Generalized connections and Higgs fields on Lie algebroids

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Motivations

- Discovery of Higgs particle in 2012.
  ➞ need for a mathematical validation of the Higgs sector in the SM.
  👎 No clue from "traditional" schemes and tools.

- **NCG:** Higgs field is part of a "generalized connection".
  Dubois-Violette, M., Kerner, R., and Madore, J. (1990). Noncommutative Differential Geometry and New Models of Gauge Theory. *J. Math. Phys.* 31, p. 323
  Connes, A. and Lott, J. (1991). Particle models and noncommutative geometry. *Nucl. Phys. B Proc. Suppl.* 18.2, pp. 29–47

  👍 Models in NCG can reproduce the Standard Model
  up to the excitement connected to the diphoton resonance at 750 GeV "seen" by ATLAS and CMS!

  👎 Mathematical structures difficult to master by particle physicists.

- **Transitive Lie algebroids:**
  ➞ generalized connections, gauge symmetries, Yang-Mills-Higgs models…
  ➞ Direct filiation from Dubois-Violette, Kerner, and Madore (1990).

  👍 Mathematics close to "usual" mathematics of Yang-Mills theories.

  👎 No realistic theory yet.

How to construct a gauge field theory?

The basic ingredients are:

1. A space of local symmetries (space-time dependence):
   ➙ a **gauge group**.
2. An implementation of the symmetry on matter fields:
   ➙ a **representation theory**.
3. A notion of derivation:
   ➙ some **differential structures**.
4. A (gauge compatible) replacement of ordinary derivations:
   ➙ a **covariant derivative**.
5. A way to write a gauge invariant Lagrangian density:
   ➙ **action functional**.

At least three mathematical schemes to construct gauge field theories:

- Ordinary differential geometry of principal fiber bundles.
- Noncommutative geometry.
- Transitive Lie algebroids (to be explained in this talk).

Ordinary differential geometry

Given a G-principal fiber bundle $\mathcal{P}$ over $\mathcal{M}$, the ingredients are

gauge group: $\mathcal{G}(\mathcal{P})$ is the group of vertical automorphisms of $\mathcal{P}$.

representation theory: sections of associated vector bundles.
➡ Natural action of $\mathcal{G}(\mathcal{P})$.

differential structures: (ordinary) de Rham differential calculus.

covariant derivative: connection 1-form ω on $\mathcal{P}$.
➡ covariant derivative on sections of any associated vector bundles.

action functional: integration on the base manifold $\mathcal{M}$, Killing form on the Lie algebra $\mathfrak{g}$ of G, Hodge star operator, curvature of ω.

Noncommutative geometry

Given an associative algebra $\mathbf{A}$, the ingredients are

representation theory: a right module $\boldsymbol{M}$ over $\mathbf{A}$.

gauge group: $\mathrm{Aut}(\boldsymbol{M})$, the group of automorphisms of the right module.

differential structures: any differential calculus defined on top of $\mathbf{A}$.

➡ many choices: spectral triples, derivations, twisted derivations...

covariant derivatives: noncommutative connections on $\boldsymbol{M}$,

(need a differential calculus).

action functional: depends on the differential calculus.

- spectral triples: spectral action...
- derivation-based differential calculus: noncommutative integration, Hodge star operator, curvature of the connection...

Outline

Lazzarini, S. and Masson, T. (2012). Connections on Lie algebroids and on derivation-based non-commutative geometry. *J. Geom. Phys.* 62, pp. 387–402
Fournel, C., Lazzarini, S., and Masson, T. (2013). Formulation of gauge theories on transitive Lie algebroids. *J. Geom. Phys.* 64, pp. 174–191

Lie algebroids and their representations

Generalities on Lie algebroids

$\mathcal{M}$ a smooth manifold, $\Gamma(T\mathcal{M})$ the Lie algebra and $C^\infty(\mathcal{M})$-module of vector fields.

Definition in terms of algebras and modules (as in NCG).

Definition (Lie algebroids)

A Lie algebroid A is a finite projective module over $C^\infty(\mathcal{M})$ equipped with a Lie bracket $[-,-]$ and a $C^\infty(\mathcal{M})$-linear Lie morphism $\rho : A \to \Gamma(T\mathcal{M})$ such that

$$[\mathfrak{X}, f\mathfrak{Y}] = f[\mathfrak{X}, \mathfrak{Y}] + (\rho(\mathfrak{X})\cdot f)\mathfrak{Y}$$

for any $\mathfrak{X}, \mathfrak{Y} \in A$ and $f \in C^\infty(\mathcal{M})$.

ρ is the anchor of A.

The usual definition uses the vector bundle $\mathcal{A}$ such that $A = \Gamma(\mathcal{A})$.
$\mathcal{A}$ is viewed as a generalization of the tangent bundle.
➡ We will never use this point of view.

Natural notion of morphisms of Lie algebroids...

Transitive Lie algebroids

A Lie algebroid $A \xrightarrow{\rho} \Gamma(T\mathcal{M})$ is **transitive** if ρ is surjective.

Proposition (The kernel of a transitive Lie algebroid)

Let A be a transitive Lie algebroid.

- $L = \operatorname{Ker} \rho$ *is a Lie algebroid with null anchor on* $\mathcal{M}$.
  ➡ L *is called the **kernel** of* A.
- *The vector bundle* $\mathcal{L}$ *such that* $L = \Gamma(\mathcal{L})$ *is a locally trivial bundle in Lie algebras.*
  ➡ *This gives the Lie structure on* L.

One has the short exact sequence of Lie algebras and $C^\infty(\mathcal{M})$-modules

$$0 \longrightarrow L \xrightarrow{\iota} A \xrightarrow{\rho} \Gamma(T\mathcal{M}) \longrightarrow 0$$

This short exact sequence is the key structure of what follows...

Very trivial example: $A = \Gamma(T\mathcal{M})$ ➡ $L = 0$.

Example 1: Derivations of a vector bundle

$\mathcal{E}$ a vector bundle over $\mathcal{M}$.

$\text{Diff}^1(\mathcal{E})$ the space of first order differential operators on $\mathcal{E}$.

Symbol map:

$$\sigma : \text{Diff}^1(\mathcal{E}) \to \text{Hom}_{C^\infty(\mathcal{M})}(T^*\mathcal{M}, \text{End}(\mathcal{E})) \simeq \Gamma(T\mathcal{M} \otimes \text{End}(\mathcal{E})) \supset \Gamma(T\mathcal{M})$$

$$\mathfrak{D}(\mathcal{E}) = \sigma^{-1}(\Gamma(T\mathcal{M}))$$

is the transitive Lie algebroid of derivations of $\mathcal{E}$:

$$0 \longrightarrow \mathbf{A}(\mathcal{E}) \xrightarrow{\iota} \mathfrak{D}(\mathcal{E}) \xrightarrow{\sigma} \Gamma(T\mathcal{M}) \longrightarrow 0$$

with $\mathbf{A}(\mathcal{E}) = \Gamma(\text{End}(\mathcal{E}))$ (0^{th}-order diff. op.).

$\mathbf{A}(\mathcal{E})$ is an associative algebra (Lie structure is the commutator).

Representation of a Lie algebroid

$A \xrightarrow{\rho} \Gamma(T\mathcal{M})$ a Lie algebroid and $\mathcal{E} \to \mathcal{M}$ a vector bundle.

Definition (Representation of a Lie algebroid)

A representation of A on $\mathcal{E}$ is a morphism of Lie algebroids $\phi : A \to \mathfrak{D}(\mathcal{E})$.

When A is transitive, one has the commutative diagram of exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{\iota} & A & \xrightarrow{\rho} & \Gamma(T\mathcal{M}) & \longrightarrow & 0 \\ & & \downarrow \phi_L & & \downarrow \phi & & \| & & \\ 0 & \longrightarrow & \mathbf{A}(\mathcal{E}) & \xrightarrow{\iota} & \mathfrak{D}(\mathcal{E}) & \xrightarrow{\sigma} & \Gamma(T\mathcal{M}) & \longrightarrow & 0 \end{array}$$

$\phi_L : L \to \mathbf{A}(\mathcal{E})$ is a morphism of Lie algebras.

