

An introduction to noncommutative geometry

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Abstract

Noncommutative geometry is a thirty years old research program, which takes its roots and motivations in mathematics and physics. I will give an introductory overview on the main ideas, the main concepts, and some of the most important results of NCG.

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NCG: the mathematical side

- NCG is motivated by deep results on correspondences $spaces \leftrightarrow algebras$.
 - Measurable spaces \Rightarrow abelian von Neumann algebras.
 - Topological spaces \Rightarrow commutative C^* -algebras.
- **Fact 1:** some tools used to study these spaces have algebraic counterpart.
- **Fact 2:** these algebraic tools can be applied to NC algebras.

**Main idea of NCG:
 replace commutative algebras of functions
 by NC algebras in an identified category.**

- Replace the geometric approach by an algebraic one.
- Give new light on difficult problems:
 - \Rightarrow foliations and quotient spaces (NC torus)...
- “Differentiability” has been investigated in the 1980’s (Connes).
 - \Rightarrow Cyclic homology, Chern character, index theorems...
- NC “riemannian manifolds”: spectral triples.
 - \Rightarrow reconstruction theorem in 2008.

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NCG: the physical side

- Physics in crisis:
 - Geometrical theories:** general relativity, gauge field theories...
 - Algebraic theories:** quantum mechanics, QFT...
- How to unify them?
- NCG is not a *theory* in physics.
- NCG is a framework in which to think about new theories.
 - \Rightarrow different conceptualisations, unification...
- NCG has been constructed in relation to physics.
 - \Rightarrow NC gauge field theories, NC space-times, quantum groups...
- Some NC topological invariants are been used to explained (partially) the Quantum Hall Effect.
- QFT on NC spaces \Rightarrow new renormalizable non local models...
 - (ϕ^4 theories on Moyal space)
- NCG gauge field theories contains naturally Higgs-like particles.

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Abelian von Neumann algebras

Theorem 1 (Dixmier, *von Neumann algebras*, 1981). *Let H be a complex Hilbert space, and Z an abelian von Neumann algebra in H . There exists a locally compact space z , a positive measure ν on z , with support z , and an isometric isomorphism of the normed $*$ -algebra Z onto the normed $*$ -algebra $L_{\mathbb{C}}^{\infty}(z, \nu)$.*

- ➔ von Neumann algebras are “NC measurable spaces”.
 - Every von Neumann algebra on a separable Hilbert space is isomorphic to a direct integral of factors. (factor = von Neumann algebra with trivial center).
 - NC integration (weights), NC probability theory (states).
 - Any locally compact group defines a von Neumann algebra (and a C^* -algebra).
 - abelian group ➔ abelian von Neumann (C^* -) algebra.
 - ➔ Fourier transformation (Pontryagin dual gr.).
 - ➔ NC harmonic analysis...
 - Tomita-Takesaki theory of von Neumann algebra:
 - ➔ extends to von Neumann algebra the non-modularity of groups;
 - ➔ relation to KMS states in statistical physics.

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Commutative C^* -algebras

Theorem 2 (Gelfand-Naimark). *The category of locally compact Hausdorff spaces is anti-equivalent to the category of commutative C^* -algebras.*

Space $X \leftrightarrow$ algebra of continuous functions $C_0(X)$ vanishing at infinity.

This leads to the correspondences:

Spaces	Algebras
point	irreducible representation
compact	unital
1-point compactification	unitarization
Stone-Ćech compactification	multiplier algebra
homeomorphism	automorphism
Borel measure	positive functional
probability measure	state

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Finite projective modules

Theorem 3 (Serre-Swan). *The category of complex vector bundles on a compact Hausdorff space X is equivalent to the category of finite projective modules over the algebra $C(X)$ (continuous functions).*

Vector bundle $E \leftrightarrow$ Space of continuous sections $\Gamma(E)$.

→ projection in some $M_N(C(X))$.

This works also in the category of smooth manifolds.

- Notion of “vector bundles” in NCG: finite projective modules over A .
- Covariant derivatives have a NC generalisation (explained later).
 - This permits to define NC gauge field theories.

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Origin of common NC spaces

NC spaces are in general defined as von Neumann algebras or C^* -algebras.

Many constructions can give very interesting examples:

Deformation: the idea is to deform a commutative algebra (+ extra structure...).

→ Moyal algebra, related to the canonical commutation relations in QM.

→ κ -Minkowski space, (co)-representation space of a quantum group.

Group algebras: any locally compact group defines a C^* -algebra.

→ Study of the representation theory of the group.

→ More generally: C^* -algebra of a smooth groupoid.

Cross products: action of a locally compact group on a given algebra.

→ Compatible with semidirect product of groups and C^* -alg. of groups.

Quantum groups: Hopf algebra structures.

→ usually a deformation of the matrix entries of an ordinary group.

→ representation theory, new “symmetries”...

Quotients by equivalence relation: general construction gives an algebra which is Morita equivalent to the “expected” one if the quotient space is good enough.

