Geometric and algebraic structures from the computation of Heat asymptotics for Laplace type operators

(Join work with Bruno Iochum)

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The initial data

- (M, g) a compact d-dimensional boundaryless Riemannian manifold.
- V a smooth hermitean vector bundle (fiber $\mathbb{C}^N$) over M with covariant derivative $\nabla_\mu = \partial_\mu + A_\mu$.
- P a nonminimal Laplace type operator acting on smooth sections $\Gamma(V)$:

$$P = -H^{\mu\nu}\partial_\mu\partial_\nu - v^\mu\partial_\nu - w = -|g|^{-1/2}\nabla_\mu|g|^{1/2}H^{\mu\nu}\nabla_\nu - p^\mu\nabla_\mu - q$$
$$= -H^{\mu\nu}\nabla_\mu\nabla_\nu - (p^\mu + \tfrac{1}{2}(\partial_\nu \ln|g|)H^{\mu\nu} + \nabla_\nu H^{\mu\nu})\nabla_\mu - q = -H^{\mu\nu}\nabla_\mu\nabla_\nu - L^\mu\nabla_\mu - q$$

  - $H^{\mu\nu}, p^\mu, L^\mu := p^\mu + \frac{1}{2}(\partial_\nu \ln|g|)H^{\mu\nu} + \nabla_\nu H^{\mu\nu}, q$ are local sections of $\mathrm{End}(V)$,
  - $H^{\mu\nu}, p^\mu, q$ have homogeneous diffeomorphism transformations,
  - $H^{\mu\nu}(x)\xi_\mu\xi_\nu$ positive and invertible for any x and any $\xi \in \mathbb{R}^N\backslash\{0\}$. Principal symbol of P.

- **Special case:** $H^{\mu\nu} = g^{\mu\nu}u$ with $u(x)$ positive and invertible. P can be written as

$$P = -g^{\mu\nu}u\nabla_\mu\nabla_\nu - \big(p^\mu + g^{\mu\nu}(\nabla_\nu u) - \Gamma^\mu u\big)\nabla_\mu - q$$

  with $\Gamma^\mu := g^{\nu\rho}\Gamma^\mu_{\nu\rho}$ where the $\Gamma^\mu_{\nu\rho}$ = Christoffel symbols of g. Here $L^\mu = p^\mu + g^{\mu\nu}(\nabla_\nu u) - \Gamma^\mu u$.

- **Special case:** $H^{\mu\nu} = g^{\mu\nu}\mathbb{1}_N$ is the “scalar” case, $L^\mu = p^\mu - \Gamma^\mu$.

The goal

- Consider P as before.

- For any $a \in \Gamma(\mathrm{End}(V))$, consider the asymptotics of the heat-trace

$$\mathrm{Tr}[a e^{-tP}] \underset{t \downarrow 0^+}{\sim} \sum_{r=0}^{\infty} a_r(a, P)\, t^{(r-d)/2}$$

- A general result shows that one can write

$$a_r(a, P) = \int_M a_r(a, P)(x)\, \mathrm{dvol}_g(x) = \int_M \mathrm{tr}[a(x)\mathcal{R}_r(x)]\, \mathrm{dvol}_g(x).$$

where $\mathrm{dvol}_g(x) := |g|^{1/2} dx$ locally, $|g| := \det(g_{\mu\nu})$ and $\mathcal{R}_r \in \Gamma(\mathrm{End}(V))$.

- The goal is to **compute $\mathcal{R}_r$**.

What I will talk about...

- I will present a method to compute $\mathcal{R}_r$.
  - This method was initiated by Avramidi and collaborators.
  - With Bruno Iochum, we have completed, refined and fully studied this method.

- This method relies on a separation between some operators and theirs arguments.
  - The algebraic part concerns mainly the operator part.
  - The geometric part concerns mainly the arguments.

- I will avoid the "analytic" part.
  - Existence of the asymptotics, convergence of integrals...
  - The formal expressions will be justified *a posteriori* by showing that they are related to usual approaches.

References

Method mainly based on:

- Avramidi, I. G. and Branson, T. P. (2001). Heat kernel asymptotics of operators with non-Laplace principal part. *Reviews in Mathematical Physics* 13.07, pp. 847–890
- Avramidi, I. G. (2004). Gauged gravity via spectral asymptotics of non-Laplace type operators. *Journal of High Energy Physics* 2004.07, p. 030

Talk based on the papers:

- Iochum, B. and Masson, T. (2017). Heat trace for Laplace type operators with non-scalar symbols. *Journal of Geometry and Physics* 116, pp. 90–118
- Iochum, B. and Masson, T. (2018). Heat asymptotics for nonminimal Laplace type operators and application to noncommutative tori. *Journal of Geometry and Physics* 129, pp. 1–24
- Iochum, B. and Masson, T. (2019). Heat coefficient a_4 for nonminimal Laplace type operators. *Journal of Geometry and Physics* 141, pp. 120–146
- (Pre-Covid) Work in progress with B. Iochum.

Comparison with other methods in:

- Gilkey, P. B. (1995). *Invariance theory, the heat equation and the Atiyah-Singer index theorem.* 2nd ed. Studies in Advanced Mathematics, CRC Press, Inc

Outline

The trace and the kernel

- Let $K(t, x, y)$ be the kernel of e^{-tP}:

$$(e^{-tP}s)(x) = \int_M K(t, x, y)s(y)\,\mathrm{dvol}_g(y), \text{ for any } s \in \Gamma(V)$$

- One has

$$\mathrm{Tr}[ae^{-tP}] = \int_M \mathrm{tr}[a(x)K(t, x, x)]\,\mathrm{dvol}_g(x)$$

- "Locally" on M, using a Fourier transform, one gets:

$$K(t, x, y) = (2\pi)^{-d}|g|^{-1/2}(y)\int_{\mathbb{R}^d} e^{-iy\cdot\xi}(e^{-tP}e^{ix\cdot\xi})\,d\xi \;:\; \mathbb{C}^N \to \mathbb{C}^N.$$

- So, one has to get an asymptotics expansion

$$K(t, x, y) \underset{t\downarrow 0^+}{\sim} \sum_{r=0}^{\infty} \mathcal{R}_r(x)\, t^{(r-d)/2}$$

First step

Use the presentation $P = -H^{\mu\nu}\nabla_\mu\nabla_\nu - L^\mu\nabla_\mu - q$.

