An introduction to noncommutative geometry

Aix-en-Provence, June 26, 2014

Thierry Masson

Centre de Physique Théorique
Campus de Luminy, Marseille

NCG: the mathematical side

- NCG is motivated by deep results on correspondences *spaces* ⟷ *algebras*.
  - Measurable spaces ➔ abelian von Neumann algebras.
  - Topological spaces ➔ commutative C^*-algebras.
- **Fact 1:** some tools used to study these spaces have algebraic counterpart.
- **Fact 2:** these algebraic tools can be applied to NC algebras.

Main idea of NCG:
replace commutative algebras of functions
by NC algebras in an identified category.

- Replace the geometric approach by an algebraic one.
- Give new light on difficult problems:
  ➔ foliations and quotient spaces (NC torus)...
- "Differentiability" has been investigated in the 1980's (Connes).
  ➔ Cyclic homology, Chern character, index theorems...
- NC "riemannian manifolds": spectral triples.
  ➔ reconstruction theorem in 2008.

NCG: the physical side

- Physics in crisis:

  Geometrical theories: general relativity, gauge field theories…
  Algebraic theories: quantum mechanic, QFT…

  How to unify them?

- NCG is not a *theory* in physics.
- NCG is a framework in which to think about new theories.
  ➡ different conceptualisations, unification…
- NCG has been constructed in relation to physics.
  ➡ NC gauge field theories, NC space-times, quantum groups…
- Some NC topological invariants are been used to explained (partially) the Quantum Hall Effect.
- QFT on NC spaces ➡ new renormalizable non local models…
  (ϕ^4 theories on Moyal space)
- NCG gauge field theories contains naturally Higgs-like particles.

Abelian on von Neuman algebras

Theorem (Dixmier, *von Neumann algebras*, 1981)

Let H be a complex Hilbert space, and Z an abelian von Neumann algebra in H. There exists a locally compact space z, a positive measure ν on z, with support z, and an isometric isomorphism of the normed $*$-algebra Z onto the normed $*$-algebra $L^\infty_{\mathbb{C}}(z, \nu)$.

➡ von Neumann algebras are "NC measurable spaces".

- Every von Neumann algebra on a separable Hilbert space is isomorphic to a direct integral of factors.
  (factor = von Neumann algebra with trivial center).
- NC integration (weights), NC probability theory (states).
- Any locally compact group defines a von Neumann algebra (and a C^*-algebra).
  abelian group ➡ abelian von Neumann (C^*-) algebra.
  ➡ Fourier transformation (Pontryagin dual gr.).
  ➡ NC harmonic analysis...
- Tomita-Takesaki theory of von Neumann algebra:
  ➡ extends to von Neumann algebra the non-modularity of groups;
  ➡ relation to KMS states in statistical physics.

Commutative C^*-algebras

Theorem (Gelfand-Naimark)

The category of locally compact Hausdorff spaces is anti-equivalent to the category of commutative C^*-algebras.

Space $X \longleftrightarrow$ algebra of continuous functions $C_0(X)$ vanishing at infinity.

This leads to the correspondences:

Spaces	Algebras
point	irreducible representation
compact	unital
1-point compactification	unitarization
Stone-Čech compactification	multiplier algebra
homeomorphism	automorphism
Borel measure	positive functional
probability measure	state

Finite projective modules

Theorem (Serre-Swan)

The category of complex vector bundles on a compact Hausdorff space X is equivalent to the category of finite projective modules over the algebra $C(X)$ (continuous functions).

Vector bundle $E \longleftrightarrow$ Space of continuous sections $\Gamma(E)$.

➞ projection in some $M_N(C(X))$.

This works also in the category of smooth manifolds.

- Notion of “vector bundles” in NCG: finite projective modules over $\mathbf{A}$.
- Covariant derivatives have a NC generalisation (explained later).
  ➞ This permits to define NC gauge field theories.

Origin of common NC spaces

NC spaces are in general defined as von Neumann algebras or C^*-algebras.

Many constructions can give very interesting examples:

Deformation: the idea is to deform a commutative algebra (+ extra structure…).
- ➞ Moyal algebra, related to the canonical commutation relations in QM.
- ➞ κ-Minkowski space, (co)-representation space of a quantum group.

Group algebras: any locally compact group defines a C^*-algebra.
- ➞ Study of the representation theory of the group.
- ➞ More generally: C^*-algebra of a smooth groupoid.

Cross products: action of a locally compact group on a given algebra.
- ➞ Compatible with semidirect product of groups and C^*-alg. of groups.

