Noncommutative generalization of $SU(n)$-principal fiber bundles

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Based on work with M. Dubois-Violette and E. Sérié

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The algebra $\mathbf{A} = \Gamma(\mathrm{End}(E))$

E a $SU(n)$-vector bundle over a smooth manifold M (fiber $\mathbb{C}^n$), $\mathrm{End}(E) = E^* \otimes E$ the fiber bundle of endomorphisms of E, $\mathbf{A}$ the algebra of sections of $\mathrm{End}(E)$.
The trivial case is the situation where $E = M \times \mathbb{C}^n$ is the trivial bundle
➞ $\mathbf{A} = C^\infty(M) \otimes M_n(\mathbb{C})$. But in general, $\mathbf{A}$ is (globally) more complicated.
Locally, using trivialisations of E, the algebra $\mathbf{A}$ looks like $C^\infty(U) \otimes M_n(\mathbb{C})$.
$\mathrm{End}(E)$ is associated to a $SU(n)$-principal fibre bundle P: fiber $M_n(\mathbb{C})$ and representation Ad.

Proposition 1 (Basic properties)

The center of $\mathbf{A}$ *is* $\mathcal{Z}(\mathbf{A}) = C^\infty(M)$.
Involution, trace map and determinant ($\mathrm{Tr}, \det : \mathbf{A} \to C^\infty(M)$*) are well defined fiberwise.*
Define $SU(\mathbf{A})$ *as the unitaries in* $\mathbf{A}$ *of determinant* 1*, and* $\mathfrak{su}(\mathbf{A})$ *as the traceless antihermitean elements. Then the gauge group of* P *is* $SU(\mathbf{A})$ *and its Lie algebra is* $\mathfrak{su}(\mathbf{A})$.

One associates to $\mathbf{A}$ its derivation-based differential calculus introduced by M. Dubois-Violette, $(\Omega^\bullet_{\mathrm{Der}}(\mathbf{A}), \hat{d})$, where $\Omega^n_{\mathrm{Der}}(\mathbf{A})$ is the $\mathcal{Z}(\mathbf{A})$-multilinear antisymmetric maps from $\mathrm{Der}(\mathbf{A})^n$ to $\mathbf{A}$, and $\hat{d}$ is defined by a Koszul-like formula.

Derivations of A

Recall that for any associative algebra $\mathbf{A}$ one has the s.e.s. of Lie algebras and $\mathcal{Z}(\mathbf{A})$-modules

$$0 \longrightarrow \mathrm{Int}(\mathbf{A}) \longrightarrow \mathrm{Der}(\mathbf{A}) \xrightarrow{\rho} \mathrm{Der}(\mathbf{A})/\mathrm{Int}(\mathbf{A}) = \mathrm{Out}(\mathbf{A}) \longrightarrow 0$$

Let ∇^E be any (usual) connection on E and ∇ the induced connection on $\mathrm{End}(E)$.

Proposition 2 (Derivations of A and ordinary connections on E)

$\rho : \mathrm{Der}(\mathbf{A}) \to \mathrm{Out}(\mathbf{A})$ *is the restriction of derivations* $\mathfrak{X} \in \mathrm{Der}(\mathbf{A})$ *to* $\mathcal{Z}(\mathbf{A}) = C^\infty(M)$.

➡ $\mathrm{Out}(\mathbf{A}) \simeq \mathrm{Der}(C^\infty(M)) = \Gamma(M)$ *and* $\mathrm{Int}(\mathbf{A}) \simeq \mathbf{A}_0$ *(traceless elements in* $\mathbf{A}$*).*

For any $X \in \Gamma(M)$, ∇_X *is a derivation of* $\mathbf{A}$ *and* $X \mapsto \nabla_X$ *is a splitting as* $C^\infty(M)$*-modules of the s.e.s.*

$$0 \longrightarrow \mathbf{A}_0 \longrightarrow \mathrm{Der}(\mathbf{A}) \underset{\overset{\nabla}{\longleftarrow}}{\longrightarrow} \Gamma(M) \longrightarrow 0$$

The obstruction to be a splitting of Lie algebras is the curvature of ∇*:*
$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$.

For any $\mathfrak{X} \in \mathrm{Der}(\mathbf{A})$*, let* $X = \rho(\mathfrak{X})$*, then* $\nabla_X - \mathfrak{X} \in \mathrm{Int}(\mathbf{A})$*. Let* $\nabla_X - \mathfrak{X} = \mathrm{ad}_{\alpha(\mathfrak{X})}$.

Then $\mathfrak{X} \mapsto \alpha(\mathfrak{X}) \in \mathbf{A}_0$ *defines a n.c. 1-form* $\alpha \in \Omega^1_{\mathrm{Der}}(\mathbf{A})$ *with* $\alpha(\mathrm{ad}_\gamma) = -\gamma$, $\forall \gamma \in \mathbf{A}_0$.