- One has

$$-(Pe^{ix\cdot\xi}s)(x) = -e^{ix\cdot\xi}([H+K+P]s)(x), \text{ with } \begin{cases} K := \xi_\mu K^\mu = -i\xi_\mu\big(L^\mu + 2H^{\mu\nu}\nabla_\nu\big) \\ H := H^{\mu\nu}\,\xi_\mu\xi_\nu \end{cases}$$

- For any $v \in \mathbb{C}^N$, at fixed x, y, one then has:

$$\begin{aligned} K(t,x,y)v &= (2\pi)^{-d}|\mathrm{g}|^{-1/2}(y)\int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi}\, e^{-t[H+K+P]}v\,\mathrm{d}\xi \\ &= t^{-d/2}(2\pi)^{-d}|\mathrm{g}|^{-1/2}(y)\int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi}\, e^{-H-\sqrt{t}K-tP}v\,\mathrm{d}\xi \end{aligned}$$

  (use the change of variables $\xi_\mu \mapsto \sqrt{t}\xi_\mu$).

- For any $v \in \mathbb{C}^N$ and x:

$$K(t,x,x)v = t^{-d/2}(2\pi)^{-d}|\mathrm{g}|^{-1/2}\int_{\mathbb{R}^d} e^{-H-\sqrt{t}K-tP}v\,\mathrm{d}\xi.$$

Second step

- One uses the Volterra series

$$e^{A+B} = e^{A} + \sum_{k=1}^{\infty} \int_{\Delta_k} e^{(s_0-s_1)A}\, B\, e^{(s_1-s_2)A} \dots e^{(s_{k-1}-s_k)A}\, B\, e^{(s_k-s_{k+1})A}\, \mathrm{d}s$$

where $\Delta_k := \{ s = (s_1, \dots, s_k) \in \mathbb{R}_+^k \mid 0 = s_{k+1} \leq s_k \leq s_{k-1} \leq \dots \leq s_2 \leq s_1 \leq s_0 = 1 \}$ and $\Delta_0 := \emptyset$.

- Then, with $A = -H(\xi)$ and $B = -\sqrt{t} K(\xi) - tP$, for any $v \in \mathbb{C}^N$,

$$e^{-H(\xi) - \sqrt{t}K(\xi) - tP} v = f_0(\xi)[1]v + \sum_{k=1}^{\infty} (-1)^k f_k(\xi)[(\sqrt{t}K(\xi) + tP) \otimes \dots \otimes (\sqrt{t}K(\xi) + tP)]v$$

where, for any $k \in \mathbb{N}$, the map $f_k(\xi) : M_N[\xi, \nabla]^{\otimes^k} \to M_N[\xi]$ is defined by

$$f_k(\xi)[B_1 \otimes \dots \otimes B_k]v := \int_{\Delta_k} e^{(s_1-s_0)H(\xi)}\, B_1\, e^{(s_2-s_1)H(\xi)}\, B_2\, e^{(s_3-s_2)H(\xi)} \dots B_k\, e^{(s_{k+1}-s_k)H(\xi)} v \, \mathrm{d}s$$

$$f_0(\xi)[\lambda] := \lambda\, e^{-H(\xi)}, \quad \text{for } \lambda \in \mathbb{C} =: M_N(\mathbb{C})^{\otimes^0}.$$

- The B_i are matrix-valued differential operators in ∇_μ depending on x and (polynomials in) ξ.

Coefficients of the asymptotics of the heat-trace

- To get $a_r(a, P)(x)$, one collects all the terms in $t^{(r-d)/2}$ in

$$t^{-d/2} \frac{|g|^{-1/2}}{(2\pi)^d} \sum_{k=0}^{\infty} (-1)^k \int_{\mathbb{R}^d} a(x) f_k(\xi)[(\sqrt{t}K(\xi) + tP)^{\otimes^k}] \,\mathrm{d}\xi$$

  $\sqrt{t}$ appears in front of $K(\xi)$ (linear in ξ) and P.

- For r odd, the integration along ξ vanishes (odd power in ξ).

- Then one has for instance, for any $v \in \mathbb{C}^N$:

$$a_0(a, P)(x)v = \operatorname{tr} \frac{|g|^{-1/2}}{(2\pi)^d} a(x) \int_{\mathbb{R}^d} f_0[1]v \,\mathrm{d}\xi,$$

$$a_2(a, P)(x)v = \operatorname{tr} \frac{|g|^{-1/2}}{(2\pi)^d} a(x) \int_{\mathbb{R}^d} (f_2[K \otimes K] - f_1[P])v \,\mathrm{d}\xi,$$

$$\begin{aligned} a_4(a, P)(x)v = \operatorname{tr} \frac{|g|^{-1/2}}{(2\pi)^d} a(x) \int_{\mathbb{R}^d} & (f_2[P \otimes P] - f_3[K \otimes K \otimes P] - f_3[K \otimes P \otimes K] \\ & - f_3[P \otimes K \otimes K] + f_4[K \otimes K \otimes K \otimes K])v \,\mathrm{d}\xi, \end{aligned}$$

The computation of $\mathcal{R}_r$

Proposition

From the previous computations, $\mathcal{R}_r$ is the following (local) section of $\text{End}(V)$*:*

$$\mathbb{C}^N \ni v \mapsto \mathcal{R}_r v = \frac{|g|^{-1/2}}{(2\pi)^d} \sum_{r/2 \leq k \leq r} (-1)^k \sum_{\substack{S \subset \{1,\dots,k\} \\ |S| = 2k-r}} \int_{\mathbb{R}^d} f_k(\xi)[B_1 \otimes \cdots \otimes B_k] v \, d\xi \quad \textit{with} \quad \begin{cases} B_i = P & \textit{if } i \notin S \\ B_i = K & \textit{if } i \in S \end{cases}$$

The second sum is over all subsets S of $\{1, \cdots, k\}$ of cardinality $2k - r$.

- B_i may depends on ξ (linearly for K).
- B_i may contains covariant derivatives ∇ (1 for K, 2 for P).

The challenges

- Compute and characterize the operators $f_k(\xi) : M_N[\xi, \nabla]^{\otimes^k} \to M_N[\xi]$.
- Manage the integration along ξ.
- Manage the derivations ∇_μ in the arguments of the $f_k(\xi)$'s.

These problems are related.

This is the object of the "algebraic" part of the talk...

Integration along ξ

For $p \in \mathbb{N}$, use the compact notation $\boldsymbol{\mu}(p) := (\mu_1, \dots, \mu_p)$ for $\mu_\ell \in \{1, \dots, d\}$ and $\boldsymbol{\mu}(0) := \varnothing$.