Quantum groups: Hopf algebra structures.
- ➞ usually a deformation of the matrix entries of an ordinary group.
- ➞ representation theory, new “symmetries”…

Quotients by equivalence relation: general construction gives an algebra which is Morita equivalent to the “expected” one if the quotient space is good enough.

Generators and relations: the algebra is directly defined by some elements.
- ➞ Compatible with C^*-alg. of groups presented as generators and relations.

K-theories

X a compact topological space:

- $K^0(X)$: Grothendieck group of the semigroup $V(X)$ of isomorphic classes of vector bundles over X.
  (Reminder: $\mathbb{Z}$ is the Grothendieck group of the semigroup $\mathbb{N}$...)
- Definition extends to non compact spaces.
- $K^{-n}(X) = K^0(X \times \mathbb{R}^n)$.

$\mathbf{A}$ a unital C^*-algebra:

- $K_0(\mathbf{A})$: Grothendieck group of the semigroup $V(\mathbf{A})$ of homotopy classes of projections in $M_\infty(\mathbf{A})$ (finite proj. mod. on $\mathbf{A}$).
- Definition extends to non unital C^*-algebras.
- $K_n(\mathbf{A}) = K_0(C_0(\mathbb{R}^n, \mathbf{A}))$.

Gelfand-Naimark + Serre-Swan theorems ➞ $K^{-n}(X) = K_n(C_0(X))$.

The topological invariants detected by K-groups are the same.

➞ K-groups of C^*-algebras are more general.

K-theories (cont'd)

- *K*-theory of C^*-algebras works for noncommutative algebras.
  ➙ Essential tool to study NC spaces in NCG.
- *K*-theory of Fréchet, pre-C^*, or Banach algebras is well defined.
  ➙ *K*-groups of a C^*-algebra can be computed using a dense subalgebra.
  (stable by holomorphic functional calculus)

$\mathbf{A}$ a unital associative algebra:

- Algebraic *K*-theory: $K_\nu^{\text{alg}}(\mathbf{A})$, $\nu = 0, 1$, for associative algebra without topology.
  (Projectors in $M_\infty(\mathbf{A})$ ($p \sim u^{-1}qu$) and abelianization of $GL_\infty(\mathbf{A})$)
- Definitions for higher degrees are very involved…

Other "*K*-theories":

- *K*-homology, dual to *K*-theory.
  ➙ Based on Fredholm operators on Hilbert spaces.
- *KK*-theory, contains *K*-theory and *K*-homology.
  ➙ Based on Hilbert C^*-modules = generalization of Hilbert spaces.

K-theories: some properties

Theorem (Bott periodicity)

For any C^*-algebra $\mathbf{A}$*, one has*

$$K_2(\mathbf{A}) \simeq K_0(\mathbf{A})$$

➡ Only two K-groups: $K_0(\mathbf{A})$ and $K_1(\mathbf{A})$.

Proposition (Six terms exact sequence)

For any short exact sequence of C^*-algebras $0 \longrightarrow \mathbf{I} \longrightarrow \mathbf{A} \longrightarrow \mathbf{A}/\mathbf{I} \longrightarrow 0$, *one has the six terms exact sequence*

$$\begin{array}{ccccc} K_0(\mathbf{I}) & \longrightarrow & K_0(\mathbf{A}) & \longrightarrow & K_0(\mathbf{A}/\mathbf{I}) \\ {\scriptstyle\delta}\uparrow & & & & \downarrow{\scriptstyle\delta} \\ K_1(\mathbf{A}/\mathbf{I}) & \longleftarrow & K_1(\mathbf{A}) & \longleftarrow & K_1(\mathbf{I}) \end{array}$$

➡ δ are index maps (as in Atiyah-Singer index theorem…).

Proposition (Algebraic K-theory)

$\mathbf{A}$ *a C^*-algebra:* $K_0^{alg}(\mathbf{A}) = K_0(\mathbf{A})$ *and* $K_1^{alg}(\mathbf{A}) \to K_1(\mathbf{A})$ *(not an isomorphism).*

Worth mentioning Morita invariance also…

Cyclic homology

A unital associative algebra.

Hochschild complex with values in **A**: $\mathbf{A} \xleftarrow{b} \cdots \xleftarrow{b} \mathbf{A}^{\otimes n} \xleftarrow{b} \mathbf{A}^{\otimes n+1} \xleftarrow{b} \cdots$

$$b(a_0 \otimes \cdots \otimes a_n) = \sum_{p=0}^{n-1} (-1)^p a_0 \otimes \cdots \otimes a_p a_{p+1} \otimes \cdots \otimes a_n + (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}$$

Hochschild homology: $HH_\bullet(\mathbf{A})$ homology of this complex.