Example 2: Atiyah Lie algebroids

$\mathcal{P} \xrightarrow{\pi} \mathcal{M}$ a G-principal fiber bundle, $\mathfrak{g}$ the Lie algebra of G.

$\mathsf{R}_g : \mathcal{P} \to \mathcal{P}$, $\mathsf{R}_g(p) = p{\cdot}g$, the right action of G on $\mathcal{P}$.

$$\Gamma_G(T\mathcal{P}) = \{\mathfrak{X} \in \Gamma(T\mathcal{P}) \,/\, \mathsf{R}_{g*}\mathfrak{X} = \mathfrak{X} \text{ for all } g \in G\}$$

$$\Gamma_G(\mathcal{P}, \mathfrak{g}) = \{v : \mathcal{P} \to \mathfrak{g} \,/\, v(p{\cdot}g) = \mathrm{Ad}_{g^{-1}} v(p) \text{ for all } g \in G\}$$

Both are Lie algebras and $C^\infty(\mathcal{M})$-modules.

$\Gamma_G(T\mathcal{P}) = \pi_*$-projectable vector fields in $\Gamma(T\mathcal{P})$ ➙ $\pi_* : \Gamma_G(T\mathcal{P}) \to \Gamma(T\mathcal{M})$.

$\iota : \Gamma_G(\mathcal{P}, \mathfrak{g}) \to \Gamma_G(T\mathcal{P})$ defined by $\iota(v)_{|p} = v(p)^{\mathcal{P}}_{|p}$,
($\mathfrak{g} \ni v \mapsto v^{\mathcal{P}}$ fundamental vector field on $\mathcal{P}$).

S.E.S. of Lie algebras and $C^\infty(\mathcal{M})$-modules:

$$0 \longrightarrow \Gamma_G(\mathcal{P}, \mathfrak{g}) \xrightarrow{\iota} \Gamma_G(T\mathcal{P}) \xrightarrow{\pi_*} \Gamma(T\mathcal{M}) \longrightarrow 0$$

$\Gamma_G(T\mathcal{P})$ is the transitive **Atiyah Lie algebroid** associated to $\mathcal{P}$

The representations of $\Gamma_G(T\mathcal{P})$ are given by the associated vector bundles to $\mathcal{P}$.

Example 3: Trivial Lie algebroids

Trivial Lie algebroid = Atiyah Lie algebroid of a trivial principal bundle $\mathcal{M} \times G$.

Concrete description in terms of the bundle $T\mathcal{M} \oplus (\mathcal{M} \times \mathfrak{g})$:

- $C^\infty(\mathcal{M})$-module: $\mathrm{TLA}(\mathcal{M}, \mathfrak{g}) \equiv A = \Gamma(T\mathcal{M} \oplus (\mathcal{M} \times \mathfrak{g}))$.
- Bracket: $[X \oplus \gamma, Y \oplus \eta] = [X, Y] \oplus (X{\cdot}\eta - Y{\cdot}\gamma + [\gamma, \eta])$
- Anchor: $\rho(X \oplus \gamma) = X$.
- Kernel: $L = \Gamma(\mathcal{M} \times \mathfrak{g})$ (section of a trivial bundle).

Proposition

Every transitive Lie algebroid A is locally of the form $\mathrm{TLA}(\mathcal{U}, \mathfrak{g})$ *for $\mathcal{U} \subset \mathcal{M}$ open subset.*

Trivialization of an Atiyah Lie algebroid $\Gamma_G(T\mathcal{P}) \leftrightarrow$ Trivialization of $\mathcal{P}$.

Differential structures

Differential forms: general definition

A Lie algebroid, $\phi : A \to \mathfrak{D}(\mathcal{E})$ a representation of A on $\mathcal{E}$.

Definition (Differential forms)

For $p \in \mathbb{N}$, let $\Omega^p(A, \mathcal{E})$ be the linear space of $C^\infty(\mathcal{M})$-multilinear antisymmetric maps $A^p \to \Gamma(\mathcal{E})$.

For $p = 0$, let $\Omega^0(A, \mathcal{E}) = \Gamma(\mathcal{E})$.

$\Omega^\bullet(A, \mathcal{E}) = \bigoplus_{p \geq 0} \Omega^p(A, \mathcal{E})$ is equipped with the natural differential

$$(\widehat{\mathrm{d}}_\phi \widehat{\omega})(\mathfrak{X}_1, \dots, \mathfrak{X}_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} \phi(\mathfrak{X}_i) \cdot \widehat{\omega}(\mathfrak{X}_1, \dots \overset{i}{\vee} \dots, \mathfrak{X}_{p+1}) + \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} \widehat{\omega}([\mathfrak{X}_i, \mathfrak{X}_j], \mathfrak{X}_1, \dots \overset{i}{\vee} \dots \overset{j}{\vee} \dots, \mathfrak{X}_{p+1})$$

$\phi(\mathfrak{X}) \cdot \varphi$ is the action of the first order diff. op. $\phi(\mathfrak{X})$ on $\varphi \in \Gamma(\mathcal{E})$.

One has $\widehat{\mathrm{d}}_\phi^2 = 0$ (since ϕ is a morphism of Lie algebras).

Differential forms: two examples

$\mathcal{E} = \mathcal{M} \times \mathbb{C} \longrightarrow \Gamma(\mathcal{E}) = C^\infty(\mathcal{M})$.
The anchor map is a representation of A on $C^\infty(\mathcal{M})$ *via* vector fields.

Definition (Forms with values in $C^\infty(\mathcal{M})$)

$(\Omega^\bullet(A), \widehat{\mathrm{d}}_A)$ is the graded commutative differential algebra of forms on A with values in $C^\infty(\mathcal{M})$ associated to the anchor as a representation.

$0 \longrightarrow L \xrightarrow{\iota} A \xrightarrow{\rho} \Gamma(T\mathcal{M}) \longrightarrow 0$ a transitive Lie algebroid.
$\mathcal{E} = \mathcal{L}$ the vector bundle such that $L = \Gamma(\mathcal{L})$.
For $\mathfrak{X} \in A$ and $\ell \in L$, define $\mathrm{ad}_{\mathfrak{X}}(\ell) \in L$ such that $\iota(\mathrm{ad}_{\mathfrak{X}}(\ell)) = [\mathfrak{X}, \iota(\ell)]$ (adjoint representation of A on $\mathcal{L}$).

Definition (Forms with values in the kernel)

$(\Omega^\bullet(A, L), \widehat{\mathrm{d}})$ is the graded differential Lie algebra of forms on A with values in the kernel L associated to the adjoint representation.

This differential space is a graded Lie algebra and a graded differential module on the graded commutative differential algebra $\Omega^\bullet(A)$.

Differential forms on trivial Lie algebroids

$A = \mathrm{TLA}(\mathcal{M}, \mathfrak{g})$ a trivial Lie algebroid.

$\Omega^\bullet(A)$ is the total complex of the bigraded commutative algebra $\Omega^\bullet(\mathcal{M}) \otimes \bigwedge^\bullet \mathfrak{g}^*$.

$\widehat{\mathrm{d}}_A = \mathrm{d} + \mathrm{s}$ with

$\mathrm{d} : \Omega^\bullet(\mathcal{M}) \otimes \bigwedge^\bullet \mathfrak{g}^* \to \Omega^{\bullet+1}(\mathcal{M}) \otimes \bigwedge^\bullet \mathfrak{g}^*$ de Rham differential

$\mathrm{s} : \Omega^\bullet(\mathcal{M}) \otimes \bigwedge^\bullet \mathfrak{g}^* \to \Omega^{\bullet}(\mathcal{M}) \otimes \bigwedge^{\bullet+1} \mathfrak{g}^*$ Chevalley-Eilenberg differential

$\Omega^\bullet(A, L)$ is the total complex of the bigraded Lie algebra $\Omega^\bullet(\mathcal{M}) \otimes \bigwedge^\bullet \mathfrak{g}^* \otimes \mathfrak{g}$.