- We define the family of operators, for $k \in \mathbb{N}$, $p \in \mathbb{N}$ and $\mu_\ell \in \{1, \dots, d\}$,

$$\mathbf{X}_{k,(\mu_1,\dots,\mu_p)} = \mathbf{X}_{k,\boldsymbol{\mu}(p)} := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \xi_{\mu_1} \cdots \xi_{\mu_p} f_k(\xi)\, d\xi \qquad \mathbf{X}_k := \mathbf{X}_{k,\boldsymbol{\mu}(0)}.$$

- $\mathbf{X}_{k,\boldsymbol{\mu}(p)}$ is completely symmetric in the μ_ℓ's.
- $|g|^{-1/2}\mathbf{X}_{k,\boldsymbol{\mu}(p)}$ is well-behaved under a change of coordinates system.
- Recall that

$$f_k(\xi)[B_1 \otimes \cdots \otimes B_k] = \int_{\Delta_k} e^{(s_1-s_0)H(\xi)}\, B_1\, e^{(s_2-s_1)H(\xi)}\, B_2\, e^{(s_3-s_2)H(\xi)} \dots B_k\, e^{(s_{k+1}-s_k)H(\xi)}\, ds$$

  with $H(\xi) = H^{\mu\nu}\xi_\mu\xi_\nu$, so that

$$\mathbf{X}_{k,\boldsymbol{\mu}(p)} = 0 \quad \text{for } p \text{ odd.}$$

- **Open question: how to compute $\mathbf{X}_{k,\mu(p)}$ in general?**

Integration along ξ: a new expression for $\mathcal{R}_r$

From now on, let $U \subset M$ be an open subset which trivializes V.

- For $B_\ell : U \to M_N[\xi, \nabla]$, $1 \leq \ell \leq k$, write (summation over the μ_ℓ's)
$$B_1 \otimes \cdots \otimes B_k =: \xi_{\mu_1} \cdots \xi_{\mu_r} \{B_1 \otimes \cdots \otimes B_k\}^{\boldsymbol{\mu}(r)} = \xi_{\mu_1} \cdots \xi_{\mu_r} \{B_1 \otimes \cdots \otimes B_k\}^{(\mu_1,\dots,\mu_r)}$$
where $\{B_1 \otimes \cdots \otimes B_k\}^{\boldsymbol{\mu}(r)} = \{B_1 \otimes \cdots \otimes B_k\}^{(\mu_1 \cdots \mu_r)}$ does not depend on ξ.

- One has
$$\mathbb{C}^N \ni v \mapsto \mathcal{R}_r v = |g|^{-1/2} \sum_{r/2 \leq k \leq r} (-1)^k \sum_{\substack{S \subset \{1,\dots,k\} \\ |S| = 2k-r}} \mathbf{X}_{k,\boldsymbol{\mu}(2k-r)}[\{B_1 \otimes \cdots \otimes B_k\}^{\boldsymbol{\mu}(2k-r)}] v \quad \text{with} \quad \begin{cases} B_i = P & \text{if } i \notin S \\ B_i = K & \text{if } i \in S, \end{cases}$$

Some combinatorial properties of the $\mathrm{X}_{k,\mu(p)}$

Let $B_\ell : U \to M_N[\xi]$, for $1 \leq \ell \leq k$.

Proposition

$$\sum_{\ell=1}^{k+1} \mathbf{X}_{k+1,\boldsymbol{\mu}(2p+2)}[\{B_1 \otimes \cdots \otimes B_{\ell-1} \otimes H^{\mu_{2p+1}\mu_{2p+2}} \otimes B_\ell \otimes \cdots \otimes B_k\}^{\boldsymbol{\mu}(2p)}] = \left(\frac{d}{2} + p\right) \mathbf{X}_{k,\boldsymbol{\mu}(2p)}[\{B_1 \otimes \cdots \otimes B_k\}^{\boldsymbol{\mu}(2p)}].$$

By convention, for $\ell = 1$, $H^{\mu_{2p+1}\mu_{2p+2}}$ is placed before B_1 and for $\ell = k+1$, $H^{\mu_{2p+1}\mu_{2p+2}}$ is after B_k.

Let $\boldsymbol{\mu}(p)^{\check{\ell}} := (\mu_1, \dots, \mu_{\ell-1}, \mu_{\ell+1}, \dots, \mu_p)$

Proposition

For any $\nu \in \{1, \dots, d\}$, one has

$$\sum_{\ell=1}^{k+1} \mathbf{X}_{k+1,\boldsymbol{\mu}(2p+2)}[\{B_1 \otimes \cdots \otimes B_{\ell-1} \otimes H^{\mu_\ell \nu} \otimes B_\ell \otimes \cdots \otimes B_k\}^{\boldsymbol{\mu}(2p+2)^{\check{\ell}}}] = \frac{1}{2} \sum_{i=1}^{2p+1} \mathbf{X}_{k,(\mu_1 \dots \mu_{i-1} \nu \mu_i \dots \mu_{2p})}[\{B_1 \otimes \cdots \otimes B_k\}^{(\mu_1 \dots \mu_{i-1} \nu \mu_i \dots \mu_{2p})}].$$

Propagation of derivations for the $f_k(\xi)$

Let $s : U \to \mathbb{C}^N$ be the trivialization of a section in $\Gamma(V)$.
Let $B_\ell : U \to M_N[\xi, \nabla]$, for $1 \leq \ell \leq i$ and $B_\ell : U \to M_N[\xi]$, for $i+1 \leq \ell \leq k$.

Proposition

$$
\begin{aligned}
f_k(\xi)[B_1 \otimes \cdots \otimes B_i \nabla_\mu \otimes \cdots \otimes B_k]\, s = {} & \textstyle\sum_{j=i+1}^{k} f_k(\xi)[B_1 \otimes \cdots \otimes (\nabla_\mu B_j) \otimes \cdots \otimes B_k]\, s \\
& - \textstyle\sum_{j=i}^{k} f_{k+1}(\xi)[B_1 \otimes \cdots \otimes B_j \otimes (\nabla_\mu H) \otimes B_{j+1} \otimes \cdots \otimes B_k]\, s \\
& + f_k(\xi)[B_1 \otimes \cdots \otimes B_i \otimes \cdots \otimes B_k](\nabla_\mu s).
\end{aligned}
$$

- The proof uses the relation, for ∂ a derivation on an algebra $\mathbf{A}$, $h \in \mathbf{A}$, and $a, b \in \mathbb{R}$,

$$
\partial e^{(a-b)\,h} = -\int_a^b e^{(s-b)\,h} (\partial h) e^{(a-s)\,h}\, \mathrm{d}s
$$

  This extra integration change Δ_k to Δ_{k+1}.
- Repeating this “propagation” for all the ∇_μ in the B_ℓ gives a relation with only $B_\ell : U \to M_N[\xi]$.

Propagation of derivations for the $\mathbf{X}_{k,\mu(p)}$

Let $s : U \to \mathbb{C}^N$ be the trivialization of a section in $\Gamma(V)$.
Let $B_\ell : U \to M_N[\xi, \nabla]$, for $1 \leq \ell \leq i$ and $B_\ell : U \to M_N[\xi]$, for $i + 1 \leq \ell \leq k$.