$t : \mathbf{A}^{\otimes n} \to \mathbf{A}^{\otimes n}$ cyclic operator: $t(a_1 \otimes \cdots \otimes a_n) = (-1)^{n+1} a_n \otimes a_1 \otimes \cdots \otimes a_{n-1}$.
b well-defined on $C_n^\lambda(\mathbf{A}) = \mathbf{A}^{\otimes n+1} / \operatorname{Ran}(1 - t)$.
Cyclic homology: $HC_\bullet(\mathbf{A})$ homology of $(C_\bullet^\lambda(\mathbf{A}), b)$.

Proposition (Connes long exact sequence)

There are morphisms I and S which induce the following long exact sequence

$$\cdots \longrightarrow HH_n(\mathbf{A}) \xrightarrow{I} HC_n(\mathbf{A}) \xrightarrow{S} HC_{n-2}(\mathbf{A}) \xrightarrow{B} HH_{n-1}(\mathbf{A}) \xrightarrow{I} \cdots$$

For non unital algebras, need bicomplexes...
Cyclic cohomology is defined using **Hochschild cohomology** with values in $\mathbf{A}^*$.
➞ similar operators I and S, and long exact sequence.

Periodic cyclic (co)homology

Periodic cyclic cohomology: $HP^\bullet(\mathbf{A})$ is defined using S.

Only 2 groups: $HP^0(\mathbf{A}) = \varinjlim HC^{2n}(\mathbf{A})$ and $HP^1(\mathbf{A}) = \varinjlim HC^{2n+1}(\mathbf{A})$.

In the same way, one can define the **periodic cyclic homology:** $HP_\bullet(\mathbf{A})$.

Proposition (Six terms exact sequence)

For any short exact sequence of associative algebras $0 \longrightarrow \mathbf{I} \longrightarrow \mathbf{A} \longrightarrow \mathbf{A}/\mathbf{I} \longrightarrow 0$*, one has the six terms exact sequence*

$$\begin{array}{ccccc} HP_0(\mathbf{I}) & \longrightarrow & HP_0(\mathbf{A}) & \longrightarrow & HP_0(\mathbf{A}/\mathbf{I}) \\ \delta \uparrow & & & & \downarrow \delta \\ HP_1(\mathbf{A}/\mathbf{I}) & \longleftarrow & HP_1(\mathbf{A}) & \longleftarrow & HP_1(\mathbf{I}) \end{array}$$

Proposition (Diffeotopic invariance)

$\mathbf{A}$ *and* $\mathbf{B}$ *two associative algebras. If* $\varphi_0, \varphi_1 : \mathbf{A} \to \mathbf{B}$ *are diffeotopic, then they induce the same morphism* $HP_\nu(\mathbf{A}) \to HP_\nu(\mathbf{B})$.

$\varphi : \mathbf{A} \to \mathbf{B} \otimes C^\infty([0,1])$ s.t. φ_t is φ_0 (resp. φ_1) at $t = 0$ (resp. $t = 1$)

➡ Does not work for homotopy!

Worth mentioning Morita invariance also…

Cyclic homology: examples

Example A = $\mathbb{C}$:

$$HH_0(\mathbb{C}) = \mathbb{C} \qquad HH_n(\mathbb{C}) = 0 \text{ for } n \geq 1 \qquad HP_0(\mathbb{C}) = \mathbb{C} \qquad HP_1(\mathbb{C}) = 0$$

Example A = $\mathbb{C}[z, z^{-1}]$ (Laurent polynomials):

$$HP_0(\mathbb{C}[z, z^{-1}]) = \mathbb{C} \qquad HP_1(\mathbb{C}[z, z^{-1}]) = \mathbb{C}$$

Hochschild and cyclic homologies can be defined for topological algebras.

Example A = $C(X)$, continuous functions on a compact space X:

$$HP_0^{\text{cont}}(C(X)) = \{\text{bounded measures on } X\} \qquad HP_1^{\text{cont}}(C(X)) = 0$$

M a finite dim. loc. compact manifold.

Example A = $C^\infty(M)$, Fréchet algebra of smooth functions on M:

Theorem (Connes, 1985)

$$HH_\bullet^{Cont}(C^\infty(M)) = \Omega_{\mathbb{C}}^\bullet(M) \text{ (complexified de Rham forms)}$$

$$HP_0^{cont}(C^\infty(M)) = H_{dR}^{even}(M) \qquad HP_1^{cont}(C^\infty(M)) = H_{dR}^{odd}(M)$$

➞ Cyclic homology is the NC generalization of de Rham cohomology.