$\widehat{\mathrm{d}} = \mathrm{d} + \mathrm{s}'$ with

s' the Chevalley-Eilenberg differential on $\bigwedge^\bullet \mathfrak{g}^* \otimes \mathfrak{g}$ (for the ad rep.).

Compact notation $(\Omega^\bullet_{\mathrm{TLA}}(\mathcal{M}, \mathfrak{g}), \widehat{\mathrm{d}}_{\mathrm{TLA}})$.

This is the model for trivializations of forms on any transitive Lie algebroid.

⚠ Mathematical structure similar to the one used in BRST differential algebras.

➜ work in progress to understand possible relations...

Differential forms on Atiyah Lie algebroids

A the Atiyah Lie algebroid of the G-principal fiber bundle $\mathcal{P} \xrightarrow{\pi} \mathcal{M}$.

$(\Omega^\bullet_{\text{Lie}}(\mathcal{P}, \mathfrak{g}), \widehat{\mathrm{d}})$ the complex of forms with values in the kernel.

$$\mathfrak{g}_{\text{equ}} = \{\xi^{\mathcal{P}} \oplus \xi \ / \ \xi \in \mathfrak{g}\} \quad \subset \quad \text{TLA}(\mathcal{P}, \mathfrak{g}) = \Gamma(T\mathcal{P} \oplus (\mathcal{P} \times \mathfrak{g}))$$

is a Lie sub-algebra, which defines a Cartan operation on $(\Omega^\bullet_{\text{TLA}}(\mathcal{P}, \mathfrak{g}), \widehat{\mathrm{d}}_{\text{TLA}})$.

$(\Omega^\bullet_{\text{TLA}}(\mathcal{P}, \mathfrak{g})_{\mathfrak{g}_{\text{equ}}}, \widehat{\mathrm{d}}_{\text{TLA}})$ the differential graded subcomplex of basic elements.

Theorem (S. Lazzarini, T.M.)

If G is connected and simply connected then
$(\Omega^\bullet_{\text{Lie}}(\mathcal{P}, \mathfrak{g}), \widehat{\mathrm{d}})$ *is isomorphic to* $(\Omega^\bullet_{\text{TLA}}(\mathcal{P}, \mathfrak{g})_{\mathfrak{g}_{\text{equ}}}, \widehat{\mathrm{d}}_{\text{TLA}})$

➞ $\Omega^\bullet_{\text{Lie}}(\mathcal{P}, \mathfrak{g}) \quad \subset \quad \Omega^\bullet_{\text{TLA}}(\mathcal{P}, \mathfrak{g}) \simeq \Omega^\bullet(\mathcal{P}) \otimes \bigwedge^\bullet \mathfrak{g}^* \otimes \mathfrak{g}.$

The global picture so far

- Transitive Lie algebroids = s.e.s. of Lie algebras and $C^\infty(\mathcal{M})$-modules

$$0 \longrightarrow L \xrightarrow{\iota} A \xrightarrow{\rho} \Gamma(T\mathcal{M}) \longrightarrow 0$$

- Generalized forms: $(\Omega^\bullet(A, L), \widehat{\mathrm{d}})$, graded differential Lie algebra.
  ➞ "Contains" ordinary de Rham calc. on $\mathcal{M}$ (basic elements for op. of L).
- Local description of transitive Lie algebroids and diff. calc. using TLA.
  ➞ $(\Omega^\bullet(\mathcal{M}) \otimes \bigwedge^\bullet \mathfrak{g}^* \otimes \mathfrak{g}, \widehat{\mathrm{d}} = \mathrm{d} + \mathrm{s}')$.
  ➞ useful for computations and definitions of structures...
- Representation theory on derivations of a vector bundle.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{\iota} & A & \xrightarrow{\rho} & \Gamma(T\mathcal{M}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi_L} & & \downarrow{\scriptstyle \phi} & & \| & & \\ 0 & \longrightarrow & \mathbf{A}(\mathcal{E}) & \xrightarrow{\iota} & \mathfrak{D}(\mathcal{E}) & \xrightarrow{\sigma} & \Gamma(T\mathcal{M}) & \longrightarrow & 0 \end{array}$$

- Principal fiber bundle ➞ canonical Atiyah Lie algebroid.

Connections and covariant derivatives

Ordinary connections

$0 \longrightarrow L \xrightarrow{\iota} A \xrightarrow{\rho} \Gamma(T\mathcal{M}) \longrightarrow 0$ a transitive Lie algebroid.

Definition (Connection on a transitive Lie algebroid)

A connection on A is a splitting $\nabla : \Gamma(T\mathcal{M}) \to A$ as $C^\infty(\mathcal{M})$-modules of the s.e.s.

$$0 \longrightarrow L \underset{\iota}{\overset{\omega^\nabla}{\leftrightarrows}} A \underset{\rho}{\overset{\nabla}{\leftrightarrows}} \Gamma(T\mathcal{M}) \longrightarrow 0$$

The curvature of ∇ is defined as the obstruction to be a morphism of Lie algebras:

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \in \iota(L)$$

∇ defines $\omega^\nabla : A \to L$ (s.e.s. properties) s.t. $\mathfrak{X} = \nabla_X - \iota \circ \omega^\nabla(\mathfrak{X})$, $\mathfrak{X} \in A$, $X = \rho(\mathfrak{X})$.

Proposition

One has $\omega^\nabla \in \Omega^1(A, L)$ and $\omega^\nabla \circ \iota(\ell) = -\ell$ for any $\ell \in L$ (normalization on L).

The 2-form $R^\nabla \in \Omega^2(A, L)$ defined by $R^\nabla(\mathfrak{X}, \mathfrak{Y}) = (\widehat{\mathrm{d}}\omega^\nabla)(\mathfrak{X}, \mathfrak{Y}) + [\omega^\nabla(\mathfrak{X}), \omega^\nabla(\mathfrak{Y})]$ vanishes when $\mathfrak{X}$ or $\mathfrak{Y}$ in $\iota(L)$, and one has $\iota \circ R^\nabla(\mathfrak{X}, \mathfrak{Y}) = R(X, Y)$.

ω^∇ is the connection 1-form associated to ∇.

Ordinary connections on Atiyah Lie algebroid

$$0 \longrightarrow \Gamma_G(\mathcal{P}, \mathfrak{g}) \xrightarrow{\iota} \Gamma_G(T\mathcal{P}) \xrightarrow{\pi_*} \Gamma(T\mathcal{M}) \longrightarrow 0$$

Proposition (Connections)

Ordinary connection on the Atiyah Lie algebroid = connection on $\mathcal{P}$.
The notions of curvature coincide.

➡ This example explains the terminology "ordinary connection".

The geometric equivalence:

A connection on $\mathcal{P}$ defines the horizontal lift $\Gamma(T\mathcal{M}) \to \Gamma_G(T\mathcal{P}), X \mapsto X^h$.

The algebraic equivalence:

Suppose G is connected and simply connected.
$\omega^{\mathcal{P}} \in \Omega^1(\mathcal{P}) \otimes \mathfrak{g}$ a connection 1-form on $\mathcal{P}$.
$\theta \in \mathfrak{g}^* \otimes \mathfrak{g}$ the Maurer-Cartan 1-form on G.
$\widehat{\omega}_{\mathfrak{g}_{\text{equ}}} = \omega^{\mathcal{P}} - \theta \quad \in \quad \Omega^1_{\text{TLA}}(\mathcal{P}, \mathfrak{g}) \quad \subset \quad \Omega^\bullet(\mathcal{P}) \otimes \textstyle\bigwedge^\bullet \mathfrak{g}^* \otimes \mathfrak{g}$ is $\mathfrak{g}_{\text{equ}}$-basic.
It corresponds to the connection 1-form $\omega^\nabla \in \Omega^1_{\text{Lie}}(\mathcal{P}, \mathfrak{g})$ associated to $\omega^{\mathcal{P}}$.

