

Generalized connections and Higgs fields on Lie algebroids

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Abstract

Connections in noncommutative geometry are well known to naturally produce, in many situations, Higgs fields. We show that this feature is shared with a correct notion of generalized connections on transitive Lie algebroids. Comparison with noncommutative geometry will be given. This leads to a better understanding of the key mathematical structure at the heart of the construction of gauge field models with Higgs fields and spontaneous symmetry breaking.

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Motivations

- Discovery of Higgs particle in 2012.
 - ➔ need for a mathematical validation of the Higgs sector in the SM.
 - ☞ No clue from “traditional” schemes and tools.
- **NCG:** Higgs field is part of a “generalized connection”.

Dubois-Violette, M., Kerner, R., and Madore, J. (1990). Noncommutative Differential Geometry and New Models of Gauge Theory. *J. Math. Phys.* 31, p. 323

Connes, A. and Lott, J. (1991). Particle models and noncommutative geometry. *Nucl. Phys. B Proc. Suppl.* 18.2, pp. 29–47

 - ☞ Models in NCG can reproduce the Standard Model up to the excitement connected to the diphoton resonance at 750 GeV “seen” by ATLAS and CMS!
 - ☞ Mathematical structures difficult to master by particle physicists.
- **Transitive Lie algebroids:**
 - ➔ generalized connections, gauge symmetries, Yang-Mills-Higgs models...
 - ➔ Direct filiation from Dubois-Violette, Kerner, and Madore (1990).
 - ☞ Mathematics close to “usual” mathematics of Yang-Mills theories.
 - ☞ No realistic theory yet.

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How to construct a gauge field theory?

The basic ingredients are:

1. A space of local symmetries (space-time dependence):
 - ➔ a **gauge group**.
2. An implementation of the symmetry on matter fields:
 - ➔ a **representation theory**.
3. A notion of derivation:
 - ➔ some **differential structures**.
4. A (gauge compatible) replacement of ordinary derivations:
 - ➔ a **covariant derivative**.
5. A way to write a gauge invariant Lagrangian density:
 - ➔ **action functional**.

At least three mathematical schemes to construct gauge field theories:

- Ordinary differential geometry of principal fiber bundles.
- Noncommutative geometry.
- Transitive Lie algebroids (to be explained in this talk).

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Ordinary differential geometry

Given a G -principal fiber bundle \mathcal{P} over \mathcal{M} , the ingredients are

gauge group: $\mathcal{G}(\mathcal{P})$ is the group of vertical automorphisms of \mathcal{P} .

representation theory: sections of associated vector bundles.

→ Natural action of $\mathcal{G}(\mathcal{P})$.

differential structures: (ordinary) de Rham differential calculus.

covariant derivative: connection 1-form ω on \mathcal{P} .

→ covariant derivative on sections of any associated vector bundles.

action functional: integration on the base manifold \mathcal{M} , Killing form on the Lie algebra \mathfrak{g} of G , Hodge star operator, curvature of ω .

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Noncommutative geometry

Given an associative algebra \mathbf{A} , the ingredients are

representation theory: a right module M over \mathbf{A} .

gauge group: $\text{Aut}(M)$, the group of automorphisms of the right module.

differential structures: any differential calculus defined on top of \mathbf{A} .

→ many choices: spectral triples, derivations, twisted derivations...

covariant derivatives: noncommutative connections on M ,
(need a differential calculus).

action functional: depends on the differential calculus.

- spectral triples: spectral action...
- derivation-based differential calculus: noncommutative integration, Hodge star operator, curvature of the connection...

1 – Lie algebroids and their representations

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Generalities on Lie algebroids

\mathcal{M} a smooth manifold, $\Gamma(T\mathcal{M})$ the Lie algebra and $C^\infty(\mathcal{M})$ -module of vector fields.

Definition in terms of algebras and modules (as in NCG).

Definition 1 (Lie algebroids). A Lie algebroid A is a finite projective module over $C^\infty(\mathcal{M})$ equipped with a Lie bracket $[-, -]$ and a $C^\infty(\mathcal{M})$ -linear Lie morphism $\rho : A \rightarrow \Gamma(T\mathcal{M})$ such that

$$[\mathfrak{X}, f\mathfrak{Y}] = f[\mathfrak{X}, \mathfrak{Y}] + (\rho(\mathfrak{X}) \cdot f)\mathfrak{Y}$$

for any $\mathfrak{X}, \mathfrak{Y} \in A$ and $f \in C^\infty(\mathcal{M})$.

ρ is the anchor of A .

The usual definition uses the vector bundle \mathcal{A} such that $A = \Gamma(\mathcal{A})$.

\mathcal{A} is viewed as a generalization of the tangent bundle.

→ We will never use this point of view.

Natural notion of morphisms of Lie algebroids...

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Transitive Lie algebroids

A Lie algebroid $A \xrightarrow{\rho} \Gamma(T\mathcal{M})$ is **transitive** if ρ is surjective.

Proposition 2 (The kernel of a transitive Lie algebroid). *Let A be a transitive Lie algebroid.*

- $L = \text{Ker } \rho$ is a Lie algebroid with null anchor on \mathcal{M} .
 → L is called the **kernel** of A .
- The vector bundle \mathcal{L} such that $L = \Gamma(\mathcal{L})$ is a locally trivial bundle in Lie algebras.
 → This gives the Lie structure on L .

One has the short exact sequence of Lie algebras and $C^\infty(\mathcal{M})$ -modules

$$0 \longrightarrow L \xrightarrow{i} A \xrightarrow{\rho} \Gamma(T\mathcal{M}) \longrightarrow 0$$

This short exact sequence is the key structure of what follows...

Very trivial example: $A = \Gamma(T\mathcal{M}) \Rightarrow L = 0$.

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Example 1: Derivations of a vector bundle

\mathcal{E} a vector bundle over \mathcal{M} .

$\text{Diff}^1(\mathcal{E})$ the space of first order differential operators on \mathcal{E} .

Symbol map:

$$\sigma : \text{Diff}^1(\mathcal{E}) \rightarrow \text{Hom}_{C^\infty(\mathcal{M})}(T^*\mathcal{M}, \text{End}(\mathcal{E})) \simeq \Gamma(T\mathcal{M} \otimes \text{End}(\mathcal{E})) \supset \Gamma(T\mathcal{M})$$

$$\mathfrak{D}(\mathcal{E}) = \sigma^{-1}(\Gamma(T\mathcal{M}))$$

is the transitive Lie algebroid of derivations of \mathcal{E} :

$$0 \longrightarrow \mathbf{A}(\mathcal{E}) \xrightarrow{\iota} \mathfrak{D}(\mathcal{E}) \xrightarrow{\sigma} \Gamma(T\mathcal{M}) \longrightarrow 0$$

with $\mathbf{A}(\mathcal{E}) = \Gamma(\text{End}(\mathcal{E}))$ (0th-order diff. op.).

$\mathbf{A}(\mathcal{E})$ is an associative algebra (Lie structure is the commutator).

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Representation of a Lie algebroid

$A \xrightarrow{\rho} \Gamma(T\mathcal{M})$ a Lie algebroid and $\mathcal{E} \rightarrow \mathcal{M}$ a vector bundle.

Definition 3 (Representation of a Lie algebroid). A representation of A on \mathcal{E} is a morphism of Lie algebroids $\phi : A \rightarrow \mathfrak{D}(\mathcal{E})$.

When A is transitive, one has the commutative diagram of exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & L & \xrightarrow{\iota} & A & \xrightarrow{\rho} & \Gamma(T\mathcal{M}) \longrightarrow 0 \\ & & \downarrow \phi_L & & \downarrow \phi & & \parallel \\ 0 & \longrightarrow & \mathbf{A}(\mathcal{E}) & \xrightarrow{\iota} & \mathfrak{D}(\mathcal{E}) & \xrightarrow{\sigma} & \Gamma(T\mathcal{M}) \longrightarrow 0 \end{array}$$

$\phi_L : L \rightarrow \mathbf{A}(\mathcal{E})$ is a morphism of Lie algebras.