The Chern character

Theorem (The (geometric) Chern character)

The (usual) Chern character $\mathrm{ch}(E) = \mathrm{tr} \circ \exp\left(\frac{iF}{2\pi}\right)$ *realizes an isomorphism*

$$\mathrm{ch} : K^0(M) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\simeq} H^{even}(M; \mathbb{Q})$$

for locally compact finite dimensional (smooth) manifolds M.
It can be extended to an isomorphism $\mathrm{ch} : K^1(M) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\simeq} H^{odd}(M; \mathbb{Q})$.

Proposition (The (algebraic) Chern character)

The Chern character realizes a map $\mathrm{ch} : K_\nu^{alg}(\mathbf{A}) \to HP_\nu(\mathbf{A})$ *for* $\nu = 0, 1$.

➡ Defined by the generators of $HP_0(\mathbb{C}) = \mathbb{C}$ and $HP_1(\mathbb{C}[z, z^{-1}]) = \mathbb{C}$.

The algebraic Chern character factorizes through K-theory of topological algebras.

Theorem (The (NC) Chern character)

For a large class of Fréchet algebras, the Chern character realizes an isomorphism

$$\mathrm{ch} : K_\nu(\mathbf{A}) \otimes \mathbb{C} \xrightarrow{\simeq} HP_\nu(\mathbf{A})$$

➡ The Fréchet algebras $C^\infty(M)$ for locally compact manifolds M are in this class.

Pairing and Fredholm modules

$\mathbf{A}$ a topological algebra (Fréchet, pre-C^*, Banach).

- We have introduced the Chern character as a map $\mathrm{ch} : K_\nu(\mathbf{A}) \to HP_\nu(\mathbf{A})$.
- It is also a pairing $K_\nu(\mathbf{A}) \times HP^\nu(\mathbf{A}) \to \mathbb{C}$.
- It is also a map $K^\nu(\mathbf{A}) \to HP^\nu(\mathbf{A})$, where $K^\nu(\mathbf{A})$ is the K-homology of $\mathbf{A}$.
  - Elements in $K^1(\mathbf{A})$ are classes of (odd) Fredholm modules $(\mathbf{A}, \mathcal{H}, F)$;
  - $\mathcal{H}$ is a Hilbert space which supports an involutive representation π of $\mathbf{A}$;
  - F is a bounded operator on $\mathcal{H}$ such that $F = F^*$, $F^2 = 1$;
  - $[F, \pi(a)]$ is compact for any $a \in \mathbf{A}$;
  - then (Connes, 1985)

$$\tau(a^0, \dots, a^n) = \mathrm{Tr}(a^0[F, a^1] \cdots [F, a^n])$$

    defines an element in $HP^1(\mathbf{A})$.
  - Need trace-class operators: "summability" of Fredholm modules.
    ➞ Schatten ideals $\mathcal{L}^p(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) \ / \ \mathrm{Tr}(|T|^p) < \infty\}$.
    ➞ "Infinitesimals of order p in this quantum calculus" (Connes).

Remark

K-theory is a theory for NC topological spaces.
Periodic cyclic homology is a theory for algebras with "differentiable structures".

Spectral triples

Spectral triples are "unbounded Fredholm modules".

$\mathbf{A}$ an involutive unital associative algebra.

Definition (Spectral triple)

A spectral triple on $\mathbf{A}$ is a triple $(\mathbf{A}, \mathcal{H}, \mathcal{D})$ where

- $\mathcal{H}$ is a Hilbert space on which an involutive representation ρ of $\mathbf{A}$ is given;
- $\mathcal{D}$ is a (unbounded) self-adjoint operator on $\mathcal{H}$;
- the resolvant of $\mathcal{D}$ is compact;
- $[\mathcal{D}, \rho(a)]$ is bounded for any $a \in \mathbf{A}$.

Many more axioms for complete description:

- Grading ➞ charge conjugaison in physics.
- Reality operator ➞ Tomita-Takesaki theory.
- Regularity condition
  ➞ defines the "smooth" algebra $\mathbf{A}$ as a dense subalgebra of a C^*-algebra

Spectral triples (cont'd)

- $(\mathbf{A}, \mathcal{H}, \mathcal{D}(1+\mathcal{D}^2)^{-1/2})$ defines a Fredholm modules (except for $F^2 = 1$):
  - class in $K^\nu(\mathbf{A})$;
  - element in $HP^\nu(\mathbf{A})$ by the Chern character;
  - Summability conditions on $(\mathbf{A}, \mathcal{H}, \mathcal{D})$.
- The operator $\mathcal{D}$ is called a **Dirac operator**.
  ➞ This comes from $(\mathbf{A}, \mathcal{H}, \mathcal{D}) = (C^\infty(M), L^2(\$), \not{\partial})$ where
  - M is a smooth compact Riemannian spin manifold,
  - $\$$ is a spin bundle,
  - $\not{\partial}$ is the usual Dirac operator.