Proposition

$$\begin{aligned}
&\mathbf{X}_{k,\mu(2p)}[\{B_1 \otimes \cdots \otimes B_i\nabla_\nu \otimes \cdots \otimes B_k\}^{\mu(2p)}]\, s \\
&\qquad = \textstyle\sum_{j=i+1}^{k} \mathbf{X}_{k,\mu(2p)}[\{B_1 \otimes \cdots \otimes (\nabla_\nu B_j) \otimes \cdots \otimes B_k\}^{\mu_1\cdots\mu_{2p}}]\, s \\
&\qquad\quad - \textstyle\sum_{j=i}^{k} \mathbf{X}_{k+1,\mu(2p+2)}[\{B_1 \otimes \cdots \otimes B_j \otimes (\nabla_\nu H) \otimes B_{j+1} \otimes \cdots \otimes B_k\}^{\mu(2p+2)}]\, s \\
&\qquad\quad + \mathbf{X}_{k,\mu(2p)}[\{B_1 \otimes \cdots \otimes B_i \otimes \cdots \otimes B_k\}^{\mu(2p)}](\nabla_\nu s).
\end{aligned}$$

This is the “integrated” version of the previous relation.

Computation of $\mathcal{R}_r$ using the operators $\mathbf{X}_{k,\mu(p)}$

Recall that

$$\mathbb{C}^N \ni v \mapsto \mathcal{R}_r v = |g|^{-1/2} \sum_{r/2 \leq k \leq r} (-1)^k \sum_{\substack{S \subset \{1,\dots,k\} \\ |S| = 2k-r}} \mathbf{X}_{k,\boldsymbol{\mu}(2k-r)}[\{B_1 \otimes \cdots \otimes B_k\}^{\boldsymbol{\mu}(2k-r)}]v \quad \text{with} \quad \begin{cases} B_i = P & \text{if } i \notin S \\ B_i = K & \text{if } i \in S, \end{cases}$$

- The propagations of all the ∇_μ's produce terms $\mathbf{X}_{k',\boldsymbol{\mu}(2p')}[\{B_1 \otimes \cdots \otimes B_{k'}\}^{\boldsymbol{\mu}(2p')}]\, Q[A]\, v$.
  - The new B_ℓ are $H^{\mu\nu}$, L^μ, q and their derivatives.
  - The $Q[A]$'s are matrix-valued functions written as a polynomial expressions in the A_μ and their derivatives. (One has $\nabla_\nu v = A_\nu\, v$ and $\nabla_{\nu_1}\nabla_{\nu_2} v = (\partial_{\nu_1} A_{\nu_2} + A_{\nu_1} A_{\nu_2})\, v$, etc, since v is constant.)
- This gives an "expression" for $\mathcal{R}_r$, but what about the $\mathbf{X}_{k,\boldsymbol{\mu}(p)}$'s?

Special case $H^{\mu\nu} = g^{\mu\nu} u$: the operators $f_k(\xi)$

Let $\mathbf{A}$ be an associative algebra.

For $0 \leq \ell \leq k$ and $a \in \mathbf{A}$, let $R_\ell(a) : \mathbf{A}^{\otimes^{k+1}} \to \mathbf{A}^{\otimes^{k+1}}$ defined by $R_\ell(a)[b_0 \otimes \cdots \otimes b_k] := b_0 \otimes \cdots \otimes b_\ell a \otimes \cdots \otimes b_k$.

Let $\mathbf{m} : \mathbf{A}^{\otimes^{k+1}} \to \mathbf{A}$ be the multiplication $\mathbf{m}(b_0 \otimes \cdots \otimes b_k) := b_0 \cdots b_k$.

For any $s \in \Delta_k$, let

$$C_k(s, a) := \sum_{\ell=0}^{k} (s_\ell - s_{\ell+1})\, R_\ell(a) : \mathbf{A}^{\otimes^{k+1}} \to \mathbf{A}^{\otimes^{k+1}}$$

In the following, $\mathbf{A} = C^\infty(U) \otimes M_N(\mathbb{C})$.

For $B_\ell : U \to M_N[\xi]$, one has

$$f_k(\xi)[B_1 \otimes \cdots \otimes B_k] = \mathbf{m} \circ \int_{\Delta_k} e^{-C_k(s,H(\xi))}[\mathbb{1} \otimes B_1 \otimes \cdots \otimes B_k]\,\mathrm{d}s = \mathbf{m} \circ \int_{\Delta_k} e^{-|\xi|_g^2 C_k(s,u)}[\mathbb{1} \otimes B_1 \otimes \cdots \otimes B_k]\,\mathrm{d}s$$

with $|\xi|_g^2 = g^{\mu\nu}\xi_\mu\xi_\nu$.

The integration along ξ is a Gaussian integral.

Special case $H^{\mu\nu} = g^{\mu\nu} u$: the operators $\mathrm{X}_{k,\mu(p)}$

Let $g_d(x) := \sqrt{\frac{|g|(x)}{(4\pi)^d}}$.

Define the completely symmetric tensor

$$\mathbf{g}_{\mu_1\dots\mu_{2p}}(x) := \frac{1}{(2\pi)^d g_d(x)} \int_{\mathbb{R}^d} \xi_{\mu_1} \cdots \xi_{\mu_{2p}} \, e^{-|\xi|^2_g(x)} \, \mathrm{d}\xi$$
$$= \frac{1}{2^{2p}\, p!} \Big(\sum_{\rho \in \mathbb{S}_{2p}} g_{\mu_{\rho(1)}\mu_{\rho(2)}} \cdots g_{\mu_{\rho(2p-1)}\mu_{\rho(2p)}} \Big)(x) = \frac{(2p)!}{2^{2p}\, p!} g_{(\mu_1\mu_2} \cdots g_{\mu_{2p-1}\mu_{2p})}(x),$$

Then

$$\mathbf{X}_{k,\boldsymbol{\mu}(2p)} = g_d(x)\, \mathbf{g}_{\mu_1\dots\mu_{2p}}(x)\, \mathbf{m} \circ \int_{\Delta_k} C_k(s,u)^{-d/2-p} \, \mathrm{d}s = g_d(x)\, \mathbf{g}_{\mu_1\dots\mu_{2p}}(x)\, X_{d/2+p,k}$$

with

$$X_{\alpha,k} := \mathbf{m} \circ \int_{\Delta_k} C_k(s,u)^{-\alpha} \, \mathrm{d}s$$

$\mathrm{X}_{k,\mu(2p)}$ factors into two parts: $g_d\, \mathbf{g}_{\mu_1\dots\mu_{2p}}$ depends only on g; $X_{\alpha,k}$ depends only on u.

The operators $X_{\alpha,k}$ have been studied in our first paper.

Special case $H^{\mu\nu} = g^{\mu\nu}\mathbb{1}_N$

Lemma

Let $\alpha \in \mathbb{R}$, $k \in \mathbb{N}$ and $B_\ell : U \to M_N(\mathbb{C})$ for $1 \leq \ell \leq k$.
Suppose that $[u, B_\ell] = 0$ for any ℓ, then

$$X_{\alpha,k}[B_1 \otimes \cdots \otimes B_k] = \frac{1}{k!} u^{-\alpha} B_1 \cdots B_k.$$

In particular, for $u = \mathbb{1}_N$ and any $B_\ell : U \to M_N(\mathbb{C})$, one has

$$X_{\alpha,k}[B_1 \otimes \cdots \otimes B_k] = \frac{1}{k!} B_1 \cdots B_k.$$

For $u = \mathbb{1}_N$, the operators $X_{\alpha,k}$ do not appear in the results.