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Example 2: Atiyah Lie algebroids

$\mathcal{P} \xrightarrow{\pi} \mathcal{M}$ a G -principal fiber bundle, \mathfrak{g} the Lie algebra of G .

$R_g : \mathcal{P} \rightarrow \mathcal{P}$, $R_g(p) = p \cdot g$, the right action of G on \mathcal{P} .

$$\Gamma_G(T\mathcal{P}) = \{\mathfrak{X} \in \Gamma(T\mathcal{P}) / R_{g*}\mathfrak{X} = \mathfrak{X} \text{ for all } g \in G\}$$

$$\Gamma_G(\mathcal{P}, \mathfrak{g}) = \{v : \mathcal{P} \rightarrow \mathfrak{g} / v(p \cdot g) = \text{Ad}_{g^{-1}}v(p) \text{ for all } g \in G\}$$

Both are Lie algebras and $C^\infty(\mathcal{M})$ -modules.

$\Gamma_G(T\mathcal{P}) = \pi_*$ -projectable vector fields in $\Gamma(T\mathcal{P}) \xrightarrow{\pi_*} \Gamma_G(T\mathcal{P}) \rightarrow \Gamma(T\mathcal{M})$.

$\iota : \Gamma_G(\mathcal{P}, \mathfrak{g}) \rightarrow \Gamma_G(T\mathcal{P})$ defined by $\iota(v)|_p = v(p)|_p^{\mathcal{P}}$,

($\mathfrak{g} \ni v \mapsto v^{\mathcal{P}}$ fundamental vector field on \mathcal{P}).

S.E.S. of Lie algebras and $C^\infty(\mathcal{M})$ -modules:

$$0 \longrightarrow \Gamma_G(\mathcal{P}, \mathfrak{g}) \xrightarrow{\iota} \Gamma_G(T\mathcal{P}) \xrightarrow{\pi_*} \Gamma(T\mathcal{M}) \longrightarrow 0$$

$\Gamma_G(T\mathcal{P})$ is the transitive **Atiyah Lie algebroid** associated to \mathcal{P}

The representations of $\Gamma_G(T\mathcal{P})$ are given by the associated vector bundles to \mathcal{P} .

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Example 3: Trivial Lie algebroids

Trivial Lie algebroid = Atiyah Lie algebroid of a trivial principal bundle $\mathcal{M} \times G$.

Concrete description in terms of the bundle $T\mathcal{M} \oplus (\mathcal{M} \times \mathfrak{g})$:

- $C^\infty(\mathcal{M})$ -module: $\text{TLA}(\mathcal{M}, \mathfrak{g}) \equiv A = \Gamma(T\mathcal{M} \oplus (\mathcal{M} \times \mathfrak{g}))$.
- Bracket: $[X \oplus \gamma, Y \oplus \eta] = [X, Y] \oplus (X \cdot \eta - Y \cdot \gamma + [\gamma, \eta])$
- Anchor: $\rho(X \oplus \gamma) = X$.
- Kernel: $L = \Gamma(\mathcal{M} \times \mathfrak{g})$ (section of a trivial bundle).

Proposition 4. *Every transitive Lie algebroid A is locally of the form $\text{TLA}(\mathcal{U}, \mathfrak{g})$ for $\mathcal{U} \subset \mathcal{M}$ open subset.*

Trivialization of an Atiyah Lie algebroid $\Gamma_G(T\mathcal{P}) \leftrightarrow$ Trivialization of \mathcal{P} .

2 – Differential structures

Differential forms: general definition

A Lie algebroid, $\phi : A \rightarrow \mathcal{D}(\mathcal{E})$ a representation of A on \mathcal{E} .

Definition 5 (Differential forms). For $p \in \mathbb{N}$, let $\Omega^p(A, \mathcal{E})$ be the linear space of $C^\infty(\mathcal{M})$ -multilinear antisymmetric maps $A^p \rightarrow \Gamma(\mathcal{E})$.

For $p = 0$, let $\Omega^0(A, \mathcal{E}) = \Gamma(\mathcal{E})$.

$\Omega^\bullet(A, \mathcal{E}) = \bigoplus_{p \geq 0} \Omega^p(A, \mathcal{E})$ is equipped with the natural differential

$$\begin{aligned} (\widehat{d}_\phi \widehat{\omega})(\mathfrak{X}_1, \dots, \mathfrak{X}_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i+1} \phi(\mathfrak{X}_i) \cdot \widehat{\omega}(\mathfrak{X}_1, \dots, \overset{i}{\checkmark} \dots, \mathfrak{X}_{p+1}) \\ &\quad + \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} \widehat{\omega}([\mathfrak{X}_i, \mathfrak{X}_j], \mathfrak{X}_1, \dots, \overset{i}{\checkmark} \dots \overset{j}{\checkmark} \dots, \mathfrak{X}_{p+1}) \end{aligned}$$

$\phi(\mathfrak{X}) \cdot \varphi$ is the action of the first order diff. op. $\phi(\mathfrak{X})$ on $\varphi \in \Gamma(\mathcal{E})$.

One has $\widehat{d}_\phi^2 = 0$ (since ϕ is a morphism of Lie algebras).

Differential forms: two examples

$\mathcal{E} = \mathcal{M} \times \mathbb{C} \rightarrow \Gamma(\mathcal{E}) = C^\infty(\mathcal{M})$.

The anchor map is a representation of A on $C^\infty(\mathcal{M})$ via vector fields.

Definition 6 (Forms with values in $C^\infty(\mathcal{M})$). $(\Omega^\bullet(A), \widehat{d}_A)$ is the graded commutative differential algebra of forms on A with values in $C^\infty(\mathcal{M})$ associated to the anchor as a representation.

$0 \rightarrow L \xrightarrow{\iota} A \xrightarrow{\rho} \Gamma(T\mathcal{M}) \rightarrow 0$ a transitive Lie algebroid.

$\mathcal{E} = \mathcal{L}$ the vector bundle such that $L = \Gamma(\mathcal{L})$.

For $\mathfrak{X} \in A$ and $\ell \in L$, define $\text{ad}_\mathfrak{X}(\ell) \in L$ such that $\iota(\text{ad}_\mathfrak{X}(\ell)) = [\mathfrak{X}, \iota(\ell)]$

(adjoint representation of A on \mathcal{L}).

Definition 7 (Forms with values in the kernel). $(\Omega^\bullet(A, L), \widehat{d})$ is the graded differential Lie algebra of forms on A with values in the kernel L associated to the adjoint representation.

This differential space is a graded Lie algebra and a graded differential module on the graded commutative differential algebra $\Omega^\bullet(A)$.

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Differential forms on trivial Lie algebroids

$A = \text{TLA}(\mathcal{M}, \mathfrak{g})$ a trivial Lie algebroid.

$\Omega^\bullet(A)$ is the total complex of the bigraded commutative algebra $\Omega^\bullet(\mathcal{M}) \otimes \wedge^\bullet \mathfrak{g}^*$.

$\widehat{d}_A = d + s$ with

$$\begin{aligned} d : \Omega^\bullet(\mathcal{M}) \otimes \wedge^\bullet \mathfrak{g}^* &\rightarrow \Omega^{\bullet+1}(\mathcal{M}) \otimes \wedge^\bullet \mathfrak{g}^* && \text{de Rham differential} \\ s : \Omega^\bullet(\mathcal{M}) \otimes \wedge^\bullet \mathfrak{g}^* &\rightarrow \Omega^\bullet(\mathcal{M}) \otimes \wedge^{\bullet+1} \mathfrak{g}^* && \text{Chevalley-Eilenberg differential} \end{aligned}$$

$\Omega^\bullet(A, L)$ is the total complex of the bigraded Lie algebra $\Omega^\bullet(\mathcal{M}) \otimes \wedge^\bullet \mathfrak{g}^* \otimes \mathfrak{g}$.