  ➞ This is the commutative prototype of spectral triples.
- **Reconstruction theorem by Connes (2008):**
  Commutative spectral triples (with additional axioms) are of the form $(C^\infty(M), L^2(\$), \not{\partial})$.
- A spectral triple $(\mathbf{A}, \mathcal{H}, \mathcal{D})$ encodes some metric properties of the "NC spaces".
  ➞ distance on the space of states of $\mathbf{A}$.
- Behavior of the eigenvalues of $|\mathcal{D}|$ ➞ dimension of the spectral triple.

Differential structures

$\mathbf{A}$ an associative algebra.

Definition (Differential calculus on an algebra)

A differential calculus on $\mathbf{A}$ is a graded differential algebra $(\Omega^\bullet, \mathrm{d})$ such that $\Omega^0 = \mathbf{A}$.

➠ Many differential calculi can be constructed on a given algebra.

Example (Universal unital differential calculus)

$\mathbf{A}$ a unital associative algebra.
$(\Omega^\bullet_U(\mathbf{A}), \mathrm{d}_U)$ is the free unital graded diff. alg. generated by $\mathbf{A}$ in degree 0.
Elements are finite sum of $a\mathrm{d}_U b_1 \cdots \mathrm{d}_U b_n$ for $a, b_1, \dots, b_n \in \mathbf{A}$.

Universal property: *for any unital diff. calc.* $(\Omega^\bullet, \mathrm{d})$ *on* $\mathbf{A}$*, there exists a unique morphism of unital diff. calc.* $\phi : \Omega^\bullet_U(\mathbf{A}) \to \Omega^\bullet$ *(of degree* 0*) such that* $\phi(a) = a$ *for any* $a \in \mathbf{A} = \Omega^0_U(\mathbf{A}) = \Omega^0$.

➠ Many diff. calc. are quotients of $(\Omega^\bullet_U(\mathbf{A}), \mathrm{d}_U)$.

Derivation based differential calculus

$\mathbf{A}$ an associative algebra with unit $\mathbb{1}$.

- $\mathcal{Z}(\mathbf{A}) = \{a \in \mathbf{A} \;/\; ab = ba, \forall b \in \mathbf{A}\}$ its center.
- Space of derivations of $\mathbf{A}$:
  $$\mathrm{Der}(\mathbf{A}) = \{\mathfrak{X} : \mathbf{A} \to \mathbf{A} \;/\; \mathfrak{X} \text{ linear}, \mathfrak{X}\cdot(ab) = (\mathfrak{X}\cdot a)b + a(\mathfrak{X}\cdot b), \forall a, b \in \mathbf{A}\}.$$
  ➞ Lie algebra: $[\mathfrak{X}, \mathfrak{Y}]a = \mathfrak{X}\mathfrak{Y}a - \mathfrak{Y}\mathfrak{X}a$ for all $\mathfrak{X}, \mathfrak{Y} \in \mathrm{Der}(\mathbf{A})$,
  ➞ $\mathcal{Z}(\mathbf{A})$-module: $(f\mathfrak{X})\cdot a = f(\mathfrak{X}\cdot a)$ for all $f \in \mathcal{Z}(\mathbf{A})$ and $\mathfrak{X} \in \mathrm{Der}(\mathbf{A})$.
- $\underline{\Omega}^0_{\mathrm{Der}}(\mathbf{A}) = \mathbf{A}$.
- $\underline{\Omega}^n_{\mathrm{Der}}(\mathbf{A})$ space of $\mathcal{Z}(\mathbf{A})$-mult. antisym. maps from $\mathrm{Der}(\mathbf{A})^n$ to $\mathbf{A}$, $n \geq 1$.
- $\underline{\Omega}^\bullet_{\mathrm{Der}}(\mathbf{A}) = \bigoplus_{n\geq 0} \underline{\Omega}^n_{\mathrm{Der}}(\mathbf{A})$.
- $\mathbb{N}$-graded differential algebra (product by antisymmetrization):
  $$\widehat{\mathrm{d}}\omega(\mathfrak{X}_1, \dots, \mathfrak{X}_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} \mathfrak{X}_i\cdot\omega(\mathfrak{X}_1, \dots \overset{i}{\vee} \dots, \mathfrak{X}_{n+1})$$
  $$+ \sum_{1\leq i<j\leq n+1} (-1)^{i+j} \omega([\mathfrak{X}_i, \mathfrak{X}_j], \dots \overset{i}{\vee} \dots \overset{j}{\vee} \dots, \mathfrak{X}_{n+1}).$$
- $\mathbf{A} = C^\infty(M)$ ➞ $(\underline{\Omega}^\bullet_{\mathrm{Der}}(\mathbf{A}), \widehat{\mathrm{d}})$ is the de Rham calculus.