Only the factor g_d and the tensors $\mathbf{g}_{\mu_1 \dots \mu_{2p}}$ can participate to the computations.

What is the “geometric” part?

- Expressions depending on the gauge structure:
  covariant derivative ∇_μ, connection A_μ, curvature $F_{\mu\nu}$…

- Expressions depending on the metric g on M:
  Levi-Civita connection ${}^g\nabla$, Riemann tensor $R_{\mu\nu\,\rho\sigma}$, Ricci tensor $\mathrm{Ric}_{\mu\nu}$, scalar curvature $\mathfrak{R}$…

How to reveal the geometric part?

- Expressions depending on the gauge structure:
  - They appear by default since ∇ is in the game.
  - The $Q[A]$ generate well-behaved gauge expressions in terms of the curvature of A_μ and its derivatives.
  - Nothing special to do: just propagate the derivations...

- Expressions depending on the metric g on M:
  - If $H^{\mu\nu}$, L^μ, and q do not depend on g, no Riemannian invariant produced by P.
  - But since L^μ is not well-behaved under a change of coordinates system, it is suitable to introduce g and so p^μ.

$$P = -H^{\mu\nu}\nabla_\mu\nabla_\nu - L^\mu\nabla_\mu - q = -H^{\mu\nu}\nabla_\mu\nabla_\nu - (p^\mu + \tfrac{1}{2}(\partial_\nu \ln|g|)H^{\mu\nu} + \nabla_\nu H^{\mu\nu})\nabla_\mu - q$$

  - Introduce the total covariant derivative $\widehat{\nabla}$: gauge connection ∇ + Levi-Civita connection ${}^g\nabla$.
  - Use normal coordinates to map ∇ to $\widehat{\nabla}$: for instance

$$\nabla_{\mu_1} H^{\mu_2\mu_3} \xrightarrow{\text{n.c.}} \widehat{\nabla}_{\mu_1} H^{\mu_2\mu_3}, \qquad \nabla^2_{\mu_1\mu_2} H^{\mu_3\mu_4} \xrightarrow{\text{n.c.}} \widehat{\nabla}^2_{\mu_1\mu_2} H^{\mu_3\mu_4} - \tfrac{1}{3}\Big(\sum_{\mu_2\nu_1}\mathrm{P}\, R_{\mu_1\mu_2\,\nu_1}{}^{\mu_3}\Big)H^{\nu_1\mu_4} - \tfrac{1}{3}\Big(\sum_{\mu_2\nu_1}\mathrm{P}\, R_{\mu_1\mu_2\,\nu_1}{}^{\mu_4}\Big)H^{\nu_1\mu_3},$$

$$L^{\mu_1} \xrightarrow{\text{n.c.}} p^{\mu_1} + \widehat{\nabla}_{\nu_1} H^{\nu_1\mu_1}, \qquad \nabla_{\mu_1} L^{\mu_2} \xrightarrow{\text{n.c.}} \widehat{\nabla}_{\mu_1} p^{\mu_2} + \widehat{\nabla}^2_{\mu_1\nu_1} H^{\mu_2\nu_1} - \tfrac{2}{3} R_{\mu_1\nu_1\,\nu_2}{}^{\mu_2} H^{\nu_1\nu_2}.$$

Example: result for $\mathcal{R}_2$ in terms of ∇

Let us first consider $P = -H^{\mu\nu}\nabla_\mu\nabla_\nu - L^\mu\nabla_\mu - q$.

$$\begin{aligned}
|g|^{1/2}\mathcal{R}_2 = {} & + \mathbf{X}_1[q] - 2\mathbf{X}_{2,\boldsymbol{\mu}(2)}[H^{\mu_1\nu_1} \otimes H^{\mu_2\nu_2}]\, F_{\nu_1\nu_2} \\
& - \mathbf{X}_{2,\boldsymbol{\mu}(2)}[H^{\nu_1\nu_2} \otimes (\nabla^2_{\nu_1\nu_2} H^{\mu_1\mu_2})] + 4\mathbf{X}_{3,\boldsymbol{\mu}(4)}[H^{\mu_1\nu_1} \otimes H^{\mu_2\nu_2} \otimes (\nabla^2_{\nu_1\nu_2} H^{\mu_3\mu_4})] \\
& - 4\mathbf{X}_{3,\boldsymbol{\mu}(4)}[H^{\mu_1\nu_1} \otimes (\nabla_{\nu_1} H^{\mu_2\mu_3}) \otimes (\nabla_{\nu_2} H^{\mu_4\nu_2})] \\
& + 4\mathbf{X}_{4,\boldsymbol{\mu}(6)}[H^{\mu_1\nu_1} \otimes (\nabla_{\nu_1} H^{\mu_2\mu_3}) \otimes (\nabla_{\nu_2} H^{\mu_4\mu_5}) \otimes H^{\mu_5\nu_2}] \\
& + 2\mathbf{X}_{2,\boldsymbol{\mu}(2)}[L^{\mu_1} \otimes (\nabla_{\nu_1} H^{\nu_1\mu_2})] \\
& - 2\mathbf{X}_{3,\boldsymbol{\mu}(4)}[L^{\mu_1} \otimes (\nabla_{\nu_1} H^{\mu_2\mu_3}) \otimes H^{\mu_4\nu_1}] + 2\mathbf{X}_{3,\boldsymbol{\mu}(4)}[H^{\mu_1\nu_1} \otimes (\nabla_{\nu_1} H^{\mu_2\mu_3}) \otimes L^{\mu_4}] \\
& - \mathbf{X}_{2,\boldsymbol{\mu}(2)}[L^{\mu_1} \otimes L^{\mu_2}] - 2\mathbf{X}_{2,\boldsymbol{\mu}(2)}[H^{\mu_1\nu_1} \otimes (\nabla_{\nu_1} L^{\mu_2})]
\end{aligned}$$

Geometric object: $F_{\nu_1\nu_2}$.

The arguments are not well-behaved under a change of coordinates system.