$\widehat{d} = d + s'$ with

s' the Chevalley-Eilenberg differential on $\wedge^\bullet \mathfrak{g}^* \otimes \mathfrak{g}$ (for the ad rep.).

Compact notation $(\Omega_{\text{TLA}}^\bullet(\mathcal{M}, \mathfrak{g}), \widehat{d}_{\text{TLA}})$.

This is the model for trivializations of forms on any transitive Lie algebroid.

⚠ Mathematical structure similar to the one used in BRST differential algebras.

→ work in progress to understand possible relations...

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Differential forms on Atiyah Lie algebroids

A the Atiyah Lie algebroid of the G -principal fiber bundle $\mathcal{P} \xrightarrow{\pi} \mathcal{M}$.

$(\Omega_{\text{Lie}}^\bullet(\mathcal{P}, \mathfrak{g}), \widehat{d})$ the complex of forms with values in the kernel.

$$\mathfrak{g}_{\text{equ}} = \{\xi^{\mathcal{P}} \oplus \xi / \xi \in \mathfrak{g}\} \subset \text{TLA}(\mathcal{P}, \mathfrak{g}) = \Gamma(T\mathcal{P} \oplus (\mathcal{P} \times \mathfrak{g}))$$

is a Lie sub-algebra, which defines a Cartan operation on $(\Omega_{\text{TLA}}^\bullet(\mathcal{P}, \mathfrak{g}), \widehat{d}_{\text{TLA}})$.

$(\Omega_{\text{TLA}}^\bullet(\mathcal{P}, \mathfrak{g})_{\mathfrak{g}_{\text{equ}}}, \widehat{d}_{\text{TLA}})$ the differential graded subcomplex of basic elements.

Theorem 8 (S. Lazzarini, T.M.). *If G is connected and simply connected then*

$(\Omega_{\text{Lie}}^\bullet(\mathcal{P}, \mathfrak{g}), \widehat{d})$ *is isomorphic to* $(\Omega_{\text{TLA}}^\bullet(\mathcal{P}, \mathfrak{g})_{\mathfrak{g}_{\text{equ}}}, \widehat{d}_{\text{TLA}})$

→ $\Omega_{\text{Lie}}^\bullet(\mathcal{P}, \mathfrak{g}) \subset \Omega_{\text{TLA}}^\bullet(\mathcal{P}, \mathfrak{g}) \simeq \Omega^\bullet(\mathcal{P}) \otimes \wedge^\bullet \mathfrak{g}^* \otimes \mathfrak{g}$.

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The global picture so far

- Transitive Lie algebroids = s.e.s. of Lie algebras and $C^\infty(\mathcal{M})$ -modules

$$0 \longrightarrow L \xrightarrow{\iota} A \xrightarrow{\rho} \Gamma(T\mathcal{M}) \longrightarrow 0$$

- Generalized forms: $(\Omega^\bullet(A, L), \widehat{d})$, graded differential Lie algebra.
 - ➔ “Contains” ordinary de Rham calc. on \mathcal{M} (basic elements for op. of L).
- Local description of transitive Lie algebroids and diff. calc. using TLA.
 - ➔ $(\Omega^\bullet(\mathcal{M}) \otimes \wedge^\bullet \mathfrak{g}^* \otimes \mathfrak{g}, \widehat{d} = d + s')$.
 - ➔ useful for computations and definitions of structures...
- Representation theory on derivations of a vector bundle.

$$\begin{array}{ccccccc} 0 & \longrightarrow & L & \xrightarrow{\iota} & A & \xrightarrow{\rho} & \Gamma(T\mathcal{M}) \longrightarrow 0 \\ & & \downarrow \phi_L & & \downarrow \phi & & \parallel \\ 0 & \longrightarrow & \mathbf{A}(\mathcal{E}) & \xrightarrow{\iota} & \mathfrak{D}(\mathcal{E}) & \xrightarrow{\sigma} & \Gamma(T\mathcal{M}) \longrightarrow 0 \end{array}$$

- Principal fiber bundle ➔ canonical Atiyah Lie algebroid.

3 – Connections and covariant derivatives

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Ordinary connections

$0 \longrightarrow L \xrightarrow{\iota} A \xrightarrow{\rho} \Gamma(T\mathcal{M}) \longrightarrow 0$ a transitive Lie algebroid.

Definition 9 (Connection on a transitive Lie algebroid). A connection on A is a splitting $\nabla : \Gamma(T\mathcal{M}) \rightarrow A$ as $C^\infty(\mathcal{M})$ -modules of the s.e.s.

$$0 \longrightarrow L \xleftarrow{\omega^\nabla} A \xleftarrow{\rho} \Gamma(T\mathcal{M}) \longrightarrow 0$$

The curvature of ∇ is defined as the obstruction to be a morphism of Lie algebras:

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]} \in \iota(L)$$

∇ defines $\omega^\nabla : A \rightarrow L$ (s.e.s. properties) s.t. $\mathfrak{X} = \nabla_X - \iota \circ \omega^\nabla(\mathfrak{X})$, $\mathfrak{X} \in A, X = \rho(\mathfrak{X})$.

Proposition 10. One has $\omega^\nabla \in \Omega^1(A, L)$ and $\omega^\nabla \circ \iota(\ell) = -\ell$ for any $\ell \in L$ (normalization on L).

The 2-form $R^\nabla \in \Omega^2(A, L)$ defined by $R^\nabla(\mathfrak{X}, \mathfrak{Y}) = (\widehat{d\omega^\nabla})(\mathfrak{X}, \mathfrak{Y}) + [\omega^\nabla(\mathfrak{X}), \omega^\nabla(\mathfrak{Y})]$ vanishes when \mathfrak{X} or \mathfrak{Y} in $\iota(L)$, and one has $\iota \circ R^\nabla(\mathfrak{X}, \mathfrak{Y}) = R(X, Y)$.

ω^∇ is the connection 1-form associated to ∇ .

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Ordinary connections on Atiyah Lie algebroid

$$0 \longrightarrow \Gamma_G(\mathcal{P}, \mathfrak{g}) \xrightarrow{\iota} \Gamma_G(T\mathcal{P}) \xrightarrow{\pi_*} \Gamma(T\mathcal{M}) \longrightarrow 0$$

Proposition 11 (Connections). Ordinary connection on the Atiyah Lie algebroid = connection on \mathcal{P} . The notions of curvature coincide.

→ This example explains the terminology “ordinary connection”.

The geometric equivalence: A connection on \mathcal{P} defines the horizontal lift $\Gamma(T\mathcal{M}) \rightarrow \Gamma_G(T\mathcal{P}), X \mapsto X^h$.

The algebraic equivalence:

Suppose G is connected and simply connected.

$\omega^\mathcal{P} \in \Omega^1(\mathcal{P}) \otimes \mathfrak{g}$ a connection 1-form on \mathcal{P} .

$\theta \in \mathfrak{g}^* \otimes \mathfrak{g}$ the Maurer-Cartan 1-form on G .

$\widehat{\omega}_{\mathfrak{g}_{\text{equ}}} = \omega^\mathcal{P} - \theta \in \Omega_{\text{TLA}}^1(\mathcal{P}, \mathfrak{g}) \subset \Omega^\bullet(\mathcal{P}) \otimes \wedge^\bullet \mathfrak{g}^* \otimes \mathfrak{g}$ is $\mathfrak{g}_{\text{equ}}$ -basic.

It corresponds to the connection 1-form $\omega^\nabla \in \Omega_{\text{Lie}}^1(\mathcal{P}, \mathfrak{g})$ associated to $\omega^\mathcal{P}$.

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Generalized connections

Definition 12 (Generalized connection). A generalized connection on a transitive Lie algebroid A is a 1-form $\widehat{\omega} \in \Omega^1(A, L)$.