Generators and relations: the algebra is directly defined by some elements.

→ Compatible with C^* -alg. of groups presented as generators and relations.

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K-theories

X a compact topological space:

- $K^0(X)$: Grothendieck group of the semigroup $V(X)$ of isomorphic classes of vector bundles over X .
 (Reminder: \mathbb{Z} is the Grothendieck group of the semigroup $\mathbb{N}\dots$)
- Definition extends to non compact spaces.
- $K^{-n}(X) = K^0(X \times \mathbb{R}^n)$.

A a unital C^* -algebra:

- $K_0(\mathbf{A})$: Grothendieck group of the semigroup $V(\mathbf{A})$ of homotopy classes of projections in $M_\infty(\mathbf{A})$ (finite proj. mod. on \mathbf{A}).
- Definition extends to non unital C^* -algebras.
- $K_n(\mathbf{A}) = K_0(C_0(\mathbb{R}^n, \mathbf{A}))$.

Gelfand-Naimark + Serre-Swan theorems $\Rightarrow K^{-n}(X) = K_n(C_0(X))$.

The topological invariants detected by K -groups are the same.

$\Rightarrow K$ -groups of C^* -algebras are more general.

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K-theories (cont'd)

- K -theory of C^* -algebras works for noncommutative algebras.
 \Rightarrow Essential tool to study NC spaces in NCG.
- K -theory of Fréchet, pre- C^* , or Banach algebras is well defined.
 $\Rightarrow K$ -groups of a C^* -algebra can be computed using a dense subalgebra.
 (stable by holomorphic functional calculus)

A a unital associative algebra:

- Algebraic K -theory: $K_v^{\text{alg}}(\mathbf{A})$, $v = 0, 1$, for associative algebra without topology.
 (Projectors in $M_\infty(\mathbf{A})$ ($p \sim u^{-1}qu$) and abelianization of $GL_\infty(\mathbf{A})$)
- Definitions for higher degrees are very involved...

Other “ K -theories”:

- K -homology, dual to K -theory.
 \Rightarrow Based on Fredholm operators on Hilbert spaces.
- KK -theory, contains K -theory and K -homology.
 \Rightarrow Based on Hilbert C^* -modules = generalization of Hilbert spaces.

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K-theories: some properties

Theorem 4 (Bott periodicity). *For any C^* -algebra \mathbf{A} , one has*

$$K_2(\mathbf{A}) \simeq K_0(\mathbf{A})$$

→ Only two K -groups: $K_0(\mathbf{A})$ and $K_1(\mathbf{A})$.

Proposition 5 (Six terms exact sequence). *For any short exact sequence of C^* -algebras $0 \rightarrow \mathbf{I} \rightarrow \mathbf{A} \rightarrow \mathbf{A}/\mathbf{I} \rightarrow 0$, one has the six terms exact sequence*

$$\begin{array}{ccccc} K_0(\mathbf{I}) & \longrightarrow & K_0(\mathbf{A}) & \longrightarrow & K_0(\mathbf{A}/\mathbf{I}) \\ \delta \uparrow & & & & \downarrow \delta \\ K_1(\mathbf{A}/\mathbf{I}) & \longleftarrow & K_1(\mathbf{A}) & \longleftarrow & K_1(\mathbf{I}) \end{array}$$

→ δ are index maps (as in Atiyah-Singer index theorem...).

Proposition 6 (Algebraic K -theory). *A C^* -algebra: $K_0^{alg}(\mathbf{A}) = K_0(\mathbf{A})$ and $K_1^{alg}(\mathbf{A}) \rightarrow K_1(\mathbf{A})$ (not an isomorphism).*

Worth mentioning Morita invariance also...

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Cyclic homology

\mathbf{A} unital associative algebra.

Hochschild complex with values in \mathbf{A} : $\mathbf{A} \xleftarrow{b} \dots \xleftarrow{b} \mathbf{A}^{\otimes n} \xleftarrow{b} \mathbf{A}^{\otimes n+1} \xleftarrow{b} \dots$

$$b(a_0 \otimes \dots \otimes a_n) = \sum_{p=0}^{n-1} (-1)^p a_0 \otimes \dots \otimes a_p a_{p+1} \otimes \dots \otimes a_n + (-1)^n a_n a_0 \otimes a_1 \otimes \dots \otimes a_{n-1}$$

Hochschild homology: $HH_\bullet(\mathbf{A})$ homology of this complex.

$t : \mathbf{A}^{\otimes n} \rightarrow \mathbf{A}^{\otimes n}$ cyclic operator: $t(a_1 \otimes \dots \otimes a_n) = (-1)^{n+1} a_n \otimes a_1 \otimes \dots \otimes a_{n-1}$.

b well-defined on $C_n^\lambda(\mathbf{A}) = \mathbf{A}^{\otimes n+1} / \text{Ran}(1 - t)$.

Cyclic homology: $HC_\bullet(\mathbf{A})$ homology of $(C_\bullet^\lambda(\mathbf{A}), b)$.