Spectral triples and forms

$(\mathbf{A}, \mathcal{H}, \mathcal{D})$ a spectral triple.

- Any form $\omega = \sum a \mathrm{d}_U b_1 \cdots \mathrm{d}_U b_n \in \Omega^\bullet_U(\mathbf{A})$ gives a bounded operator
$$\pi_\mathcal{D}(\omega) = \sum \pi(a)[\mathcal{D}, \pi(b_1)] \cdots [\mathcal{D}, \pi(b_n)]$$
- This is not a representation of the $(\Omega^\bullet_U(\mathbf{A}), \mathrm{d}_U)$ as a graded diff. alg.
- Perturbation of $\mathcal{D}$ by bounded operators does not change the K-homology class.
  ➸ $\mathcal{D} + \pi_\mathcal{D}(\omega)$ in the same class.
- Case $(C^\infty(M), L^2(\$), \not{D})$:
  $E \to M$ vector bundle, ω connection 1-form on E
  ➸ "$\not{D} + \omega$" is the twisted Dirac operator defined on $\$ \otimes E$.

NC connections

$\mathbf{A}$ an associative algebra with unit $\mathbb{1}$. $(\Omega^\bullet, \mathrm{d})$ a diff. calc. on $\mathbf{A}$.
$\boldsymbol{M}$ a finite projective right module over $\mathbf{A}$.

Definition (NC connection)

A NC connection on $\boldsymbol{M}$ is a linear map $\widehat{\nabla} : \boldsymbol{M} \to \boldsymbol{M} \otimes_{\mathbf{A}} \Omega^1$ such that $\widehat{\nabla}(ma) = (\widehat{\nabla} m)a + m \otimes \mathrm{d}a$ for any $m \in \boldsymbol{M}$ and $a \in \mathbf{A}$.

$\widehat{\nabla}$ can be extended as $\widehat{\nabla} : \boldsymbol{M} \otimes_{\mathbf{A}} \Omega^p \to \boldsymbol{M} \otimes_{\mathbf{A}} \Omega^{p+1}$, for any $p \geq 0$, using the rule

$$\widehat{\nabla}(m \otimes \omega_p) = (\widehat{\nabla} m) \otimes \omega_p + m \otimes \mathrm{d}\omega_p \quad \text{for any } \omega_p \in \Omega^p.$$

Definition (Curvature)

The curvature of $\widehat{\nabla}$ is defined as $\widehat{R} = \widehat{\nabla}^2 = \widehat{\nabla} \circ \widehat{\nabla} : \boldsymbol{M} \to \boldsymbol{M} \otimes_{\mathbf{A}} \Omega^2$.
It satisfies $\widehat{R}(ma) = (\widehat{R} m)a$ for any $m \in \boldsymbol{M}$ and $a \in \mathbf{A}$

Definition (Gauge transformations)

$\mathcal{G}$ the group of automorphisms of $\boldsymbol{M}$ as a right $\mathbf{A}$-module.
For any $\Phi \in \mathcal{G}$, $\widehat{\nabla}^\Phi = \Phi^{-1} \circ \widehat{\nabla} \circ \Phi$ is also a NC connection on $\boldsymbol{M}$.

$\mathbf{A} = C^\infty(M)$ ➞ usual theory of connections on vector bundles $\boldsymbol{M} = \Gamma(E)$.

NC connections: special case $M = \mathbf{A}$

Special example of right module: $\boldsymbol{M} = \mathbf{A}$.

- Since $\mathbf{A}$ is unital: $\widehat{\nabla}(a) = \widehat{\nabla}(\mathbb{1}a) = \widehat{\nabla}(\mathbb{1})a + \mathbb{1} \otimes \mathrm{d}a = \widehat{\nabla}(\mathbb{1})a + \mathrm{d}a$.
- $\widehat{\nabla}(\mathbb{1}) = \omega \in \Omega^1$ characterizes completely $\widehat{\nabla}$.
- ω is the NC connection 1-form.
- The curvature of $\widehat{\nabla}$ is the left multiplication by the 2-form $\Omega = \mathrm{d}\omega + \omega\omega \in \Omega^2$.
- $\Phi \in \mathcal{G}$ is completely determined by $\Phi(\mathbb{1}) = g \in \mathbf{A}$ (invertible element).
- Gauge transformation of $\widehat{\nabla}$:

$$\omega \mapsto g^{-1}\omega g + g^{-1}\mathrm{d}g, \qquad \Omega \mapsto g^{-1}\Omega g.$$