The curvature of $\widehat{\omega}$ is the 2-form $\widehat{R} = \widehat{d}\widehat{\omega} + \frac{1}{2}[\widehat{\omega}, \widehat{\omega}] \in \Omega^2(A, L)$.

A generalized connection is an ordinary connection iff $\widehat{\omega} \circ \iota = -\text{Id}_L$.

Consider a representation of A on \mathcal{E} :

$$\begin{array}{ccccccc}
 0 & \longrightarrow & L & \xleftarrow{\widehat{\omega}} & A & \xrightarrow{\rho} & \Gamma(TM) \longrightarrow 0 \\
 & & \phi_L \downarrow & & \downarrow \widehat{\nabla} \phi & & \parallel \\
 0 & \longrightarrow & \mathbf{A}(\mathcal{E}) & \xrightarrow{\iota} & \mathfrak{D}(\mathcal{E}) & \xrightarrow{\sigma} & \Gamma(TM) \longrightarrow 0
 \end{array}$$

$\widehat{\omega}$ defines $\widehat{\nabla} : A \rightarrow \mathfrak{D}(\mathcal{E})$ by $\widehat{\nabla}_{\mathfrak{X}} = \phi(\mathfrak{X}) + \iota \circ \phi_L(\widehat{\omega}(\mathfrak{X}))$.

This is the **covariant derivative** on \mathcal{E} associated to $\widehat{\omega}$.

$[\widehat{\nabla}_{\mathfrak{X}}, \widehat{\nabla}_{\mathfrak{Y}}] - \widehat{\nabla}_{[\mathfrak{X}, \mathfrak{Y}]} = \iota \circ \phi_L \circ \widehat{R}(\mathfrak{X}, \mathfrak{Y}) \rightarrow \widehat{\nabla}$ is not a representation in general.

Other terminologies for $\widehat{\nabla}$: “generalized representation”, “ A -connection”...

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Generalized connections on Atiyah Lie algebroids

$$0 \longrightarrow \Gamma_G(\mathcal{P}, \mathfrak{g}) \xrightarrow{\iota} \Gamma_G(T\mathcal{P}) \xrightarrow{\pi_*} \Gamma(TM) \longrightarrow 0$$

To simplify the presentation: suppose G is connected and simply connected.

A generalized connection $\widehat{\omega}$ on $\Gamma_G(T\mathcal{P})$ is a $\mathfrak{g}_{\text{equ}}$ -basic 1-form $\widehat{\omega}_{\mathfrak{g}_{\text{equ}}} \in \Omega_{\text{TLA}}^1(\mathcal{P}, \mathfrak{g})$.

$$\widehat{\omega}_{\mathfrak{g}_{\text{equ}}} = \omega + \varphi \in (\Omega^1(\mathcal{P}) \otimes \mathfrak{g}) \oplus (C^\infty(\mathcal{P}) \otimes \mathfrak{g}^* \otimes \mathfrak{g}).$$

Proposition 13 (Ordinary versus generalized connections). *If $\varphi = -\theta$, then $\widehat{\omega}$ is an ordinary connection on $\Gamma_G(T\mathcal{P})$.*

$\rightarrow \omega$ is an (ordinary) connection 1-form on \mathcal{P} .

Otherwise, $\varphi + \theta$ measures the deviation of $\widehat{\omega}$ from an ordinary connection.

$\rightarrow \varphi + \theta \simeq$ Higgs scalar fields...

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Connections: a summary

- Ordinary connection on a transitive Lie algebroid = splitting:

$$0 \longrightarrow L \begin{array}{c} \xleftarrow{\omega^\nabla} \\ \xrightarrow{\iota} \end{array} A \begin{array}{c} \xleftarrow{\nabla} \\ \xrightarrow{\rho} \end{array} \Gamma(TM) \longrightarrow 0$$

$\nabla \mapsto \omega^\nabla \in \Omega^1(A, L)$ connection 1-form, curvature as a 2-form.

- Generalized connections are any 1-forms $\widehat{\omega} \in \Omega^1(A, L)$.
 - Covariant derivatives on representations.
 - Notion of curvature.
- Ordinary connection = normalized generalized connection:
 - $\widehat{\omega} \circ \iota(\ell) = -\ell$ for any $\ell \in L$
- For Atiyah Lie algebroids:
 - space of ordinary connections on $\mathcal{P} \subset$ space of generalized connections;
 - connection 1-forms and curvatures are directly related in $\Omega_{\text{TLA}}^\bullet(\mathcal{P}, \mathfrak{g})$.

4 – The gauge group

Gauge group of a representation

Suppose given a representation of a transitive Lie algebroid A on \mathcal{E} :

$$\begin{array}{ccccccc} 0 & \longrightarrow & L & \xrightarrow{\iota} & A & \xrightarrow{\rho} & \Gamma(TM) \longrightarrow 0 \\ & & \phi_L \downarrow & & \downarrow \phi & & \parallel \\ 0 & \longrightarrow & \mathbf{A}(\mathcal{E}) & \xrightarrow{\iota} & \mathfrak{D}(\mathcal{E}) & \xrightarrow{\sigma} & \Gamma(TM) \longrightarrow 0 \end{array}$$

Definition 14 (Gauge group of a representation). The gauge group of \mathcal{E} is the group $\text{Aut}(\mathcal{E}) \subset \mathbf{A}(\mathcal{E})$ (vertical automorphisms of \mathcal{E}).

No (finite) gauge transformation at the level of A (similar situation in NCG).

Any $\xi \in L$ defines an infinitesimal gauge transformation on $\Gamma(\mathcal{E})$ by $\varphi \mapsto \phi_L(\xi)\varphi$.

Definition 15 (Infinitesimal gauge transformations). An infinitesimal gauge transformation on A is an element $\xi \in L$.

Gauge transformations

$$\xi \in L \mapsto g = e^{\phi_L(\xi)} \simeq 1 + \phi_L(\xi) + \dots \in \text{Aut}(\mathcal{E}) \subset \mathbf{A}(\mathcal{E})$$

$\widehat{\omega}$ generalized connection on A , and $\widehat{\nabla}$ its associated covariant derivative on \mathcal{E} :

$$\widehat{\nabla}_{\mathcal{X}}\varphi = \phi(\mathcal{X})\cdot\varphi + \phi_L(\widehat{\omega}(\mathcal{X}))\varphi$$

The first order diff. op. $\widehat{\nabla}_{\mathcal{X}}^g = g^{-1} \circ \widehat{\nabla}_{\mathcal{X}} \circ g$ on \mathcal{E} can be written as

$$\widehat{\nabla}_{\mathcal{X}}^g\varphi = \phi(\mathcal{X})\cdot\varphi + \phi_L(\widehat{\omega}(\mathcal{X}))\varphi + \phi_L(\widehat{d}\xi(\mathcal{X}) + [\widehat{\omega}(\mathcal{X}), \xi])\varphi + O(\xi^2)\varphi$$

Definition 16 (Infinitesimal gauge variation). The infinitesimal gauge variation of $\widehat{\omega}$ induced by ξ is defined to be $\widehat{d}\xi + [\widehat{\omega}, \xi]$.

→ The infinitesimal gauge variation of the curvature \widehat{R} of $\widehat{\omega}$ is $[\widehat{R}, \xi]$.

The gauge principle is implemented on A at the infinitesimal level (indep. of a rep.).

→ Similar to ordinary differential geometry.

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Gauge transformations on Atiyah Lie algebroids

$$0 \longrightarrow \Gamma_G(\mathcal{P}, \mathfrak{g}) \xrightarrow{l} \Gamma_G(T\mathcal{P}) \xrightarrow{\pi_*} \Gamma(TM) \longrightarrow 0$$

$\mathcal{G}(\mathcal{P})$ the gauge group of \mathcal{P} (vertical automorphisms of \mathcal{P}).

