structures, shapes, and patterns

- Physics uses a lot the “quantitative component” of mathematics…
- but **physics is a big consumer of mathematical structures**:
  Theories (and models) elaborated with (mathematical) structures/shapes/patterns.
- The only natural science to use this component of mathematics?

The Unreasonable Effectiveness of Mathematics in the Natural Sciences

Richard Courant Lecture in Mathematical Sciences delivered at New York University, May 11, 1959

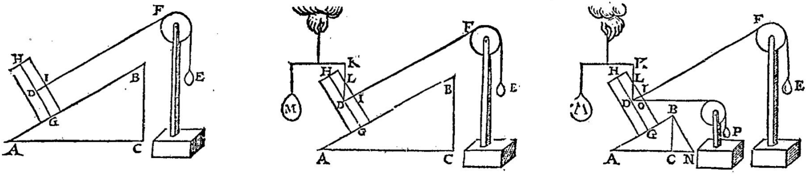
EUGENE P. WIGNER

Princeton University

- Here, Wigner describes some relations between mathematics and physics.
- But he misses the key difference between “counting” and “structuring”.
  ➙ **“effectiveness” cannot mean the same thing in both situations!**

When did it start? Forces as vectors…

- (Probably) The first relevant mathematical structure used in physics.
  (“nature” of forces is a debate since Greek science…)
- In modern words (vector analysis, ~1900):
  - The force is described by a vector ➞ **direction** and **magnitude**.
  - Forces can be added as vectors (parallelogram law).
- This structure is implicit (not explicit) in the work by Simon Stevin (1548—1620) on equilibrium in mechanics: **“vector force” as mechanical devices.**
  (ropes, weights, pulleys, simplified weighing-machines…)
  - **direction** along the ropes, **magnitude** given by the weights E, M, P…
  - implicit “addition” of forces (as vectors) to get “0” (static condition)…



Stevin, *De Beghinselen der Weeghconst*, 1586 (here $AB = 2BC$)

- Fundamental law $m\vec{a} = \vec{F}$: dynamical system + mathematical structure.

Structures in history of physics...

- XIX$^{\text{th}}$ century: emergence of the notion of fields (Faraday, Maxwell...).
  - **Maxwell equations** give the "structure" of the electromagnetic field in space-time (not its evolution but its "configuration").
  - Maxwell equations have rigid structures: **special relativity**, **gauge principle**. (both coming from these equations, now used as principles in physics)
- 1916: General Relativity describes the space-time structure (not its evolution).
- ~1925: Quantum Mechanics describes evolution in time of "abstract" objects (vectors in Hilbert spaces) that are not directly "measurable".
- ≥1930: Quantum Field Theory
  - no equation to describe any time evolution (no dynamical system!);
  - ≥1960: Lagrangian of the Standard Model of Particles Physics.
    (Known) Properties of elementary particles and fundamental interactions = algebraic structures.
- Today, we are used to many structures in modern physics:
  symmetries (groups and representations), **various geometries** (fiber bundles and connections), **abstract spaces** (Hilbert spaces, operator algebras)...

History of physics and structures...

- Dynamical systems can be adjusted using the “free parameters”.
  - Look at sea shells patterns...
  - We could change the Newton gravitational force to behave as $1/r^{2+\epsilon}$...

- **But it is not possible to change a structure by “adjusting parameters”.**
  - We can't “adjust” the relation between vectors $m\vec{a} = \vec{F}$.
  - Change Maxwell equations ➞ refute special relativity and/or gauge principle.

- **Physics evolves by “steps” by changing whole theories.**
  - Strong impact on comparisons between theories...
  - Strong relation with history of mathematical structures...

  ➞ Key point to keep in mind when studying history of physics.

A real change of theory is not a change of equations – it is a change of mathematical structure, and only a fragments of competing theories [...] admit comparison with each other within a limited range of phenomena. The “gravitational potential” of Newton and the “curvature of the Einstein metric” describe different worlds in different languages.

Manin, *Mathematics and physics*, 1981, p. 46

Conclusions

- Mathematics is first the **science of quantities.**
  - Any quantitative science uses this component of mathematics.
  - Effectiveness of Mathematics in natural sciences is mainly based on this. (*e.g.* dynamical systems).
- But study of **abstract relations between objects** is another component.
  - Physics is the only natural science which deeply uses this component.
  - Effectiveness of Mathematics here is a more intriguing question…
- Physics “describes” (and “explains”) Nature using mathematical structures.
  - Strong impact on its historical evolution.
  - History of physics is strongly related to history of mathematics.
  - Makes history of physics “different” compared to other natural sciences.
- Another relations between physics and mathematics have not been mentioned: beauty, axiomatisation of physics, reality in physics *versus* in mathematics…

Thank you for your attention

Let us say that in mathematics there are two inexhaustible sources of raw phenomena which are, on the one hand, arithmetic, and on the other, physics.
Alain Connes