➞ This permits to construct NC gauge field theories:

- Need an appropriate Lagrangian ➞ different approaches.
- For $\mathbf{A} = C^\infty(M) \otimes \mathbf{A}_F$ where $\mathbf{A}_F$ = finite dim. alg. $\simeq M_n(\mathbb{C}), \mathbb{C}^N \dots$
  ➞ NC connections split in two parts:
  - geometric along $C^\infty(M)$ ➞ "ordinary" gauge field (Yang-Mills).
  - algebraic along $\mathbf{A}_F$ ➞ new fields $\simeq$ scalar fields of the Higgs mechanism.

NC connections and spectral triples

Definition

Two spectral triples $(\mathbf{A}, \mathcal{H}, \mathcal{D})$ and $(\mathbf{A}', \mathcal{H}', \mathcal{D}')$ are unitary equivalent if there exists a unitary operator $U : \mathcal{H} \to \mathcal{H}'$ and an algebra isomorphism $\phi : \mathbf{A} \to \mathbf{A}'$ such that $\pi' \circ \phi = U \pi U^{-1}$, $\mathcal{D}' = U \mathcal{D} U^{-1}$.

Inner fluctuations:

- Case $\mathcal{H}' = \mathcal{H}$, $\mathbf{A}' = \mathbf{A}$, $\pi' = \pi$ and $\phi_u(a) = u^* a u$ for a unitary $u \in U(\mathcal{A})$.
  ➞ Then $U = \pi(u)^*$.
- $\mathcal{D}$ is transformed as $\mathcal{D}^u = U \mathcal{D} U^{-1} = \mathcal{D} + \pi(u)^* [\mathcal{D}, \pi(u)]$.
- Let $\omega \in \Omega^1_U(\mathbf{A})$ be a NC connection 1-form for the module $\boldsymbol{M} = \mathbf{A}$.
  - Interpret $u \in U(\mathcal{A})$ as a (unitary) gauge transformation.
  - $\omega^u = u^* \omega u + u^* \mathrm{d} u$
  - $\pi_{\mathcal{D}}(\omega^u) = \pi(u)^* \pi_{\mathcal{D}}(\omega) \pi(u) + \pi(u)^* [\mathcal{D}, \pi(u)]$.
- Define $\mathcal{D}_\omega = \mathcal{D} + \pi_{\mathcal{D}}(\omega)$, then

$$(\mathcal{D}_\omega)^u = \mathcal{D}_{\omega^u} \quad (= \mathcal{D} + \pi(u)^* \pi_{\mathcal{D}}(\omega) \pi(u) + \pi(u)^* [\mathcal{D}, \pi(u)])$$

To compensate the action of inner symmetries on $\mathcal{D}$, a NC connection is necessary.

➞ Implementation of the gauge principle.

NC torus

θ a real number.

- On the Hilbert space $L^2(\mathbb{S}^1)$, consider the two unitary operators
  $$(Uf)(t) = e^{2\pi i t} f(t) \qquad (Vf)(t) = f(t-\theta)$$
  where $f : \mathbb{S}^1 \to \mathbb{C}$ is a periodic function in the variable $t \in \mathbb{R}$.
- Then $UV = e^{2\pi i \theta} VU \in \mathcal{B}(L^2(\mathbb{S}^1))$.
- $\mathcal{A}_\theta$ the C^*-algebra generated by U and V in $\mathcal{B}(L^2(\mathbb{S}^1))$ ➞ **NC torus**.

Suppose θ is irrational:

- $\mathcal{A}_\theta$ is the **irrational rotation algebra**.
  It is simple and universal for the relation $UV = e^{2\pi i \theta} VU$.
- $\mathcal{A}_\theta = \mathcal{A}_{1-\theta}$.
- $\alpha : \mathbb{S}^1 \to \mathbb{S}^1, z \mapsto e^{2i\pi\theta} z$ ➞ action of $\mathbb{Z}$ on $C(\mathbb{S}^1)$
  ➞ $\mathcal{A}_\theta = C(\mathbb{S}^1) \rtimes_\alpha \mathbb{Z}$ (cross product C^*-algebra).
- $\mathcal{A}_\theta$ is also associated to the Kronecker foliation $\mathrm{d}x = \theta \mathrm{d}y$ of the 2-torus.
- Finite projective modules are classified by two integers (p, q) s.t. $p + q\theta \geq 0$.