$u \in \mathcal{G}(\mathcal{P})$ is a G -equivariant map $u : \mathcal{P} \rightarrow G$, $u(p \cdot g) = g^{-1}u(p)g$.

- **Finite gauge transformations are defined.**
- $L = \Gamma_G(\mathcal{P}, \mathfrak{g})$ is the Lie algebra of $\mathcal{G}(\mathcal{P})$.
- Infinitesimal (usual) gauge transformations are elements in L .

$\widehat{\omega} \in \Omega_{\text{Lie}}^1(\mathcal{P}, \mathfrak{g})$ and $u \in \mathcal{G}(\mathcal{P})$.

Define $\widehat{\omega}^u(\mathfrak{X}) = u^{-1}\widehat{\omega}(\mathfrak{X})u + u^{-1}(\mathfrak{X} \cdot u)$ for any $\mathfrak{X} \in \Gamma_G(T\mathcal{P})$.

- $\widehat{\omega}^u \in \Omega_{\text{Lie}}^1(\mathcal{P}, \mathfrak{g})$;
- induced by $\widehat{\nabla} \mapsto \widehat{\nabla}^u$;
- infinitesimal gauge transformations on $\widehat{\omega}$ are induced by $\widehat{\omega} \mapsto \widehat{\omega}^u$;
- restricts to (ordinary) gauge transformation on ordinary connections...
 - ➔ preserves the decomposition $\omega^{\mathcal{P}} - \theta$.

5 – Structures to construct an action functional

Metrics on transitive Lie algebroids

All the structures rely on a notion of metric...

Definition 17 (Metric on a Lie algebroid). A metric on A is a symmetric $C^\infty(\mathcal{M})$ -linear map $\widehat{g} : A \otimes_{C^\infty(\mathcal{M})} A \rightarrow C^\infty(\mathcal{M})$.

\widehat{g} defines a metric $h = \iota^* \widehat{g}$ on L given by $h(\gamma, \eta) = \widehat{g}(\iota(\gamma), \iota(\eta))$ for any $\gamma, \eta \in L$.

→ \widehat{g} is **inner non degenerate** if h is non degenerate on L .

Proposition 18 (C. Fournel, S. Lazzarini, T.M.). An inner non degenerate metric \widehat{g} on A is equivalent to a triple (g, h, ∇) where

- g is a (possibly degenerate) metric on \mathcal{M} ;
- h is a non degenerate metric on L ;
- ∇ is an ordinary connection on A , with $\mathring{\omega} \in \Omega^1(A, L)$ its connection 1-form;
- $\widehat{g}(\mathfrak{X}, \mathfrak{Y}) = g(\rho(\mathfrak{X}), \rho(\mathfrak{Y})) + h(\mathring{\omega}(\mathfrak{X}), \mathring{\omega}(\mathfrak{Y}))$.
- $\widehat{g}(\nabla_X, \iota(\gamma)) = 0$ for any $X \in \Gamma(TM)$ and $\gamma \in L$.

Given \widehat{g} , look at $\mathring{\omega}$ as a background connection...

Integration along the kernel

∇ a connection on A , $\mathring{\omega} \in \Omega^1(A, L)$ its connection 1-form.

h a metric on L .

Suppose \mathcal{L} is orientable where $L = \Gamma(\mathcal{L})$ (→ A is called inner orientable), let $n = \text{rank}(\mathcal{L})$.

Proposition 19 (Volume form along L and inner integration). h and $\mathring{\omega}$ define a global form in $\Omega^\bullet(A)$ of maximal degree in the L direction.

This volume form defines integrations

$$\int_{\text{inner}} : \Omega^\bullet(A) \rightarrow \Omega^{\bullet-n}(\mathcal{M}) \qquad \int_{\text{inner}} : \Omega^\bullet(A, L) \rightarrow \Omega^{\bullet-n}(\mathcal{M}, \mathcal{L}).$$

They do not depend on ∇ .

→ After integration, only geometrical structures (de Rham).

.....
Integration on A

Suppose also that \mathcal{M} is orientable ($\Rightarrow A$ is called orientable) and g non degenerate.

Definition 20 (Integration on A). Using g , the integration on A of a form $\widehat{\omega} \in \Omega^\bullet(A)$ is defined by

$$\int_A \widehat{\omega} = \int_{\mathcal{M}} \int_{\text{inner}} \widehat{\omega} \in \mathbb{C}.$$

Definition 21 (Scalar product of forms). The scalar product of any 2 forms $\widehat{\omega}_1, \widehat{\omega}_2 \in \Omega^\bullet(A, L)$ is defined by

$$\langle \widehat{\omega}_1, \widehat{\omega}_2 \rangle = \int_A h(\widehat{\omega}_1, \widehat{\omega}_2) \in \mathbb{C}$$

.....
Metrics: a summary

A non degenerate metric $\widehat{g} = (g, h, \nabla)$, with $\nabla \leftrightarrow \widehat{\omega} \in \Omega^1(A, L)$, gives us:

- $h \Rightarrow$ scalar product on L ;
- $h, \widehat{\omega} \Rightarrow$ integration along L ;
- $g \Rightarrow$ integration on \mathcal{M} ;
- $g, h, \widehat{\omega} \Rightarrow$ integration on A ;
- $g, h, \widehat{\omega} \Rightarrow$ Hodge star operator (straightforward to define)

$$\star : \Omega^p(A, L) \rightarrow \Omega^{m+n-p}(A, L)$$

6 – Gauge theories

.....
Gauge invariant action

A orientable transitive Lie algebroid, $\widehat{g} = (g, h, \nabla)$ non degenerate metric.

Suppose h is a Killing metric: $h([\xi, \gamma], \eta) + h(\gamma, [\xi, \eta]) = 0$ for any $\gamma, \eta, \xi \in L$.

$\widehat{\omega} \in \Omega^1(A, L)$ a connection on A and \widehat{R} its curvature 2-form.

Proposition 22 (C. Fournel, S. Lazzarini, T.M.). *The action functional*

$$\mathcal{S}_{\text{Gauge}}[\widehat{\omega}] = \langle \widehat{R}, \star \widehat{R} \rangle = \int_A h(\widehat{R}, \star \widehat{R}).$$

is invariant under infinitesimal gauge transformations in L .

Example 23 (Atiyah Lie algebroid). A the Atiyah Lie algebroid of a G -principal fiber bundle \mathcal{P} . h induced by the Killing form on \mathfrak{g} (semisimple).

$\widehat{\omega}$ any generalized connection $\rightarrow \mathcal{S}_{\text{Gauge}}[\widehat{\omega}]$ is $\mathcal{G}(\mathcal{P})$ -gauge invariant.

$\widehat{\omega}$ an ordinary connection on $A \rightarrow \mathcal{S}_{\text{Gauge}}[\widehat{\omega}]$ is the ordinary Yang-Mills action.

Possible to define $\mathcal{S}_{\text{Matter}}[\varphi, \widehat{\omega}]$ for $\varphi \in \Gamma(\mathcal{E})$ where $A \rightarrow \mathfrak{D}(\mathcal{E})$ is a representation of A .

.....
Decomposition of a connection

$\widehat{\omega} \in \Omega^1(A, L)$ a generalized connection on A .

$\tau \in \text{End}(\mathcal{L})$ defined by

$$\tau = \widehat{\omega} \circ \iota + \text{Id}_L.$$

τ vanishes iff $\widehat{\omega}$ is an ordinary connection on A

\rightarrow measures the “non Yang-Mills” part (Higgs scalar fields).

$\triangle!$ τ is not a Lie morphism.

$\widehat{g} = (g, h, \nabla)$ metric with $\nabla \leftrightarrow \widehat{\omega} \in \Omega^1(A, L)$.

Proposition 24. $\omega = \widehat{\omega} + \tau(\widehat{\omega}) \in \Omega^1(A, L)$ is an ordinary connection on A .

The induced infinitesimal gauge action of L is the one on ordinary connections.