Ordinary and n.c. connections

Proposition 3 (Ordinary connections and n.c. forms)

The map $\nabla^E \mapsto \alpha$ is an isomorphism between the affine spaces of $SU(n)$-connections on E and the traceless antihermitean n.c. 1-forms on $\mathbf{A}$ s.t. $\alpha(\mathrm{ad}_\gamma) = -\gamma$.

The n.c. 2-form $(\mathfrak{X}, \mathfrak{Y}) \mapsto \Omega(\mathfrak{X}, \mathfrak{Y}) = \hat{\mathrm{d}}\alpha(\mathfrak{X}, \mathfrak{Y}) + [\alpha(\mathfrak{X}), \alpha(\mathfrak{Y})]$ depends only on the projections X and Y of $\mathfrak{X}$ and $\mathfrak{Y}$: it is the curvature R^E of ∇^E as a section of $\bigwedge^2 T^*M \otimes \mathrm{Ad}P \subset \bigwedge^2 T^*M \otimes \mathrm{End}(E)$ where $\mathrm{Ad}P = P \times_{\mathrm{Ad}} \mathfrak{g}$.

Recall that a n.c. connection $\hat{\nabla}$ on the particular right $\mathbf{A}$-module $\mathbf{A}$ is completely given by the n.c. 1-form $\omega \in \Omega^1_{\mathrm{Der}}(\mathbf{A})$ defined by $\hat{\nabla}_{\mathfrak{X}} \mathbb{1} = \omega(\mathfrak{X})$.

Use the relation $\hat{\nabla}_{\mathfrak{X}} a = \mathfrak{X} a + \omega(\mathfrak{X}) a$.

∇^E a connection on E ➞ α its n.c. 1-form ➞ $\hat{\nabla}^\alpha$ its n.c. connection.

Theorem 4 (Ordinary connections as n.c. connections)

The map $\nabla^E \mapsto \hat{\nabla}^\alpha$ embeds the space of ordinary connections on E into the space of n.c. connections on the right $\mathbf{A}$-module $\mathbf{A}$ compatible with the Hermitean structure $(a, b) \mapsto a^* b$. This inclusion is compatible with the corresponding definitions of curvature and gauge transformations.

Relations with the principal fiber bundle

Consider the associative algebra $\mathbf{B} = C^\infty(P) \otimes M_n(\mathbb{C})$.

- $\mathcal{Z}(\mathbf{B}) = C^\infty(P)$, $\mathrm{Der}(\mathbf{B}) = \Gamma(P) \oplus [C^\infty(P) \otimes \mathfrak{sl}_n]$ and $\Omega^\bullet_{\mathrm{Der}}(\mathbf{B}) = \Omega^\bullet(P) \otimes \Omega^\bullet_{\mathrm{Der}}(M_n(\mathbb{C}))$ with the differential $\widehat{\mathrm{d}} = \mathrm{d} + \mathrm{d}'$.
- $\mathfrak{su}(n)$ is a real Lie subalgebra of $\mathrm{Der}(\mathbf{B})$ for two inclusions:
  $\xi \mapsto \xi^v$ vertical vector field on P $\xi \mapsto \mathrm{ad}_\xi$ inner derivation
- $\mathfrak{g}_{\mathrm{ad}} = \{\mathrm{ad}_\xi \ / \ \xi \in \mathfrak{su}(n)\}$ and $\mathfrak{g}_{\mathrm{equ}} = \{\xi^v + \mathrm{ad}_\xi \ / \ \xi \in \mathfrak{su}(n)\}$ are Lie subalgebras of $\mathrm{Der}(\mathbf{B})$.

Proposition 5 (Relations between the ordinary and n.c. geometries)

The Lie subalgebras $\mathfrak{g}_{\mathrm{ad}}$ and $\mathfrak{g}_{\mathrm{equ}}$ define Cartan operations on $(\Omega^\bullet_{\mathrm{Der}}(\mathbf{B}), \widehat{\mathrm{d}})$. One has:

$$\begin{array}{ccc} \Omega^\bullet(P) \otimes \Omega^\bullet_{\mathrm{Der}}(M_n(\mathbb{C})) & \xleftarrow[\mathfrak{su}(n) \ni \xi \mapsto \mathrm{ad}_\xi]{\textit{basic elements}} & \Omega^\bullet(P) \\ {\scriptstyle \textit{basic elements} \atop \mathfrak{su}(n) \ni \xi \mapsto \xi^v + \mathrm{ad}_\xi} \uparrow & & \uparrow {\scriptstyle \textit{basic elements} \atop \mathfrak{su}(n) \ni \xi \mapsto \xi^v} \\ \Omega^\bullet_{\mathrm{Der}}(\mathbf{A}) & \xleftarrow[\mathrm{Int}(\mathbf{A})]{\textit{basic elements}} & \Omega^\bullet(M) \end{array}$$