Generalized connections

Definition (Generalized connection)

A generalized connection on a transitive Lie algebroid A is a 1-form $\widehat{\omega} \in \Omega^1(A, L)$.
The curvature of $\widehat{\omega}$ is the 2-form $\widehat{R} = \widehat{\mathrm{d}}\widehat{\omega} + \frac{1}{2}[\widehat{\omega}, \widehat{\omega}] \in \Omega^2(A, L)$.

A generalized connection is an ordinary connection iff $\widehat{\omega} \circ \iota = -\mathrm{Id}_L$.

Consider a representation of A on $\mathcal{E}$:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & L & \overset{\iota}{\underset{\widehat{\omega}}{\rightleftarrows}} & A & \xrightarrow{\rho} & \Gamma(T\mathcal{M}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi_L} & & {\scriptstyle \widehat{\nabla}}\downarrow{\scriptstyle \phi} & & \| & & \\
0 & \longrightarrow & \mathbf{A}(\mathcal{E}) & \xrightarrow{\iota} & \mathfrak{D}(\mathcal{E}) & \xrightarrow{\sigma} & \Gamma(T\mathcal{M}) & \longrightarrow & 0
\end{array}
$$

$\widehat{\omega}$ defines $\widehat{\nabla} : A \to \mathfrak{D}(\mathcal{E})$ by $\widehat{\nabla}_{\mathfrak{X}} = \phi(\mathfrak{X}) + \iota \circ \phi_L(\widehat{\omega}(\mathfrak{X}))$.

This is the **covariant derivative** on $\mathcal{E}$ associated to $\widehat{\omega}$.

$[\widehat{\nabla}_{\mathfrak{X}}, \widehat{\nabla}_{\mathfrak{Y}}] - \widehat{\nabla}_{[\mathfrak{X},\mathfrak{Y}]} = \iota \circ \phi_L \circ \widehat{R}(\mathfrak{X}, \mathfrak{Y})$ ➞ $\widehat{\nabla}$ is not a representation in general.

Other terminologies for $\widehat{\nabla}$: “generalized representation”, “A-connection”...

Generalized connections on Atiyah Lie algebroids

$$0 \longrightarrow \Gamma_G(\mathcal{P}, \mathfrak{g}) \xrightarrow{\iota} \Gamma_G(T\mathcal{P}) \xrightarrow{\pi_*} \Gamma(T\mathcal{M}) \longrightarrow 0$$

To simplify the presentation: suppose G is connected and simply connected.

A generalized connection $\widehat{\omega}$ on $\Gamma_G(T\mathcal{P})$ is a $\mathfrak{g}_{\text{equ}}$-basic 1-form $\widehat{\omega}_{\mathfrak{g}_{\text{equ}}} \in \Omega^1_{\text{TLA}}(\mathcal{P}, \mathfrak{g})$.

$\widehat{\omega}_{\mathfrak{g}_{\text{equ}}} = \omega + \varphi \quad \in \quad (\Omega^1(\mathcal{P}) \otimes \mathfrak{g}) \oplus (C^\infty(\mathcal{P}) \otimes \mathfrak{g}^* \otimes \mathfrak{g})$.

Proposition (Ordinary *versus* generalized connections)

If $\varphi = -\theta$, then $\widehat{\omega}$ is an ordinary connection on $\Gamma_G(T\mathcal{P})$.
➞ *ω is an (ordinary) connection 1-form on $\mathcal{P}$.*

Otherwise, $\varphi + \theta$ measures the deviation of $\widehat{\omega}$ from an ordinary connection.
➞ $\varphi + \theta \simeq$ Higgs scalar fields...

Connections: a summary

- Ordinary connection on a transitive Lie algebroid = splitting:

$$0 \longrightarrow L \underset{\iota}{\longrightarrow} A \underset{\rho}{\longrightarrow} \Gamma(T\mathcal{M}) \longrightarrow 0 \qquad \omega^\nabla : A \to L, \quad \nabla : \Gamma(T\mathcal{M}) \to A$$

  ∇ ➞ $\omega^\nabla \in \Omega^1(A, L)$ connection 1-form, curvature as a 2-form.

- Generalized connections are any 1-forms $\widehat{\omega} \in \Omega^1(A, L)$.
  - ➞ Covariant derivatives on representations.
  - ➞ Notion of curvature.
- Ordinary connection = normalized generalized connection: $\widehat{\omega} \circ \iota(\ell) = -\ell$ for any $\ell \in L$
- For Atiyah Lie algebroids:
  - space of ordinary connections on $\mathcal{P} \subset$ space of generalized connections;
  - connection 1-forms and curvatures are directly related in $\Omega^\bullet_{\text{TLA}}(\mathcal{P}, \mathfrak{g})$.

The gauge group

Gauge group of a representation

Suppose given a representation of a transitive Lie algebroid A on $\mathcal{E}$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{\iota} & A & \xrightarrow{\rho} & \Gamma(T\mathcal{M}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi_L} & & \downarrow{\scriptstyle \phi} & & \| & & \\ 0 & \longrightarrow & \mathbf{A}(\mathcal{E}) & \xrightarrow{\iota} & \mathfrak{D}(\mathcal{E}) & \xrightarrow{\sigma} & \Gamma(T\mathcal{M}) & \longrightarrow & 0 \end{array}$$

Definition (Gauge group of a representation)

The gauge group of $\mathcal{E}$ is the group $\mathrm{Aut}(\mathcal{E}) \subset \mathbf{A}(\mathcal{E})$ (vertical automorphisms of $\mathcal{E}$).

No (finite) gauge transformation at the level of A (similar situation in NCG).

Any $\xi \in L$ defines an infinitesimal gauge transformation on $\Gamma(\mathcal{E})$ by $\varphi \mapsto \phi_L(\xi)\varphi$.

Definition (Infinitesimal gauge transformations)

An infinitesimal gauge transformation on A is an element $\xi \in L$.

Gauge transformations

$\xi \in L \longrightarrow g = e^{\phi_L(\xi)} \simeq 1 + \phi_L(\xi) + \cdots \in \mathrm{Aut}(\mathcal{E}) \subset \mathbf{A}(\mathcal{E})$

$\widehat{\omega}$ generalized connection on A, and $\widehat{\nabla}$ its associated covariant derivative on $\mathcal{E}$:

$$\widehat{\nabla}_{\mathfrak{X}}\varphi = \phi(\mathfrak{X})\cdot\varphi + \phi_L(\widehat{\omega}(\mathfrak{X}))\varphi$$

The first order diff. op. $\widehat{\nabla}^g_{\mathfrak{X}} = g^{-1} \circ \widehat{\nabla}_{\mathfrak{X}} \circ g$ on $\mathcal{E}$ can be written as

$$\widehat{\nabla}^g_{\mathfrak{X}}\varphi = \phi(\mathfrak{X})\cdot\varphi + \phi_L(\widehat{\omega}(\mathfrak{X}))\varphi + \phi_L(\widehat{\mathrm{d}}\xi(\mathfrak{X}) + [\widehat{\omega}(\mathfrak{X}), \xi])\varphi + O(\xi^2)\varphi$$

Definition (Infinitesimal gauge variation)

The infinitesimal gauge variation of $\widehat{\omega}$ induced by ξ is defined to be $\widehat{\mathrm{d}}\xi + [\widehat{\omega}, \xi]$.

$\longrightarrow$ The infinitesimal gauge variation of the curvature $\widehat{R}$ of $\widehat{\omega}$ is $[\widehat{R}, \xi]$.

The gauge principle is implemented on A at the infinitesimal level (indep. of a rep.).
$\longrightarrow$ Similar to ordinary differential geometry.