$\widehat{g} = (g, h, \nabla)$ decomposes any connection $\widehat{\omega}$ on A as:

$$\widehat{\omega} \leftrightarrow (\omega, \tau) \quad \text{ordinary connection on } A \text{ } \vdash \text{ algebraic object on } L$$

$\widehat{\omega}$ ordinary connection $\rightarrow \tau = 0 \rightarrow \omega = \widehat{\omega}$.

.....
The total action functional

Using the decomposition $\widehat{\omega} \leftrightarrow (\omega, \tau)$:

$$\mathcal{S}_{\text{Gauge}}[\widehat{\omega}] + \mathcal{S}_{\text{Matter}}[\varphi, \widehat{\omega}] = \begin{aligned} & (1) \text{ Yang-Mills like term for } \omega \\ & (2) \text{ covariant derivative for } \tau \text{ along } \omega \\ & (3) \text{ potential for } \tau \\ & (4) \text{ covariant derivative for } \varphi \text{ along } \omega \\ & (5) \text{ coupling } \varphi \leftrightarrow \tau \end{aligned}$$

The potential (3) can vanish for $\tau \neq 0$.

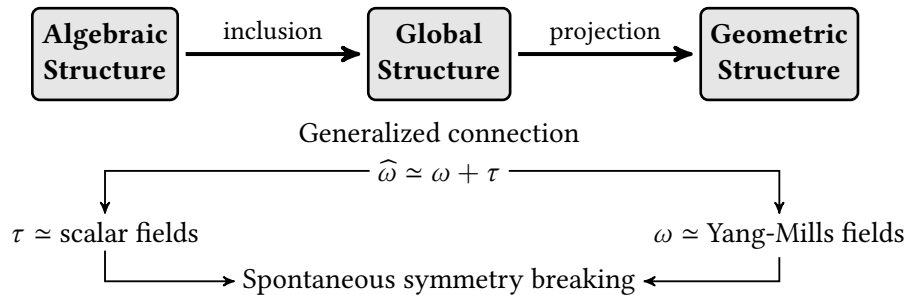
A development around a solution $\tau_0 \neq 0$ induces:

- A mass term for the ordinary connection ω in (2).
- A mass term for φ in (5).
- Massive bosons (ω) coupled to massive particles (φ).
- Yang-Mills-Higgs type gauge theory.

Conclusion

.....
Why Higgs fields?

A pattern for Yang-Mills-Higgs gauge field theories (on “ordinary space-time”):



Transitive Lie algebroids: $0 \longrightarrow L \longrightarrow A \longrightarrow \Gamma(TM) \longrightarrow 0$

NCG: $1 \longrightarrow \text{Inn}(\mathbf{A}) \longrightarrow \text{Aut}(\mathbf{A}) \longrightarrow \text{Out}(\mathbf{A}) \longrightarrow 1$

François, J., Lazzarini, S., and Masson, T. (2014). “Gauge field theories: various mathematical approaches”. In: *Mathematical Structures of the Universe*. Ed. by Eckstein, M., Heller, M., and Szybka, S. J. Kraków, Poland: Copernicus Center Press, pp. 177–225

.....
Conclusion

- (Geometric) Gauge field theories can be generalized in at least two directions:
 - Noncommutative Geometry
 - Transitive Lie Algebroids.
- Same pattern: add some purely algebraic directions to space-time.
 - ➔ Yang-Mills-Higgs type gauge theories.
- Gauge theories on Atiyah Lie algebroids are close to Yang-Mills gauge theories.
 - They contain ordinary gauge theories used in physics.
 - They share common mathematical structures.
 - No restriction on the gauge group.
- A lot more to investigate:
 - Relation to BRST structures...
 - Construction of realistic models...
 - Relation with “dressing field” method elaborated in Fournel, C., François, J., Lazzarini, S., and Masson, T. (2014). Gauge invariant composite fields out of connections, with examples. *Int. J. Geom. Methods Mod. Phys.* 11.1, p. 1450016
 - Here, generalization of Ehresman’s connections:
 - ➔ we investigate generalization of Cartan’s connections (used in gravitational and conformal theories).

Additional material

Trivialization of transitive Lie algebroids

$0 \longrightarrow L \xrightarrow{\iota} A \xrightarrow{\rho} \Gamma(TM) \longrightarrow 0$ a transitive Lie algebroid.

A local trivialization of A is a triple $(\mathcal{U}, \Psi, \nabla^0)$ where

- \mathcal{U} is an open subset of \mathcal{M} ;
- $\Psi : \Gamma(\mathcal{U} \times \mathfrak{g}) \xrightarrow{\cong} L_{\mathcal{U}} =$ isomorphism of Lie algebras and $C^\infty(\mathcal{U})$ -modules;
- $\nabla^0 : \Gamma(T\mathcal{U}) \rightarrow A_{\mathcal{U}} =$ injective morphism of Lie algebras and $C^\infty(\mathcal{U})$ -modules compatible ρ ;
- $[\nabla_X^0, \iota \circ \Psi(\gamma)] = \iota \circ \Psi(X \cdot \gamma)$ for any $X \in \Gamma(T\mathcal{U})$ and any $\gamma \in \Gamma(\mathcal{U} \times \mathfrak{g})$.

$S(X \oplus \gamma) = \nabla_X^0 + \iota \circ \Psi(\gamma)$ is a isomorphism of Lie algebroids $S : \text{TLA}(\mathcal{U}, \mathfrak{g}) \xrightarrow{\cong} A_{\mathcal{U}}$.

Atlas for $A =$ family of local trivializations $\{(\mathcal{U}_i, \Psi_i, \nabla^{0,i})\}_{i \in I}$ with $\bigcup_{i \in I} \mathcal{U}_i = \mathcal{M}$.

$\mathfrak{X} \in A$ is decomposed as $X^i \oplus \gamma^i \in \text{TLA}(\mathcal{U}_i, \mathfrak{g})$ such that $S_i(X^i \oplus \gamma^i) = \mathfrak{X}|_{\mathcal{U}_i}$.

The X^i 's are the restrictions to \mathcal{U}_i of the global vector field $X = \rho(\mathfrak{X})$.

On $\mathcal{U}_{ij} = \mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset$ one can define $\alpha_j^i = \Psi_i^{-1} \circ \Psi_j : \mathcal{U}_{ij} \rightarrow \text{Aut}(\mathfrak{g})$.

$\exists \chi_{ij} \in \Omega^1(\mathcal{U}_{ij}) \otimes \mathfrak{g}$ such that $\gamma^i = \alpha_j^i(\gamma^j) + \chi_{ij}(X)$.

Cocycle relations:

$$\alpha_k^i = \alpha_j^i \circ \alpha_k^j \quad \alpha_j^i \circ \alpha_i^j = \text{Id} \quad \chi_{ik} = \alpha_j^i \circ \chi_{jk} + \chi_{ij} \quad \alpha_j^i \circ \chi_{ji} + \chi_{ij} = 0$$

Trivialization of differential forms

$0 \longrightarrow L \xrightarrow{\iota} A \xrightarrow{\rho} \Gamma(TM) \longrightarrow 0$ a transitive Lie algebroid.

$\{(\mathcal{U}_i, \Psi_i, \nabla^{0,i})\}_{i \in I}$ a Lie algebroid atlas for A .

$\omega \in \Omega^q(A, L) \longrightarrow$ family of local q -forms $\omega_{\text{loc}}^i \in \Omega_{\text{TLA}}^q(\mathcal{U}_i, \mathfrak{g})$

$$\omega_{\text{loc}}^i = \Psi_i^{-1} \circ \omega \circ S_i$$

$s_i^j = S_j^{-1} \circ S_i : \text{TLA}(\mathcal{U}_{ij}, \mathfrak{g}) \xrightarrow{\cong} \text{TLA}(\mathcal{U}_{ij}, \mathfrak{g}) =$ isomorphism of TLA.