Example: result for $\mathcal{R}_2$ in terms of $\widehat{\nabla}$

Choose a metric and write $P = -H^{\mu\nu}\nabla_\mu\nabla_\nu - (p^\mu + \frac{1}{2}(\partial_\nu \ln|g|)H^{\mu\nu} + \nabla_\nu H^{\mu\nu})\nabla_\mu - q$.

$$
\begin{aligned}
|g|^{1/2}\mathcal{R}_2 = \; & + \mathbf{X}_1[q] - 2\mathbf{X}_{2,\boldsymbol{\mu}(2)}[H^{\mu_1\nu_1} \otimes H^{\mu_2\nu_2}]F_{\nu_1\nu_2} + 2\mathbf{X}_{3,\boldsymbol{\mu}(4)}[H^{\mu_1\nu_1} \otimes H^{\mu_2\nu_2} \otimes [F_{\nu_1\nu_2}, H^{\mu_3\mu_4}]] \\
& + \tfrac{2}{3} R_{\nu_1\nu_2\,\nu_3}{}^{\mu_1} \mathbf{X}_{2,\boldsymbol{\mu}(2)}[H^{\nu_1\nu_3} \otimes H^{\mu_2\nu_2}] + \tfrac{4}{3} R_{\nu_1\nu_2\,\nu_3}{}^{\mu_1} \mathbf{X}_{2,\boldsymbol{\mu}(2)}[H^{\mu_2\nu_1} \otimes H^{\nu_2\nu_3}] \\
& + \tfrac{4}{3} R_{\nu_1\nu_2\,\nu_3}{}^{\mu_1} \mathbf{X}_{3,\boldsymbol{\mu}(4)}[H^{\mu_2\nu_1} \otimes H^{\mu_3\nu_2} \otimes H^{\mu_4\nu_3}] - \tfrac{8}{3} R_{\nu_1\nu_2\,\nu_3}{}^{\mu_1} \mathbf{X}_{3,\boldsymbol{\mu}(4)}[H^{\mu_2\nu_1} \otimes H^{\mu_3\nu_3} \otimes H^{\mu_4\nu_2}] \\
& - 2\mathbf{X}_{3,\boldsymbol{\mu}(4)}[H^{\mu_1\nu_1} \otimes (\widehat{\nabla}^2_{\nu_1\nu_2} H^{\mu_2\mu_3}) \otimes H^{\mu_4\nu_2}] \\
& + 4\mathbf{X}_{4,\boldsymbol{\mu}(6)}[H^{\mu_1\nu_1} \otimes (\widehat{\nabla}_{\nu_1} H^{\mu_2\mu_3}) \otimes (\widehat{\nabla}_{\nu_2} H^{\mu_4\mu_5}) \otimes H^{\mu_6\nu_2}] \\
& + \mathbf{X}_{2,\boldsymbol{\mu}(2)}\big[\big((\widehat{\nabla}_{\nu_1} H^{\nu_1\mu_1}) + p^{\mu_1}\big) \otimes \big((\widehat{\nabla}_{\nu_2} H^{\nu_2\mu_2}) - p^{\mu_2}\big)\big] \\
& - 2\mathbf{X}_{3,\boldsymbol{\mu}(4)}\big[\big((\widehat{\nabla}_{\nu_1} H^{\nu_1\mu_1}) + p^{\mu_1}\big) \otimes (\widehat{\nabla}_{\nu_2} H^{\mu_2\mu_3}) \otimes H^{\mu_4\nu_2}\big] \\
& - 2\mathbf{X}_{3,\boldsymbol{\mu}(4)}\big[H^{\mu_1\nu_1} \otimes (\widehat{\nabla}_{\nu_1} H^{\mu_2\mu_3}) \otimes \big((\widehat{\nabla}_{\nu_2} H^{\nu_2\mu_4}) - p^{\mu_4}\big)\big] - 2\mathbf{X}_{2,\boldsymbol{\mu}(2)}[H^{\mu_1\nu_1} \otimes (\widehat{\nabla}_{\nu_1} p^{\mu_2})]
\end{aligned}
$$

Geometric objects: $F_{\nu_1\nu_2}$, $R_{\nu_1\nu_2\,\nu_3}{}^{\nu_4}$.

The arguments are now well-behaved under a change of coordinates system.

Example: results for $\mathcal{R}_2$ with $H^{\mu\nu} = g^{\mu\nu} u$ and $H^{\mu\nu} = g^{\mu\nu} \mathbb{1}_N$

For $H^{\mu\nu} = g^{\mu\nu} u$, we get $P = -g^{\mu\nu} u \nabla_\mu \nabla_\nu - \big(p^\mu + g^{\mu\nu}(\nabla_\nu u) - \Gamma^\mu u\big)\nabla_\mu - q$.

$$\begin{aligned}(4\pi)^{d/2}\mathcal{R}_2 = &+ \tfrac{1}{6}\mathfrak{R}\, X_{d/2,1}[u] + X_{d/2,1}[q] - X_{d/2+1,2}[u \otimes (\widehat{\nabla}_\mu p^\mu)] \\ &+ \tfrac{1}{2}\, g^{\mu\nu}\, X_{d/2+1,2}[(\widehat{\nabla}_\mu u + p_\mu) \otimes (\widehat{\nabla}_\nu\, u - p_\nu)] \\ &- \tfrac{1}{2}(d+2)\, g^{\mu\nu}\, X_{d/2+2,3}[u \otimes (\widehat{\nabla}^2_{\mu\nu} u) \otimes u] \\ &- \tfrac{1}{2}(d+2) g^{\mu\nu}\, X_{d/2+2,3}[u \otimes (\widehat{\nabla}_\mu u) \otimes (\widehat{\nabla}_\nu u - p_\nu)] \\ &- \tfrac{1}{2}(d+2) g^{\mu\nu}\, X_{d/2+2,3}[(\widehat{\nabla}_\mu u + p_\mu) \otimes (\widehat{\nabla}_\nu\, u) \otimes u] \\ &+ \tfrac{1}{2}(d+2)(d+4)\, g^{\mu\nu}\, X_{d/2+3,4}[u \otimes (\widehat{\nabla}_\mu u) \otimes (\widehat{\nabla}_\nu u) \otimes u]\end{aligned}$$

For $H^{\mu\nu} = g^{\mu\nu} \mathbb{1}_N$, we get $P = -g^{\mu\nu} \nabla_\mu \nabla_\nu - \big(p^\mu - \Gamma^\mu\big)\nabla_\mu - q$.

$$(4\pi)^{d/2}\mathcal{R}_2 = \tfrac{1}{6}\mathfrak{R} + q - \tfrac{1}{2}(\widehat{\nabla}_\mu p^\mu) - \tfrac{1}{4} p^\mu p_\mu$$

Geometric object: $\mathfrak{R}$.

The scalar curvature appears thanks to a lot of contractions with the tensor $\mathbf{g}_{\mu_1 \dots \mu_{2p}}$.

Example: results for $\mathcal{R}_2$ for the NC torus

- In our second paper we have applied the method to the NC torus.
- In order to remain in a geometrical framework we have taken the deformation parameter to be rational.
- The computation consists to consider a special case of the general result.
- We have recovered some known results: Connes and Tretkoff, Fathizadeth and Khalkhali…
  (This was the non trivial part of the computation!)