NC torus: smooth structures

- $\mathcal{S}(\mathbb{Z}^2)$ the Schwartz space of sequences $(a_{m,n})_{m,n\in\mathbb{Z}}$ of rapid decay: $(|m|+|n|)^q|a_{m,n}|$ is bounded for any $q\in\mathbb{N}$.
- $\mathcal{A}_\theta^\infty$ = elements in $\mathcal{A}_\theta$ of the form $\sum_{m,n\in\mathbb{Z}} a_{m,n}U^mV^n$ for $(a_{m,n})_{m,n\in\mathbb{Z}}\in\mathcal{S}(\mathbb{Z}^2)$.
- By Fourier analysis $\mathcal{S}(\mathbb{Z}^2)\simeq C^\infty(\mathbb{T}^2)$.
  ➞ $\mathcal{A}_\theta^\infty$ is the space of "smooth functions on the NC torus".
- $p_q(a)=\sup_{m,n\in\mathbb{Z}}\{(1+|m|+|n|)^q|a_{m,n}|\}$ is a family of semi-norms on $\mathcal{A}_\theta^\infty$.
  ➞ Fréchet algebra.
- $\mathcal{A}_\theta^\infty$ has two derivations:

$$\delta_1(U^m)=2\pi i m U^m, \qquad \delta_1(V^n)=V^n,$$
$$\delta_2(U^m)=U^m, \qquad \delta_2(V^n)=2\pi i n V^n$$

- $\tau\left(\sum_{m,n\in\mathbb{Z}} a_{m,n}U^mV^n\right)=a_{0,0}$ is a trace (unique, extends to $\mathcal{A}_\theta$).
- Spectral triple:
  - $\mathbf{A}=\mathcal{A}_\theta^\infty$,
  - $\mathcal{H}=$ (Hilbert space of the GNS representation associated to τ)$\otimes\mathbb{C}^2$,
  - $\mathcal{D}=\begin{pmatrix} 0 & \delta_1+i\delta_2 \\ \delta_1-i\delta_2 & 0\end{pmatrix}$.

NC torus: homologies

Let $\lambda = \exp(2\pi i\theta)$.
Diophantine condition: there exists an integer k such that $|1 - \lambda^n|^{-1}$ is $O(n^k)$.

Theorem (Connes, 1985)

If λ satisfies some diophantine condition:

$$HH_0^{Cont}(\mathcal{A}_\theta^\infty) = \mathbb{C} \qquad HH_1^{Cont}(\mathcal{A}_\theta^\infty) = \mathbb{C}^2$$

For any λ:

$$HH_2^{Cont}(\mathcal{A}_\theta^\infty) = \mathbb{C} \qquad HH_n^{Cont}(\mathcal{A}_\theta^\infty) = 0 \text{ for } n \geq 3$$

If λ does not satisfy some diophantine condition:

$HH_0^{Cont}(\mathcal{A}_\theta^\infty)$ and $HH_1^{Cont}(\mathcal{A}_\theta^\infty)$ are infinite dimensional.

For any λ:

$$HP_0^{cont}(\mathcal{A}_\theta^\infty) = \mathbb{C}^2 \qquad HP_1^{cont}(\mathcal{A}_\theta^\infty) = \mathbb{C}^2$$

Theorem

For any θ irrational,

$$K_0(\mathcal{A}_\theta^\infty) = K_0(\mathcal{A}_\theta) = \mathbb{Z}^2 \simeq \theta\mathbb{Z} + \mathbb{Z}, \qquad K_1(\mathcal{A}_\theta^\infty) = K_1(\mathcal{A}_\theta) = \mathbb{Z}^2$$

Some references

Connes, A. (1985). *Non-commutative differential geometry.* Publications Mathématiques de l'IHES.

Connes, A. (1994). *Noncommutative Geometry.* Academic Press.

Connes, A. and Marcolli, M. (2008). "A walk in the noncommutative garden". In: *An invitation to noncommutative geometry.* World Scientific Publishing Company, pp. 1–128.

Dixmier, J. (1981). *von Neumann algebras.* Elsevier (cit. on p. 4).

François, J., Lazzarini, S., and Masson, T. (2014). "Gauge field theories: various mathematical approaches". In: *Mathematical Structures of the Universe (to appear).* Kraków, Poland: Copernicus Center Press.

Masson, T. (2008a). "An informal introduction to the ideas and concepts of noncommutative geometry". In: *Integrables Systems and Quantum Fields Theories.* Hermann.

Masson, T. (2008b). Examples of derivation-based differential calculi related to noncommutative gauge theories. *Int. J. Geom. Methods Mod. Phys.* 5.8, pp. 1315–1336.

Masson, T. (2012). "Gauge theories in noncommutative geometry". In: *FFP11 Symposium Proceedings.* AIP.

Thanks you for your attention