$\widehat{\alpha}_j^i : \Omega_{\text{TLA}}^q(\mathcal{U}_{ij}, \mathfrak{g}) \rightarrow \Omega_{\text{TLA}}^q(\mathcal{U}_{ij}, \mathfrak{g})$ defined by $\widehat{\alpha}_j^i(\omega_{\text{loc}}^j) = \alpha_j^i \circ \omega_{\text{loc}}^j \circ s_i^j$.

Proposition 25. • A family of local forms $\omega_{\text{loc}}^i \in \Omega_{\text{TLA}}^\bullet(\mathcal{U}_i, \mathfrak{g})$ is a system of trivializations of a global form

$\omega \in \Omega^\bullet(A, L)$ if and only if $\widehat{\alpha}_j^i(\omega_{\text{loc}}^j) = \omega_{\text{loc}}^i$ on any $\mathcal{U}_{ij} \neq \emptyset$.

- For any $\omega \in \Omega^\bullet(A, L)$ trivialized on \mathcal{U} as ω_{loc} , one has $\widehat{d}_{\text{TLA}} \omega_{\text{loc}} = \Psi^{-1} \circ (\widehat{d}\omega) \circ S$.
- $\widehat{\alpha}_j^i : \Omega_{\text{TLA}}^\bullet(\mathcal{U}_{ij}, \mathfrak{g}) \rightarrow \Omega_{\text{TLA}}^\bullet(\mathcal{U}_{ij}, \mathfrak{g}) =$ isomorphism of grad. diff. Lie algebras.

.....
Local mixed basis

\mathfrak{g} the Lie algebra fiber of \mathcal{L} where $L = \Gamma(\mathcal{L})$, $\mathcal{U} \subset \mathcal{M}$ open subset which trivializes A .

$\{E_a\}_{1 \leq a \leq n}$ basis of \mathfrak{g} , $\{\theta^a\}_{1 \leq a \leq n}$ dual basis of \mathfrak{g}^* .

$\widehat{\omega} \in \Omega^p(A, L)$ and $\widehat{\omega}_{\text{loc}} \in \Omega_{\text{TLA}}^p(\mathcal{U}, \mathfrak{g})$ its local description:

$$\widehat{\omega}_{\text{loc}} = \sum_{r+s=p} \widehat{\omega}_{\mu_1 \dots \mu_r a_1 \dots a_s}^{\theta} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_r} \wedge \theta^{a_1} \wedge \dots \wedge \theta^{a_s}, \quad \widehat{\omega}_{\mu_1 \dots \mu_r a_1 \dots a_s}^{\theta} : \mathcal{U} \rightarrow \mathfrak{g}$$

$\triangleleft \theta^a$ is not convenient \rightarrow inhomogeneous transformations!

∇ ordinary connection on A , $\dot{\omega}$ its connection 1-form

$\rightarrow \dot{\omega}_{\text{loc}} = (A^a - \theta^a)E_a$ with $A^a \in \Omega^1(\mathcal{U}) \otimes \mathfrak{g}$,

\rightarrow the $\dot{\omega}^a = A^a - \theta^a \in \Omega_{\text{TLA}}^1(\mathcal{U})$ define the **mixed basis** in $\Omega_{\text{TLA}}^1(\mathcal{U})$.

Then one can write

$$\widehat{\omega}_{\text{loc}} = \sum_{r+s=p} \widehat{\omega}_{\mu_1 \dots \mu_r a_1 \dots a_s} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_r} \wedge \dot{\omega}^{a_1} \wedge \dots \wedge \dot{\omega}^{a_s}, \quad \widehat{\omega}_{\mu_1 \dots \mu_r a_1 \dots a_s} : \mathcal{U} \rightarrow \mathfrak{g}$$

Proposition 26 (Homogeneous transformations). *The $\widehat{\omega}_{\mu_1 \dots \mu_r a_1 \dots a_s}$'s have homog. transf. in a change of local trivializations.*

.....
Integration along the kernel

$\mathcal{U} \subset \mathcal{M}$ an open subset which trivializes A . $\{E_a\}$ basis of \mathfrak{g} , $\{\theta^a\}$ dual basis of \mathfrak{g}^* .

∇ a connection on A , $\dot{\omega} \in \Omega^1(A, L)$ its connection 1-form.

$\rightarrow \dot{\omega}^a = A^a - \theta^a \in \Omega_{\text{TLA}}^1(\mathcal{U})$ mixed basis in $\Omega_{\text{TLA}}^1(\mathcal{U})$.

h a metric on L .

$h_{\text{loc}} =$ trivialization of h over \mathcal{U} , $h_{ab} = h_{\text{loc}}(E_a, E_b) \in C^\infty(\mathcal{U})$, $|h_{\text{loc}}| = |\det(h_{ab})|$.

Suppose \mathcal{L} is orientable where $L = \Gamma(\mathcal{L})$ (A is called inner orientable),

Proposition 27 (Volume form along L and inner integration).

$$\widehat{\omega}_{h, \dot{\omega} \text{ loc}} = (-1)^n \sqrt{|h_{\text{loc}}|} \dot{\omega}_{\text{loc}}^1 \wedge \dots \wedge \dot{\omega}_{\text{loc}}^n$$

defines a global form $\widehat{\omega}_{h, \dot{\omega}} \in \Omega^\bullet(A)$ of maximal degree $n = \dim \mathfrak{g}$ in the L direction.

This volume form defines integrations

$$\int_{\text{inner}} : \Omega^\bullet(A, L) \rightarrow \Omega^{\bullet-n}(\mathcal{M}, \mathcal{L}) \qquad \int_{\text{inner}} : \Omega^\bullet(A) \rightarrow \Omega^{\bullet-n}(\mathcal{M}).$$

They do not depend on ∇ .

.....

Hodge star operator

A an orientable transitive Lie algebroid.

$\widehat{g} = (g, h, \nabla)$ a metric on \mathbf{A} , $\widehat{\omega}$ the connection 1-form of ∇ .

$\widehat{\omega} \in \Omega^p(A, L)$, written locally as

$$\widehat{\omega}_{\text{loc}} = \sum_{r+s=p} \widehat{\omega}_{\mu_1 \dots \mu_r a_1 \dots a_s} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_r} \wedge \widehat{\omega}^{a_1} \wedge \dots \wedge \widehat{\omega}^{a_s}$$

$\star \widehat{\omega}_{\text{loc}} \in \Omega_{\text{TLA}}^{m+n-p}(U, \mathfrak{g})$ is defined by (usual notations)

$$\begin{aligned} \star \widehat{\omega}_{\text{loc}} = \sum_{r+s=p} (-1)^{s(m-r)} \frac{1}{r!s!} \sqrt{|h_{\text{loc}}|} \sqrt{|g|} \widehat{\omega}_{\mu_1 \dots \mu_r a_1 \dots a_s} \epsilon_{v_1 \dots v_m} \epsilon_{b_1 \dots b_n} \\ \times g^{\mu_1 v_1} \dots g^{\mu_r v_r} h^{a_1 b_1} \dots h^{a_s b_s} dx^{v_{r+1}} \wedge \dots \wedge dx^{v_m} \wedge \widehat{\omega}^{b_{s+1}} \wedge \dots \wedge \widehat{\omega}^{b_n} \end{aligned}$$

Proposition 28 (Hodge star operator). *The map $\star : \Omega^p(A, L) \rightarrow \Omega^{m+n-p}(A, L)$ is well defined globally.*

This is the Hodge star operator associated to \widehat{g} on A .

.....

Gauge transformations on Atiyah Lie algebroids

Suppose G is connected and simply connected.

$\widehat{\omega} \in \Omega_{\text{Lie}}^1(\mathcal{P}, \mathfrak{g}) \mapsto \widehat{\omega}_{\text{g equ}} \in \Omega_{\text{TLA}}^1(\mathcal{P}, \mathfrak{g})$, $\mathfrak{g}_{\text{equ}}$ -basic.