Example: results for $\mathcal{R}_4$

Recall that

$$a_4(a,P)(x)v = \operatorname{tr} \frac{|g|^{-1/2}}{(2\pi)^d} a(x) \int_{\mathbb{R}^d} \big(f_2[P\otimes P] - f_3[K\otimes K\otimes P] - f_3[K\otimes P\otimes K] - f_3[P\otimes K\otimes K] + f_4[K\otimes K\otimes K\otimes K]\big)v \, d\xi,$$

- The computation is tedious: the propagation of the derivations ∇ produces thousands of terms...
- We have developed a computer program to perform the computations for $H^{\mu\nu} = g^{\mu\nu}u$:
  - it propagates the derivations;
  - it manages the normal coordinates;
  - it contracts terms with tensors $\mathbf{g}_{\mu_1\dots\mu_{2p}}$;
  - it collects and simplifies all the terms;
  - it exports in LaTeX for publication...
- The final result (our third paper) requires 3 pages to display all the terms.
  (The result is quite useless in the present form...)

Using the resolvent series

For any $v \in \mathbb{C}^N$, we have to develop $e^{-H-\sqrt{t}K-tP}v$ in powers of $\sqrt{t}$.

- Let $R(A, z) := (A - z)^{-1}$ be the resolvent of A for $z \in \mathbb{C}$. Then, for a suitable path $\Gamma_A \subset \mathbb{C}$,

$$e^{-A} = \frac{1}{2\pi i} \int_{\Gamma_A} e^{-z} R(A, z) \, \mathrm{d}z$$

- From $R(A + B, z) = R(A, z) - R(A + B, z) \, B \, R(A, z)$, we get the (formal) resolvent series

$$R(A + B, z) = \sum_{k=0}^{\infty} (-1)^k R(A, z) \left[B \, R(A, z)\right]^k$$

- Then, with $A = H(\xi)$ and $B = \sqrt{t}K(\xi) + tP$, for any $v \in \mathbb{C}^N$,

$$\begin{aligned} e^{-H(\xi)-\sqrt{t}K(\xi)-tP} v &= \frac{1}{2\pi i} \int_{\Gamma} e^{-z} R(H(\xi) + \sqrt{t}K(\xi) + tP, z) v \, \mathrm{d}z \\ &= \frac{1}{2\pi i} \int_{\Gamma} e^{-z} \sum_{k=0}^{\infty} (-1)^k R(H(\xi), z) \left[(\sqrt{t}K(\xi) + tP) \, R(H(\xi), z)\right]^k v \, \mathrm{d}z \end{aligned}$$

for a suitable path Γ that can be deformed to $(-i\infty, +i\infty)$ (properties of $H(\xi)$).

The new operators $\widetilde{f}_k(\xi)$

- For any $k \in \mathbb{N}$, define the map $\widetilde{f}_k(\xi) : M_N[\xi, \nabla]^{\otimes^k} \to M_N[\xi]$ by

$$\widetilde{f}_k(\xi)[B_1 \otimes \cdots \otimes B_k]v := \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} e^{-z}\, R(H(\xi), z)\, B_1\, R(H(\xi), z)\, B_2 \cdots B_k\, R(H(\xi), z) v \,\mathrm{d}z$$

$$\widetilde{f}_0(\xi)[\lambda] := \lambda \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} e^{-z}\, R(H(\xi), z) \,\mathrm{d}z, \quad \text{for } \lambda \in \mathbb{C} =: M_N(\mathbb{C})^{\otimes^0}.$$

- Then

$$e^{-H(\xi) - \sqrt{t}K(\xi) - tP} v = \widetilde{f}_0(\xi)[1]v + \sum_{k=1}^{\infty} (-1)^k \widetilde{f}_k(\xi)[(\sqrt{t}K(\xi) + tP) \otimes \cdots \otimes (\sqrt{t}K(\xi) + tP)]v$$

Comparison between $f_k(\xi)$ and $\widetilde{f}_k(\xi)$

We have two (formal) series:

$$e^{-H(\xi)-\sqrt{t}K(\xi)-tP}v = f_0(\xi)[1]v + \sum_{k=1}^{\infty}(-1)^k f_k(\xi)[(\sqrt{t}K(\xi)+tP)^{\otimes^k}]v$$
$$= \widetilde{f}_0(\xi)[1]v + \sum_{k=1}^{\infty}(-1)^k \widetilde{f}_k(\xi)[(\sqrt{t}K(\xi)+tP)^{\otimes^k}]v$$

They are the same:

Proposition

For any $k \in \mathbb{N}$ and any $B_i : U \to M_N[\xi, \nabla]$

$$f_k(\xi)[B_1 \otimes \cdots \otimes B_k] = \widetilde{f}_k(\xi)[B_1 \otimes \cdots \otimes B_k].$$

and

$$\mathbb{C}^N \ni v \mapsto \mathcal{R}_r v = \frac{|\mathrm{g}|^{-1/2}}{(2\pi)^d} \sum_{r/2\leq k\leq r} (-1)^k \sum_{\substack{S\subset\{1,\dots,k\}\\ |S|=2k-r}} \int_{\mathbb{R}^d} \widetilde{f}_k(\xi)[B_1 \otimes \cdots \otimes B_k]v \, \mathrm{d}\xi \quad \textit{with} \quad \begin{cases} B_i = P & \textit{if } i \notin S \\ B_i = K & \textit{if } i \in S \end{cases}$$

Propagation of derivations for the $\widetilde{f}_k(\xi)$

Let $s : U \to \mathbb{C}^N$ be the trivialization of a section in $\Gamma(V)$.
Let $B_\ell : U \to M_N[\xi, \nabla]$, for $1 \leq \ell \leq i$ and $B_\ell : U \to M_N[\xi]$, for $i < \ell \leq k$.

Proposition

$$\begin{aligned}\widetilde{f}_k(\xi)[B_1 \otimes \cdots \otimes B_i \nabla_\mu \otimes \cdots \otimes B_k]\, s = & \textstyle\sum_{j=i+1}^{k} \widetilde{f}_k(\xi)[B_1 \otimes \cdots \otimes (\nabla_\mu B_j) \otimes \cdots \otimes B_k]\, s \\ & - \textstyle\sum_{j=i}^{k} \widetilde{f}_{k+1}(\xi)[B_1 \otimes \cdots \otimes B_j \otimes (\nabla_\mu H) \otimes B_{j+1} \otimes \cdots \otimes B_k]\, s \\ & + \widetilde{f}_k(\xi)[B_1 \otimes \cdots \otimes B_i \otimes \cdots \otimes B_k](\nabla_\mu s).\end{aligned}$$

The proof uses the relation, for ∂ a derivation on an algebra $\mathbf{A}$ and $h \in \mathbf{A}$,

$$\partial R(h, z) = -R(h, z)(\partial h)R(h, z)$$

The pseudo-differential approach

The coefficients of the asymptotics expansion of $\mathrm{Tr}[ae^{-tP}]$ can be computed using a ΨD approach.