Gauge transformations on Atiyah Lie algebroids

$$0 \longrightarrow \Gamma_G(\mathcal{P}, \mathfrak{g}) \xrightarrow{\iota} \Gamma_G(T\mathcal{P}) \xrightarrow{\pi_*} \Gamma(T\mathcal{M}) \longrightarrow 0$$

$\mathcal{G}(\mathcal{P})$ the gauge group of $\mathcal{P}$ (vertical automorphisms of $\mathcal{P}$).
$u \in \mathcal{G}(\mathcal{P})$ is a G-equivariant map $u : \mathcal{P} \to G$, $u(p{\cdot}g) = g^{-1}u(p)g$.

- **Finite gauge transformations are defined.**
- $L = \Gamma_G(\mathcal{P}, \mathfrak{g})$ is the Lie algebra of $\mathcal{G}(\mathcal{P})$.
- Infinitesimal (usual) gauge transformations are elements in L.

$\widehat{\omega} \in \Omega^1_{\text{Lie}}(\mathcal{P}, \mathfrak{g})$ and $u \in \mathcal{G}(\mathcal{P})$.
Define $\widehat{\omega}^u(\mathfrak{X}) = u^{-1}\widehat{\omega}(\mathfrak{X})u + u^{-1}(\mathfrak{X}{\cdot}u)$ for any $\mathfrak{X} \in \Gamma_G(T\mathcal{P})$.

- $\widehat{\omega}^u \in \Omega^1_{\text{Lie}}(\mathcal{P}, \mathfrak{g})$;
- induced by $\widehat{\nabla} \mapsto \widehat{\nabla}^u$;
- infinitesimal gauge transformations on $\widehat{\omega}$ are induced by $\widehat{\omega} \mapsto \widehat{\omega}^u$;
- restricts to (ordinary) gauge transformation on ordinary connections...
  ➞ preserves the decomposition $\omega^{\mathcal{P}} - \theta$.

Structures to construct an action functional

Metrics on transitive Lie algebroids

All the structures rely on a notion of metric...

Definition (Metric on a Lie algebroid)

A metric on A is a symmetric $C^\infty(\mathcal{M})$-linear map $\widehat{g} : A \otimes_{C^\infty(\mathcal{M})} A \to C^\infty(\mathcal{M})$.

$\widehat{g}$ defines a metric $h = \iota^* \widehat{g}$ on L given by $h(\gamma, \eta) = \widehat{g}(\iota(\gamma), \iota(\eta))$ for any $\gamma, \eta \in L$.
➞ $\widehat{g}$ is **inner non degenerate** if h is non degenerate on L.

Proposition (C. Fournel, S. Lazzarini, T.M.)

An inner non degenerate metric $\widehat{g}$ on A is equivalent to a triple (g, h, ∇) where

- *g is a (possibly degenerate) metric on $\mathcal{M}$;*
- *h is a non degenerate metric on L;*
- *∇ is an ordinary connection on A, with $\mathring{\omega} \in \Omega^1(A, L)$ its connection 1-form;*
- $\widehat{g}(\mathfrak{X}, \mathfrak{Y}) = g(\rho(\mathfrak{X}), \rho(\mathfrak{Y})) + h(\mathring{\omega}(\mathfrak{X}), \mathring{\omega}(\mathfrak{Y}))$.
- $\widehat{g}(\nabla_X, \iota(\gamma)) = 0$ *for any* $X \in \Gamma(T\mathcal{M})$ *and* $\gamma \in L$.

Given $\widehat{g}$, look at $\mathring{\omega}$ as a background connection...

Integration along the kernel

∇ a connection on A, $\mathring{\omega} \in \Omega^1(A, L)$ its connection 1-form.
h a metric on L.

Suppose $\mathcal{L}$ is orientable where $L = \Gamma(\mathcal{L})$ (➡ A is called inner orientable), let $n = \text{rank}(\mathcal{L})$.

Proposition (Volume form along L and inner integration)

h and $\mathring{\omega}$ define a global form in $\Omega^\bullet(A)$ of maximal degree in the L direction.

This volume form defines integrations

$$\int_{\text{inner}} : \Omega^\bullet(A) \to \Omega^{\bullet - n}(\mathcal{M}) \qquad \int_{\text{inner}} : \Omega^\bullet(A, L) \to \Omega^{\bullet - n}(\mathcal{M}, \mathcal{L}).$$

They do not depend on ∇.

➡ After integration, only geometrical structures (de Rham).

Integration on A

Suppose also that $\mathcal{M}$ is orientable (➡ A is called orientable) and g non degenerate.

Definition (Integration on A)

Using g, the integration on A of a form $\widehat{\omega} \in \Omega^\bullet(A)$ is defined by

$$\int_A \widehat{\omega} = \int_{\mathcal{M}} \int_{\text{inner}} \widehat{\omega} \in \mathbb{C}.$$

Definition (Scalar product of forms)

The scalar product of any 2 forms $\widehat{\omega}_1, \widehat{\omega}_2 \in \Omega^\bullet(A, L)$ is defined by

$$\langle \widehat{\omega}_1, \widehat{\omega}_2 \rangle = \int_A h(\widehat{\omega}_1, \widehat{\omega}_2) \in \mathbb{C}$$

Metrics: a summary

A non degenerate metric $\widehat{g} = (g, h, \nabla)$, with $\nabla \leftrightarrow \mathring{\omega} \in \Omega^1(A, L)$, gives us:

- h ➡ scalar product on L;
- $h, \mathring{\omega}$ ➡ integration along L;
- g ➡ integration on $\mathcal{M}$;
- $g, h, \mathring{\omega}$ ➡ integration on A;
- $g, h, \mathring{\omega}$ ➡ Hodge star operator (straighforward to define)

$$\star : \Omega^p(A, L) \to \Omega^{m+n-p}(A, L)$$

Gauge theories

Gauge invariant action

A orientable transitive Lie algebroid, $\widehat{g} = (g, h, \nabla)$ non degenerate metric.

Suppose h is a Killing metric: $h([\xi, \gamma], \eta) + h(\gamma, [\xi, \eta]) = 0$ for any $\gamma, \eta, \xi \in L$.

$\widehat{\omega} \in \Omega^1(A, L)$ a connection on A and $\widehat{R}$ its curvature 2-form.

Proposition (C. Fournel, S. Lazzarini, T.M.)

The action functional

$$\mathcal{S}_{\text{Gauge}}[\widehat{\omega}] = \langle \widehat{R}, \star\widehat{R} \rangle = \int_A h(\widehat{R}, \star\widehat{R}).$$

is invariant under infinitesimal gauge transformations in L.

Example (Atiyah Lie algebroid)

A the Atiyah Lie algebroid of a G-principal fiber bundle $\mathcal{P}$.
h induced by the Killing form on $\mathfrak{g}$ (semisimple).
$\widehat{\omega}$ any generalized connection ➞ $\mathcal{S}_{\text{Gauge}}[\widehat{\omega}]$ is $\mathcal{G}(\mathcal{P})$-gauge invariant.
$\widehat{\omega}$ an ordinary connection on A ➞ $\mathcal{S}_{\text{Gauge}}[\widehat{\omega}]$ is the ordinary Yang-Mills action.

Possible to define $\mathcal{S}_{\text{Matter}}[\varphi, \widehat{\omega}]$ for $\varphi \in \Gamma(\mathcal{E})$ where $A \to \mathfrak{D}(\mathcal{E})$ is a representation of A.

Decomposition of a connection

$\widehat{\omega} \in \Omega^1(A, L)$ a generalized connection on A.

$\tau \in \mathrm{End}(\mathcal{L})$ defined by

$$\tau = \widehat{\omega} \circ \iota + \mathrm{Id}_L.$$

τ vanishes iff $\widehat{\omega}$ is an ordinary connection on A

➞ measures the "non Yang-Mills" part (Higgs scalar fields).

⚠ τ is not a Lie morphism.