Other constructions

- There are deep relations between vector fields on P, derivations on $\mathbf{A}$ and derivations on $\mathbf{B}$.
- The study of the cohomology of $(\Omega^{\bullet}_{\mathrm{Der}}(\mathbf{A}), \widehat{\mathrm{d}})$ shows some similarities with the cohomology of fiber bundles ➛ noncommutative Leray theorem coming from a noncommutative Čech-de Rham bicomplex.
- By an adaptation of a construction by P. Lecomte, one gets the Chern characteristic classes of E as the classes associated to the splittings of the s.e.s. $0 \longrightarrow \mathrm{Int}(\mathbf{A}) \longrightarrow \mathrm{Der}(\mathbf{A}) \xrightarrow{\rho} \Gamma(M) \longrightarrow 0$ of Lie algebras and $\mathcal{Z}(\mathbf{A})$-modules.
- There is an interpretation of lagrangians based on noncommutative connections as Yang-Mills-Higgs lagrangians in some non trivial global topology.
- Invariant noncommutative connections have been studied ➛ extension of the ordinary results.
- Generalisations of these constructions to other similar algebras, *e.g.* Clifford algebra of a spin manifold.

About the cohomology of $\Omega^\bullet_{\mathrm{Der}}(\mathbf{A})$

Denote by $\mathbf{A}(U) \simeq C^\infty(U) \otimes M_n(\mathbb{C})$ the sections of $\mathrm{End}(E)$ restricted over a local trivialisation $U \subset M$.

Lemma 6 (The presheaf $\Omega^\bullet_{\mathrm{Der}}(\mathbf{A}(U))$)

$\Omega^\bullet_{\mathrm{Der}}(\mathbf{A}(U))$ *can be given a structure of presheaf which permits one to introduce a noncommutative Čech-de Rham bicomplex*
$\boldsymbol{C}^{p,q}(\mathfrak{U}; \mathcal{F}) = \prod_{\alpha_0 < \cdots < \alpha_p} \Omega^q_{\mathrm{Der}}(\mathbf{A}(U_{\alpha_0 \dots \alpha_p}))$.

Proposition 7 (Noncommutative Leray theorem)

The cohomology of the total complex of the bicomplex $(\boldsymbol{C}^{\bullet,\bullet}(\mathfrak{U}; \mathcal{F}), \widehat{\mathrm{d}}, \delta)$ *is the cohomology of* $\Omega^\bullet_{\mathrm{Der}}(\mathbf{A})$.
The spectral sequence $\{\boldsymbol{E}_r\}$ *associated to the filtration*

$$F^p \boldsymbol{C}(\mathfrak{U}; \mathcal{F}) = \bigoplus_{s \geq p} \bigoplus_{q \geq 0} \boldsymbol{C}^{s,q}(\mathfrak{U}; \mathcal{F})$$

converges to the cohomology of $\Omega^\bullet_{\mathrm{Der}}(\mathbf{A})$ *and satisfies*

$$\boldsymbol{E}_2 = H^\bullet_{dR}(M) \otimes \mathcal{I}(\textstyle\bigwedge^\bullet \mathfrak{sl}_n^*)$$

$\mathcal{I}(\bigwedge^\bullet \mathfrak{sl}_n^*) =$ *invariant elements for the natural Lie derivative.*

Characteristic classes of E constructed from A

Let $0 \longrightarrow \mathfrak{i} \longrightarrow \mathfrak{g} \xrightarrow{\pi} \mathfrak{h} \longrightarrow 0$ be a s.e.s. of Lie algebras and $\varphi : \mathfrak{h} \rightarrow \mathfrak{g}$ be a morphism which splits it as vector spaces.

Let V a vector space and ρ a representation of $\mathfrak{h}$ on V, $S^q_\rho(\mathfrak{i}, V)$ the space of linear symmetric maps $\otimes^q \mathfrak{i} \rightarrow V$ which intertwine the adjoint representation $\mathrm{ad}^{\otimes^q}$ of $\mathfrak{g}$ on $\otimes^q \mathfrak{i}$ and the representation $\rho \circ \pi$ of $\mathfrak{g}$ on V.

Proposition 8 (Lecomte)

To any $\alpha \in S^q_\rho(\mathfrak{i}, V)$ one can associate a cohomology class $[\alpha_\varphi] \in H^{2q}(\mathfrak{h}; V)$ which does not depends on the choice of φ.
If the s.e.s. is split exact as a Lie algebra s.e.s. then these cohomology classes are zero.

Proposition 9 (Characteristic classes of E)

The construction by Lecomte can be adapted to the s.e.s.

$0 \longrightarrow \mathrm{Int}(\mathbf{A}) \longrightarrow \mathrm{Der}(\mathbf{A}) \xrightarrow{\rho} \Gamma(M) \longrightarrow 0$ *in order to take into account the extra $\mathcal{Z}(\mathbf{A})$-module structures. The Chern characteristic classes of E are obtained in this way with $V = \mathcal{Z}(\mathbf{A})$ and $(\mathfrak{X}, f) \mapsto \rho(\mathfrak{X}) \cdot f$.*