- Here P is of degree p and $\sigma(x,\xi) = \sum_{k=0}^{p} \sigma_k(x,\xi)$ is the symbol of P.
- One constructs a ΨD operator $R(\lambda)$ which approximates $R(P,\lambda) = (P-\lambda)^{-1}$.
- Then one approximates $e^{-tP} = \frac{1}{2\pi i}\int_{\Gamma_P} e^{-t\lambda}\, R(P,\lambda)\,\mathrm{d}\lambda$ by $\frac{1}{2\pi i}\int_{\Gamma_A} e^{-t\lambda}\, R(\lambda)\,\mathrm{d}\lambda$.
- The symbol of $R(\lambda)$ is $\sum_{n\geq 0} r_n(x,\xi,\lambda)$ defined by the recursive relation

$$r_0(x,\xi,\lambda) = (\sigma_p - \lambda)^{-1},$$

$$r_n(x,\xi,\lambda) = -r_0 \sum_{\substack{|\alpha|+p+j-k=n \\ j<n}} \frac{(-i)^{|\alpha|}}{\alpha!}(\partial_\xi^\alpha \sigma_k)(\partial^\alpha r_j) \quad \text{for } n>0, \quad \text{and } r_n(x,\xi,\lambda) = 0 \text{ for } n<0.$$

- In this approach, the coefficient $a_n(a,P)$ of the asymptotics is related to

$$|g|^{-1/2}\int_{\mathbb{R}^N}\left(\frac{1}{2\pi i}\int_\Gamma e^{-\lambda} r_n(x,\xi,\lambda)\,\mathrm{d}\lambda\right)\mathrm{d}\xi$$

Relation with the $\widetilde{f}_k(\xi)$'s

- Let $P = P(x,\partial)$ be of degree p. Let $\mathcal{P} = \mathcal{P}(x,\partial,\xi) := P(x, \partial + i\xi)$, so that $(P e^{ix\cdot\xi} s)(x) = e^{ix\cdot\xi}(\mathcal{P}s)(x)$.
- Decompose $\mathcal{P}(x,\partial,\xi) =: \sum_{\ell=0}^{p} \mathcal{P}_\ell(x,\partial,\xi)$ with $\mathcal{P}_\ell(x,\partial,\xi)$ homogeneous of degree ℓ in ξ.
- One has $\mathcal{P}_\ell(x,\partial,\xi) = \sum_{0\leq|\alpha|\leq p-\ell} \frac{(-i)^{|\alpha|}}{\alpha!}(\partial_\xi^\alpha \sigma_{|\alpha|+\ell})\partial^\alpha$ and in particular $\mathcal{P}_p = \sigma_p$.
- One can show that $r_n = -r_0 \sum_{\ell=0}^{p-1} \mathcal{P}_\ell(x,\partial,\xi)\, r_{n-p+\ell}$
  so that r_n is a sum of terms $(-1)^k r_0 \mathcal{P}_{\ell_1} r_0 \cdots r_0 \mathcal{P}_{\ell_k} r_0$ with $0 \leq \ell_i \leq p-1$.
- **Case $p = 2$:** $\mathcal{P} = H + K + P$, so that $\mathcal{P}_0 = P$, $\mathcal{P}_1 = K$, $\mathcal{P}_2 = \sigma_2 = H$.
  $r_0 = (\sigma_2 - \lambda)^{-1} = R(H(\xi), \lambda)$ and the sum of terms $(-1)^k r_0 \mathcal{P}_{\ell_1} r_0 \cdots r_0 \mathcal{P}_{\ell_k} r_0$, with $0 \leq \ell_i \leq 1$, reproduces

$$\sum_{r/2\leq k\leq r} (-1)^k \sum_{\substack{S\subset\{1,\dots,k\}\\ |S|=2k-r}} \widetilde{f}_k(\xi)[B_1 \otimes \cdots \otimes B_k] \quad \text{with} \quad \begin{cases} B_i = P & \text{if } i \notin S \\ B_i = K & \text{if } i \in S \end{cases}$$

  This is true for any p.

- **The ΨD approach computes $\mathcal{R}_r$ directly in terms of the $\widetilde{f}_k(\xi) = f_k(\xi)$.**

Computation using invariants from (M, g) and V

Some approaches rely on a list of geometric invariants terms.

- List all the possible expected terms, using gauge and Riemannan invariants.
- Play with parameters in $\mathrm{Tr}[a e^{-tP}]$ to collect constrains between these terms.
- Compute the coefficients in front of these terms…

From the results presented before, this seems a big challenge for the general situation.

- The list of possible terms is very extensive!
- Not easy to have enough constrains to determine all the coefficients.
- The operators $\mathbf{X}_{k,\boldsymbol{\mu}(p)}$ may not be adapted to this approach:
  their combinatorial properties permit to write the same expression in different forms!

Conclusions

- We have developed a method to compute the coefficients of the asymptotics of $\mathrm{Tr}[a e^{-tP}]$ for $P = -H^{\mu\nu}\nabla_\mu\nabla_\nu - L^\mu\nabla_\mu - q = -H^{\mu\nu}\nabla_\mu\nabla_\nu - (p^\mu + \frac{1}{2}(\partial_\nu \ln|g|)H^{\mu\nu} + \nabla_\nu H^{\mu\nu})\nabla_\mu - q$.

- $H^{\mu\nu}$ plays a crucial role in the computation.
  - It is related to the geometry of M by the indices μ, ν and to the geometry of V as a section of $\mathrm{End}(V)$.
  - It is the principal symbol of P.

- This method makes appear algebraic structures:
  - the operators $f_k(\xi) = \widetilde{f}_k(\xi)$ and $\mathbf{X}_{k,\boldsymbol{\mu}(p)}$ depend only on $H^{\mu\nu}$;
  - they are not easy to evaluate for generic $H^{\mu\nu}$ and they do not show up for scalar type operators ($H^{\mu\nu} = g^{\mu\nu}$);
  - they satisfy combinatorial properties that make the writing of the results not unique.

- This method is related to the geometries of (M, g) and V:
  - through the "arguments" of the operators and the actions of ∇ and $\widehat{\nabla}$;
  - through the tensor $\mathbf{g}_{\mu_1 \dots \mu_{2p}}$ when $H^{\mu\nu} = g^{\mu\nu}\boldsymbol{u}$;
  - the geometry of V appears naturally.
  - The Riemannian structures appear by requiring an explicit diffeomorphism invariant expression.

Perspectives

- We would like to have a better understanding of the combinatorial origin of the operators.
  Volterra series, resolvent series, ΨD approximation of $(P - z)^{-1}$...

- We would like to prove directly that the method computes the coefficients of the asymptotics.
  - Our current proof relies on the final results related to the ΨD approach.
  - This is the “analytic” part of the method!

- Adapt this method to some noncommutative situations.
  - Our computation for the NC torus suggests that this can be done (at least in specific situations).
  - The algebraic structures are well adapted to NCG.

Thank you for your attention