$\widehat{g} = (g, h, \nabla)$ metric with $\nabla \leftrightarrow \mathring{\omega} \in \Omega^1(A, L)$.

Proposition

$\omega = \widehat{\omega} + \tau(\mathring{\omega}) \in \Omega^1(A, L)$ is an ordinary connection on A.
The induced infinitesimal gauge action of L is the one on ordinary connections.

$\widehat{g} = (g, h, \nabla)$ decomposes any connection $\widehat{\omega}$ on A as:

$$\widehat{\omega} \leftrightarrow (\omega, \tau) \quad \text{ordinary connection on } A + \text{algebraic object on } L$$

$\widehat{\omega}$ ordinary connection ➞ $\tau = 0$ ➞ $\omega = \widehat{\omega}$.

The total action functional

Using the decomposition $\widehat{\omega} \leftrightarrow (\omega, \tau)$:

$$\mathcal{S}_{\text{Gauge}}[\widehat{\omega}] + \mathcal{S}_{\text{Matter}}[\varphi, \widehat{\omega}] = \text{(1) Yang-Mills like term for } \omega$$

(2) covariant derivative for τ along ω

(3) potential for τ

(4) covariant derivative for φ along ω

(5) coupling $\varphi \leftrightarrow \tau$

The potential (3) can vanish for $\tau \neq 0$.
A development around a solution $\tau_0 \neq 0$ induces:

- A mass term for the ordinary connection ω in (2).
- A mass term for φ in (5).

➙ Massive bosons (ω) coupled to massive particles (φ).
➙ Yang-Mills-Higgs type gauge theory.

Why Higgs fields?

A pattern for Yang-Mills-Higgs gauge field theories (on “ordinary space-time”):

Algebraic Structure —inclusion→ **Global Structure** —projection→ **Geometric Structure**

Generalized connection

$\widehat{\omega} \simeq \omega + \tau$

$\tau \simeq$ scalar fields $\qquad$ $\omega \simeq$ Yang-Mills fields

→ Spontaneous symmetry breaking ←

Transitive Lie algebroids: $0 \longrightarrow L \longrightarrow A \longrightarrow \Gamma(T\mathcal{M}) \longrightarrow 0$

NCG: $1 \longrightarrow \mathrm{Inn}(\mathbf{A}) \longrightarrow \mathrm{Aut}(\mathbf{A}) \longrightarrow \mathrm{Out}(\mathbf{A}) \longrightarrow 1$

François, J., Lazzarini, S., and Masson, T. (2014). “Gauge field theories: various mathematical approaches”. In: *Mathematical Structures of the Universe*. Ed. by Eckstein, M., Heller, M., and Szybka, S. J. Kraków, Poland: Copernicus Center Press, pp. 177–225

Conclusion

- (Geometric) Gauge field theories can be generalized in at least two directions:
  - Noncommutative Geometry
  - Transitive Lie Algebroids.
- Same pattern: add some purely algebraic directions to space-time.
  ➞ Yang-Mills-Higgs type gauge theories.
- Gauge theories on Atiyah Lie algebroids are close to Yang-Mills gauge theories.
  - They contain ordinary gauge theories used in physics.
  - They share common mathematical structures.
  - No restriction on the gauge group.
- A lot more to investigate:
  - Relation to BRST structures...
  - Construction of realistic models...
  - Relation with "dressing field" method elaborated in
    Fournel, C., François, J., Lazzarini, S., and Masson, T. (2014). Gauge invariant composite fields out of connections, with examples. *Int. J. Geom. Methods Mod. Phys.* 11.1, p. 1450016
  - Here, generalization of Ehresman's connections:
    ➞ we investigate generalization of Cartan's connections
    (used in gravitational and conformal theories).

Trivialization of transitive Lie algebroids

$0 \longrightarrow L \xrightarrow{\iota} A \xrightarrow{\rho} \Gamma(T\mathcal{M}) \longrightarrow 0$ a transitive Lie algebroid.

A local trivialization of A is a triple $(\mathcal{U}, \Psi, \nabla^0)$ where

- $\mathcal{U}$ is an open subset of $\mathcal{M}$;
- $\Psi : \Gamma(\mathcal{U} \times \mathfrak{g}) \xrightarrow{\simeq} L_{\mathcal{U}}$ = isomorphism of Lie algebras and $C^\infty(\mathcal{U})$-modules;
- $\nabla^0 : \Gamma(T\mathcal{U}) \to A_{\mathcal{U}}$ = injective morphism of Lie algebras and $C^\infty(\mathcal{U})$-modules compatible ρ;
- $[\nabla^0_X, \iota \circ \Psi(\gamma)] = \iota \circ \Psi(X{\cdot}\gamma)$ for any $X \in \Gamma(T\mathcal{U})$ and any $\gamma \in \Gamma(\mathcal{U} \times \mathfrak{g})$.

$S(X \oplus \gamma) = \nabla^0_X + \iota \circ \Psi(\gamma)$ is a isomorphism of Lie algebroids $S : \text{TLA}(\mathcal{U}, \mathfrak{g}) \xrightarrow{\simeq} A_{\mathcal{U}}$.

Atlas for A = family of local trivializations $\{(\mathcal{U}_i, \Psi_i, \nabla^{0,i})\}_{i\in I}$ with $\bigcup_{i\in I} \mathcal{U}_i = \mathcal{M}$.

$\mathfrak{X} \in A$ is decomposed as $X^i \oplus \gamma^i \in \text{TLA}(\mathcal{U}_i, \mathfrak{g})$ such that $S_i(X^i \oplus \gamma^i) = \mathfrak{X}_{|\mathcal{U}_i}$.
The X^i's are the restrictions to $\mathcal{U}_i$ of the global vector field $X = \rho(\mathfrak{X})$.

On $\mathcal{U}_{ij} = \mathcal{U}_i \cap \mathcal{U}_j \neq \varnothing$ one can define $\alpha^i_j = \Psi_i^{-1} \circ \Psi_j : \mathcal{U}_{ij} \to \text{Aut}(\mathfrak{g})$.
$\exists \chi_{ij} \in \Omega^1(\mathcal{U}_{ij}) \otimes \mathfrak{g}$ such that $\gamma^i = \alpha^i_j(\gamma^j) + \chi_{ij}(X)$.
Cocycle relations:

$$\alpha^i_k = \alpha^i_j \circ \alpha^j_k \qquad \alpha^i_j \circ \alpha^j_i = \text{Id} \qquad \chi_{ik} = \alpha^i_j \circ \chi_{jk} + \chi_{ij} \qquad \alpha^i_j \circ \chi_{ji} + \chi_{ij} = 0$$

Trivialization of differential forms

$0 \longrightarrow L \xrightarrow{\iota} A \xrightarrow{\rho} \Gamma(T\mathcal{M}) \longrightarrow 0$ a transitive Lie algebroid.

$\{(\mathcal{U}_i, \Psi_i, \nabla^{0,i})\}_{i\in I}$ a Lie algebroid atlas for A.

$\omega \in \Omega^q(A, L)$ ➡ family of local q-forms $\omega^i_{\text{loc}} \in \Omega^q_{\text{TLA}}(\mathcal{U}_i, \mathfrak{g})$

$$\omega^i_{\text{loc}} = \Psi_i^{-1} \circ \omega \circ S_i$$

$s^j_i = S_j^{-1} \circ S_i : \text{TLA}(\mathcal{U}_{ij}, \mathfrak{g}) \xrightarrow{\simeq} \text{TLA}(\mathcal{U}_{ij}, \mathfrak{g})$ = isomorphism of TLA.

$\widehat{\alpha}^i_j : \Omega^q_{\text{TLA}}(\mathcal{U}_{ij}, \mathfrak{g}) \to \Omega^q_{\text{TLA}}(\mathcal{U}_{ij}, \mathfrak{g})$ defined by $\widehat{\alpha}^i_j(\omega^j_{\text{loc}}) = \alpha^i_j \circ \omega^j_{\text{loc}} \circ s^j_i$.