The gauge action $\widehat{\omega} \mapsto \widehat{\omega}^u$ induces

$$\widehat{\omega}_{\text{g equ}} \mapsto u \widehat{\omega}_{\text{g equ}} u^{-1} + u \widehat{d}_{\text{TLA}} u^{-1} \in \Omega_{\text{TLA}}^1(\mathcal{P}, \mathfrak{g})$$

where

$$u \widehat{d}_{\text{TLA}} u^{-1} = u du^{-1} + u \theta u^{-1} - \theta$$

($\theta = \text{Cartan 1-form}$)

Notice that $u \theta u^{-1} - \theta = u[\theta, u^{-1}]$ is more or less “s” applied to u .

Proposition 29 (Ordinary gauge transformations). *If $\widehat{\omega}_{\text{g equ}} = \omega^{\mathcal{P}} - \theta$ is an ordinary connection on $\Gamma_G(T\mathcal{P})$, this action reduces to the usual gauge transformation $\omega^{\mathcal{P}} \mapsto u \omega^{\mathcal{P}} u^{-1} + u du^{-1}$ on the (ordinary) connection 1-form $\omega^{\mathcal{P}}$.*

.....

Decomposition of a connection

$\widehat{\omega} \in \Omega^1(A, L)$ a generalized connection on A .

Definition 30 (Reduced kernel endomorphism). The reduced kernel endomorphism $\tau \in \text{End}(\mathcal{L})$ associated to $\widehat{\omega}$ is defined by

$$\tau = \widehat{\omega} \circ \iota + \text{Id}_L.$$

τ vanishes iff $\widehat{\omega}$ is an ordinary connection on A

→ measures the “non Yang-Mills” part.

τ is not a Lie morphism. Define $R_\tau(\gamma, \eta) = [\tau(\gamma), \tau(\eta)] - \tau([\gamma, \eta])$ for any $\gamma, \eta \in L$.

Let $\check{\omega} \in \Omega^1(A, L)$ be a fixed ordinary connection on A (“background connection”).

Theorem 31. $\widehat{\omega} \in \Omega^1(A, L)$ a connection and τ its reduced kernel endomorphism.

$$\omega = \widehat{\omega} + \tau(\check{\omega})$$

is an ordinary connection on A .

The induced infinitesimal gauge action of L is the one on ordinary connections.

$\widehat{\omega}$ ordinary connection → $\tau = 0$ → $\omega = \widehat{\omega}$.

→ $\check{\omega}$ only relevant for connections which are not ordinary connections.

.....

Decomposition of curvature and covariant derivative

$\widehat{\omega} = \omega - \tau(\check{\omega})$ connection on A .

$\check{\nabla}, \nabla : \Gamma(T\mathcal{M}) \rightarrow A$ the splittings associated to the ordinary connections $\check{\omega}, \omega$.

$\check{R}, R \in \Omega^2(\mathcal{M}, \mathcal{L})$ the curvature 2-forms of $\check{\omega}, \omega$.

$\widehat{F} = R - \tau \circ \check{R} \in \Omega^2(\mathcal{M}, \mathcal{L}) \Rightarrow \rho^* \widehat{F} \in \Omega^2(A, L)$.

For $X \in \Gamma(T\mathcal{M})$, define $\mathcal{D}_X \tau \in \text{End}(\mathcal{L})$ by, for any $\gamma \in L$,

$(\mathcal{D}_X \tau)(\gamma) = [\nabla_X, \tau(\gamma)] - \tau([\check{\nabla}_X, \gamma]) \Rightarrow (\rho^* \mathcal{D} \tau) \circ \check{\omega} \in \Omega^2(A, L)$.

$\nabla^\mathcal{E}$ the (ordinary) covariant derivative induced on \mathcal{E} by the (ordinary) connection ω .

For any $\varphi \in \Gamma(\mathcal{E})$, one has $\rho^* \phi(\nabla) \cdot \varphi = \rho^* \nabla^\mathcal{E} \varphi$.

Proposition 32 (Decomposition of the curvature and the covariant derivative). The curvature $\widehat{R} \in \Omega^2(A, L)$ of $\widehat{\omega}$ can be decomposed as

$$\widehat{R} = \rho^* \widehat{F} - (\rho^* \mathcal{D} \tau) \circ \check{\omega} + R_\tau \circ \check{\omega}$$

The covariant derivative $\widehat{\nabla}^\mathcal{E} \varphi \in \Omega^1(A, \mathcal{E})$ can be decomposed as

$$\widehat{\nabla}^\mathcal{E} \varphi = \rho^* \phi(\nabla) \cdot \varphi - (\phi_L(\tau) \varphi) \circ \check{\omega}$$

Under infinitesimal gauge transformations, each term has homog. transf.

“ $\circ \check{\omega}$ ” = along the mixed basis and “ ρ^* ” = along $\Gamma(T\mathcal{M})$.

.....
Coupling to matter fields

Matter fields are sections $\varphi \in \Gamma(\mathcal{E})$ of a representation $\phi : A \rightarrow \mathfrak{D}(\mathcal{E})$ of A .

Definition 33 (ϕ_L -compatible metric). A metric $h^\mathcal{E}$ on \mathcal{E} is ϕ_L -compatible if

$$h^\mathcal{E}(\phi_L(\xi)\varphi_1, \varphi_2) + h^\mathcal{E}(\varphi_1, \phi_L(\xi)\varphi_2) = 0$$

for any $\varphi_1, \varphi_2 \in \Gamma(\mathcal{E})$ and any $\xi \in L$.

Generalization of “Killing metric”.

One can define a Hodge star operator on $\Omega^\bullet(A, \mathcal{E})$.

$\widehat{\omega}$ connection on A and $\widehat{\nabla}^\mathcal{E}$ the induced covariant derivative on $\Gamma(\mathcal{E})$.

Proposition 34. *The action functional*

$$\mathcal{S}_{\text{Matter}}[\varphi, \widehat{\omega}] = \int_A h^\mathcal{E}(\widehat{\nabla}^\mathcal{E} \varphi, \star \widehat{\nabla}^\mathcal{E} \varphi)$$

is invariant under infinitesimal gauge transformations in L .

.....
Decomposition of the action functional

$\widehat{\omega}$ connection on A , $\varphi \in \Gamma(\mathcal{E})$ matter field.

$$\mathcal{S}[\varphi, \widehat{\omega}] = \mathcal{S}_{\text{Gauge}}[\widehat{\omega}] + \mathcal{S}_{\text{Matter}}[\varphi, \widehat{\omega}] \text{ total action functional}$$

$\widehat{g} = (g, h, \nabla)$, with $\nabla \leftrightarrow \widehat{\omega} \in \Omega^1(A, L)$, metric on A .

The decomposition $\widehat{\omega} = \omega - \tau(\widehat{\omega})$ induces the decomposition:

$\mathcal{S}[\varphi, \widehat{\omega}] = \langle \rho^* \widehat{F}, \star \rho^* \widehat{F} \rangle$	(1) spatial term: Yang-Mills like
$+ \langle (\rho^* \mathcal{D}\tau) \circ \widehat{\omega}, \star (\rho^* \mathcal{D}\tau) \circ \widehat{\omega} \rangle$	(2) mixed term: covariant derivative of τ
$+ \langle R_\tau \circ \widehat{\omega}, \star R_\tau \circ \widehat{\omega} \rangle$	(3) algebraic term: potential for τ
$+ \langle \rho^* \phi(\nabla) \cdot \varphi, \star \rho^* \phi(\nabla) \cdot \varphi \rangle$	(4) spatial term: covariant derivative of φ
$+ \langle (\phi_L(\tau)\varphi) \circ \widehat{\omega}, \star (\phi_L(\tau)\varphi) \circ \widehat{\omega} \rangle$	(5) algebraic term: coupling $\varphi \leftrightarrow \tau$