Proposition

- *A family of local forms $\omega^i_{\text{loc}} \in \Omega^\bullet_{\text{TLA}}(\mathcal{U}_i, \mathfrak{g})$ is a system of trivializations of a global form $\omega \in \Omega^\bullet(A, L)$ if and only if $\widehat{\alpha}^i_j(\omega^j_{\text{loc}}) = \omega^i_{\text{loc}}$ on any $\mathcal{U}_{ij} \neq \varnothing$.*
- *For any $\omega \in \Omega^\bullet(A, L)$ trivialized on $\mathcal{U}$ as ω_{loc}, one has $\widehat{\text{d}}_{\text{TLA}}\omega_{\text{loc}} = \Psi^{-1} \circ (\widehat{\text{d}}\omega) \circ S$.*
- *$\widehat{\alpha}^i_j : \Omega^\bullet_{\text{TLA}}(\mathcal{U}_{ij}, \mathfrak{g}) \to \Omega^\bullet_{\text{TLA}}(\mathcal{U}_{ij}, \mathfrak{g})$ = isomorphism of grad. diff. Lie algebras.*

Local mixed basis

$\mathfrak{g}$ the Lie algebra fiber of $\mathcal{L}$ where $L = \Gamma(\mathcal{L}), \mathcal{U} \subset \mathcal{M}$ open subset which trivializes A.

$\{E_a\}_{1 \leq a \leq n}$ basis of $\mathfrak{g}$, $\{\theta^a\}_{1 \leq a \leq n}$ dual basis of $\mathfrak{g}^*$.

$\widehat{\omega} \in \Omega^p(A, L)$ and $\widehat{\omega}_{\text{loc}} \in \Omega^p_{\text{TLA}}(\mathcal{U}, \mathfrak{g})$ its local description:

$$\widehat{\omega}_{\text{loc}} = \sum_{r+s=p} \widehat{\omega}^{\theta}_{\mu_1 \dots \mu_r a_1 \dots a_s} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_r} \wedge \theta^{a_1} \wedge \cdots \wedge \theta^{a_s}, \quad \widehat{\omega}^{\theta}_{\mu_1 \dots \mu_r a_1 \dots a_s} : \mathcal{U} \to \mathfrak{g}$$

⚠ θ^a is not convenient ➡ inhomogeneous transformations!

∇ ordinary connection on A, $\mathring{\omega}$ its connection 1-form

➡ $\mathring{\omega}_{\text{loc}} = (A^a - \theta^a) E_a$ with $A^a \in \Omega^1(\mathcal{U}) \otimes \mathfrak{g}$,

➡ the $\mathring{\omega}^a = A^a - \theta^a \in \Omega^1_{\text{TLA}}(\mathcal{U})$ define the **mixed basis** in $\Omega^1_{\text{TLA}}(\mathcal{U})$.

Then one can write

$$\widehat{\omega}_{\text{loc}} = \sum_{r+s=p} \widehat{\omega}_{\mu_1 \dots \mu_r a_1 \dots a_s} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_r} \wedge \mathring{\omega}^{a_1} \wedge \cdots \wedge \mathring{\omega}^{a_s}, \quad \widehat{\omega}_{\mu_1 \dots \mu_r a_1 \dots a_s} : \mathcal{U} \to \mathfrak{g}$$

Proposition (Homogeneous transformations)

The $\widehat{\omega}_{\mu_1 \dots \mu_r a_1 \dots a_s}$'s have homog. transf. in a change of local trivializations.

Integration along the kernel

$\mathcal{U} \subset \mathcal{M}$ an open subset which trivializes A. $\{E_a\}$ basis of $\mathfrak{g}$, $\{\theta^a\}$ dual basis of $\mathfrak{g}^*$.

∇ a connection on A, $\mathring{\omega} \in \Omega^1(A, L)$ its connection 1-form.
➡ $\mathring{\omega}^a = A^a - \theta^a \in \Omega^1_{\mathrm{TLA}}(\mathcal{U})$ mixed basis in $\Omega^1_{\mathrm{TLA}}(\mathcal{U})$.

h a metric on L.
$h_{\mathrm{loc}} =$ trivialization of h over $\mathcal{U}$, $h_{ab} = h_{\mathrm{loc}}(E_a, E_b) \in C^\infty(\mathcal{U})$, $|h_{\mathrm{loc}}| = |\det(h_{ab})|$.

Suppose $\mathcal{L}$ is orientable where $L = \Gamma(\mathcal{L})$ (A is called inner orientable),

Proposition (Volume form along L and inner integration)

$$\widehat{\omega}_{h, \mathring{\omega}\ \mathrm{loc}} = (-1)^n \sqrt{|h_{\mathrm{loc}}|}\ \mathring{\omega}^1_{\mathrm{loc}} \wedge \cdots \wedge \mathring{\omega}^n_{\mathrm{loc}}$$

defines a global form $\widehat{\omega}_{h, \mathring{\omega}} \in \Omega^\bullet(A)$ *of maximal degree* $n = \dim \mathfrak{g}$ *in the* L *direction.*

This volume form defines integrations

$$\int_{\mathrm{inner}} : \Omega^\bullet(A, L) \to \Omega^{\bullet - n}(\mathcal{M}, \mathcal{L}) \qquad \int_{\mathrm{inner}} : \Omega^\bullet(A) \to \Omega^{\bullet - n}(\mathcal{M}).$$

They do not depend on ∇.

Hodge star operator

A an orientable transitive Lie algebroid.
$\widehat{g} = (g, h, \nabla)$ a metric on $\mathbf{A}$, $\mathring{\omega}$ the connection 1-form of ∇.

$\widehat{\omega} \in \Omega^p(A, L)$, written locally as

$$\widehat{\omega}_{\text{loc}} = \sum_{r+s=p} \widehat{\omega}_{\mu_1 \dots \mu_r a_1 \dots a_s} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_r} \wedge \mathring{\omega}^{a_1} \wedge \cdots \wedge \mathring{\omega}^{a_s}$$

$\star\widehat{\omega}_{\text{loc}} \in \Omega^{m+n-p}_{\text{TLA}}(U, \mathfrak{g})$ is defined by (usual notations)

$$\star\widehat{\omega}_{\text{loc}} = \sum_{r+s=p} (-1)^{s(m-r)} \frac{1}{r!s!} \sqrt{|h_{\text{loc}}|}\sqrt{|g|}\, \widehat{\omega}_{\mu_1 \dots \mu_r a_1 \dots a_s}\, \epsilon_{\nu_1 \dots \nu_m}\, \epsilon_{b_1 \dots b_n}$$
$$\times g^{\mu_1 \nu_1} \cdots g^{\mu_r \nu_r}\, h^{a_1 b_1} \cdots h^{a_s b_s}\, \mathrm{d}x^{\nu_{r+1}} \wedge \cdots \wedge \mathrm{d}x^{\nu_m} \wedge \mathring{\omega}^{b_{s+1}} \wedge \cdots \wedge \mathring{\omega}^{b_n}$$

Proposition (Hodge star operator)

The map $\star : \Omega^p(A, L) \to \Omega^{m+n-p}(A, L)$ is well defined globally.

This is the Hodge star operator associated to $\widehat{g}$ on A.

Gauge transformations on Atiyah Lie algebroids

Suppose G is connected and simply connected.
$\widehat{\omega} \in \Omega^1_{\text{Lie}}(\mathcal{P}, \mathfrak{g}) \rightarrow \widehat{\omega}_{\mathfrak{g}_{\text{equ}}} \in \Omega^1_{\text{TLA}}(\mathcal{P}, \mathfrak{g})$, $\mathfrak{g}_{\text{equ}}$-basic.

The gauge action $\widehat{\omega} \mapsto \widehat{\omega}^u$ induces

$$\widehat{\omega}_{\mathfrak{g}_{\text{equ}}} \mapsto u \widehat{\omega}_{\mathfrak{g}_{\text{equ}}} u^{-1} + u \widehat{\text{d}}_{\text{TLA}} u^{-1} \in \Omega^1_{\text{TLA}}(\mathcal{P}, \mathfrak{g})$$

where

$$u \widehat{\text{d}}_{\text{TLA}} u^{-1} = u \text{d} u^{-1} + u \theta u^{-1} - \theta$$

($\theta =$ Cartan 1-form)
Notice that $u \theta u^{-1} - \theta = u[\theta, u^{-1}]$ is more or less “s” applied to u.

Proposition (Ordinary gauge transformations)

If $\widehat{\omega}_{\mathfrak{g}_{\text{equ}}} = \omega^{\mathcal{P}} - \theta$ is an ordinary connection on $\Gamma_G(T\mathcal{P})$, this action reduces to the usual gauge transformation $\omega^{\mathcal{P}} \mapsto u \omega^{\mathcal{P}} u^{-1} + u \text{d} u^{-1}$ on the (ordinay) connection 1-form $\omega^{\mathcal{P}}$.

Decomposition of a connection

$\widehat{\omega} \in \Omega^1(A, L)$ a generalized connection on A.

Definition (Reduced kernel endomorphism)

The reduced kernel endomorphism $\tau \in \mathrm{End}(\mathcal{L})$ associated to $\widehat{\omega}$ is defined by

$$\tau = \widehat{\omega} \circ \iota + \mathrm{Id}_L.$$

τ vanishes iff $\widehat{\omega}$ is an ordinary connection on A

➡ measures the "non Yang-Mills" part.

τ is not a Lie morphism. Define $R_\tau(\gamma, \eta) = [\tau(\gamma), \tau(\eta)] - \tau([\gamma, \eta])$ for any $\gamma, \eta \in L$.

Let $\mathring{\omega} \in \Omega^1(A, L)$ be a fixed ordinary connection on A ("background connection").

Theorem

$\widehat{\omega} \in \Omega^1(A, L)$ a connection and τ its reduced kernel endomorphism.

$$\omega = \widehat{\omega} + \tau(\mathring{\omega})$$

is an ordinary connection on A.
The induced infinitesimal gauge action of L is the one on ordinary connections.

$\widehat{\omega}$ ordinary connection ➡ $\tau = 0$ ➡ $\omega = \widehat{\omega}$.

➡ $\mathring{\omega}$ only relevant for connections which are not ordinary connections.

Decomposition of curvature and covariant derivative

$\widehat{\omega} = \omega - \tau(\mathring{\omega})$ connection on A.
$\mathring{\nabla}, \nabla : \Gamma(T\mathcal{M}) \to A$ the splittings associated to the ordinary connections $\mathring{\omega}, \omega$.

$\mathring{R}, R \in \Omega^2(\mathcal{M}, \mathcal{L})$ the curvature 2-forms of $\mathring{\omega}, \omega$.
$\widehat{F} = R - \tau \circ \mathring{R} \in \Omega^2(\mathcal{M}, \mathcal{L}) \longrightarrow \rho^* \widehat{F} \in \Omega^2(A, L)$.

For $X \in \Gamma(T\mathcal{M})$, define $\mathcal{D}_X \tau \in \mathrm{End}(\mathcal{L})$ by, for any $\gamma \in L$,
$(\mathcal{D}_X \tau)(\gamma) = [\nabla_X, \tau(\gamma)] - \tau([\mathring{\nabla}_X, \gamma]) \longrightarrow (\rho^* \mathcal{D}\tau) \circ \mathring{\omega} \in \Omega^2(A, L)$.

$\nabla^{\mathcal{E}}$ the (ordinary) covariant derivative induced on $\mathcal{E}$ by the (ordinary) connection ω.
For any $\varphi \in \Gamma(\mathcal{E})$, one has $\rho^* \phi(\nabla) \cdot \varphi = \rho^* \nabla^{\mathcal{E}} \varphi$.

Proposition (Decomposition of the curvature and the covariant derivative)

The curvature $\widehat{R} \in \Omega^2(A, L)$ of $\widehat{\omega}$ can be decomposed as

$$\widehat{R} = \rho^* \widehat{F} - (\rho^* \mathcal{D}\tau) \circ \mathring{\omega} + R_\tau \circ \mathring{\omega}$$

The covariant derivative $\widehat{\nabla}^{\mathcal{E}} \varphi \in \Omega^1(A, \mathcal{E})$ can be decomposed as

$$\widehat{\nabla}^{\mathcal{E}} \varphi = \rho^* \phi(\nabla) \cdot \varphi - (\phi_L(\tau)\varphi) \circ \mathring{\omega}$$

Under infinitesimal gauge transformations, each term has homog. transf.

"$\circ\mathring{\omega}$" = along the mixed basis and "ρ^*" = along $\Gamma(T\mathcal{M})$.

Coupling to matter fields

Matter fields are sections $\varphi \in \Gamma(\mathcal{E})$ of a representation $\phi : A \to \mathfrak{D}(\mathcal{E})$ of A.

Definition (ϕ_L-compatible metric)

A metric $h^{\mathcal{E}}$ on $\mathcal{E}$ is ϕ_L-compatible if

$$h^{\mathcal{E}}(\phi_L(\xi)\varphi_1, \varphi_2) + h^{\mathcal{E}}(\varphi_1, \phi_L(\xi)\varphi_2) = 0$$

for any φ_1, $\varphi_2 \in \Gamma(\mathcal{E})$ and any $\xi \in L$.

Generalization of "Killing metric".
One can define a Hodge star operator on $\Omega^\bullet(A, \mathcal{E})$.

$\widehat{\omega}$ connection on A and $\widehat{\nabla}^{\mathcal{E}}$ the induced covariant derivative on $\Gamma(\mathcal{E})$.

Proposition

The action functional

$$\mathcal{S}_{\text{Matter}}[\varphi, \widehat{\omega}] = \int_A h^{\mathcal{E}}(\widehat{\nabla}^{\mathcal{E}}\varphi, \star\widehat{\nabla}^{\mathcal{E}}\varphi)$$

is invariant under infinitesimal gauge transformations in L.

Decomposition of the action functional

$\widehat{\omega}$ connection on A, $\varphi \in \Gamma(\mathcal{E})$ matter field.

$$\mathcal{S}[\varphi, \widehat{\omega}] = \mathcal{S}_{\text{Gauge}}[\widehat{\omega}] + \mathcal{S}_{\text{Matter}}[\varphi, \widehat{\omega}] \quad \text{total action functional}$$

$\widehat{g} = (g, h, \nabla)$, with $\nabla \leftrightarrow \mathring{\omega} \in \Omega^1(A, L)$, metric on A.

The decomposition $\widehat{\omega} = \omega - \tau(\mathring{\omega})$ induces the decomposition:

$$\begin{aligned}
\mathcal{S}[\varphi, \widehat{\omega}] = \quad & \langle \rho^* \widehat{F}, \star \rho^* \widehat{F} \rangle && \text{(1) spatial term: Yang-Mills like} \\
& + \langle (\rho^* \mathcal{D}\tau) \circ \mathring{\omega}, \star (\rho^* \mathcal{D}\tau) \circ \mathring{\omega} \rangle && \text{(2) mixed term: covariant derivative of } \tau \\
& + \langle R_\tau \circ \mathring{\omega}, \star R_\tau \circ \mathring{\omega} \rangle && \text{(3) algebraic term: potential for } \tau \\
& + \langle \rho^* \phi(\nabla) \cdot \varphi, \star \rho^* \phi(\nabla) \cdot \varphi \rangle && \text{(4) spatial term: covariant derivative of } \varphi \\
& + \langle (\phi_L(\tau)\varphi) \circ \mathring{\omega}, \star (\phi_L(\tau)\varphi) \circ \mathring{\omega} \rangle && \text{(5) algebraic term: coupling } \varphi \leftrightarrow \tau
\end{aligned}$$