Proposition 7 (Connes long exact sequence). *There are morphisms I and S which induce the following long exact sequence*

$$\dots \longrightarrow HH_n(\mathbf{A}) \xrightarrow{I} HC_n(\mathbf{A}) \xrightarrow{S} HC_{n-2}(\mathbf{A}) \xrightarrow{B} HH_{n-1}(\mathbf{A}) \xrightarrow{I} \dots$$

For non unital algebras, need bicomplexes...

Cyclic cohomology is defined using **Hochschild cohomology** with values in \mathbf{A}^* .

→ similar operators I and S , and long exact sequence.

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Periodic cyclic (co)homology

Periodic cyclic cohomology: $HP^*(\mathbf{A})$ is defined using S .
 Only 2 groups: $HP^0(\mathbf{A}) = \varinjlim HC^{2n}(\mathbf{A})$ and $HP^1(\mathbf{A}) = \varinjlim HC^{2n+1}(\mathbf{A})$.
 In the same way, one can define the **periodic cyclic homology:** $HP_*(\mathbf{A})$.

Proposition 8 (Six terms exact sequence). *For any short exact sequence of associative algebras $0 \rightarrow \mathbf{I} \rightarrow \mathbf{A} \rightarrow \mathbf{A}/\mathbf{I} \rightarrow 0$, one has the six terms exact sequence*

$$\begin{array}{ccccc} HP_0(\mathbf{I}) & \longrightarrow & HP_0(\mathbf{A}) & \longrightarrow & HP_0(\mathbf{A}/\mathbf{I}) \\ & & \delta \uparrow & & \downarrow \delta \\ HP_1(\mathbf{A}/\mathbf{I}) & \longleftarrow & HP_1(\mathbf{A}) & \longleftarrow & HP_1(\mathbf{I}) \end{array}$$

Proposition 9 (Diffeotopic invariance). *\mathbf{A} and \mathbf{B} two associative algebras. If $\varphi_0, \varphi_1 : \mathbf{A} \rightarrow \mathbf{B}$ are diffeotopic, then they induce the same morphism $HP_v(\mathbf{A}) \rightarrow HP_v(\mathbf{B})$.*

$\varphi : \mathbf{A} \rightarrow \mathbf{B} \otimes C^\infty([0,1])$ s.t. φ_t is φ_0 (resp. φ_1) at $t = 0$ (resp. $t = 1$)
 ➔ Does not work for homotopy!

Worth mentioning Morita invariance also...

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Cyclic homology: examples

Example $\mathbf{A} = \mathbb{C}$:

$$HH_0(\mathbb{C}) = \mathbb{C} \qquad HH_n(\mathbb{C}) = 0 \text{ for } n \geq 1 \qquad HP_0(\mathbb{C}) = \mathbb{C} \qquad HP_1(\mathbb{C}) = 0$$

Example $\mathbf{A} = \mathbb{C}[z, z^{-1}]$ (Laurent polynomials):

$$HP_0(\mathbb{C}[z, z^{-1}]) = \mathbb{C} \qquad HP_1(\mathbb{C}[z, z^{-1}]) = \mathbb{C}$$

Hochschild and cyclic homologies can be defined for topological algebras.

Example $\mathbf{A} = C(X)$, continuous functions on a compact space X :

$$HP_0^{cont}(C(X)) = \{\text{bounded measures on } X\} \qquad HP_1^{cont}(C(X)) = 0$$

M a finite dim. loc. compact manifold.

Example $\mathbf{A} = C^\infty(M)$, Fréchet algebra of smooth functions on M :

Theorem 10 (Connes, 1985).

$$HH_\bullet^{cont}(C^\infty(M)) = \Omega_\bullet^*(M) \text{ (complexified de Rham forms)}$$

$$HP_0^{cont}(C^\infty(M)) = H_{dR}^{even}(M) \qquad HP_1^{cont}(C^\infty(M)) = H_{dR}^{odd}(M)$$

➔ Cyclic homology is the NC generalization of de Rham cohomology.

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The Chern character

Theorem 11 (The (geometric) Chern character). *The (usual) Chern character $\text{ch}(E) = \text{tr} \circ \exp\left(\frac{iF}{2\pi}\right)$ realizes an isomorphism*

$$\text{ch} : K^0(M) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} H^{\text{even}}(M; \mathbb{Q})$$

for locally compact finite dimensional (smooth) manifolds M .

It can be extended to an isomorphism $\text{ch} : K^1(M) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} H^{\text{odd}}(M; \mathbb{Q})$.

Proposition 12 (The (algebraic) Chern character). *The Chern character realizes a map $\text{ch} : K_v^{\text{alg}}(\mathbf{A}) \rightarrow HP_v(\mathbf{A})$ for $v = 0, 1$.*

➔ Defined by the generators of $HP_0(\mathbb{C}) = \mathbb{C}$ and $HP_1(\mathbb{C}[z, z^{-1}]) = \mathbb{C}$.

The algebraic Chern character factorizes through K -theory of topological algebras.

Theorem 13 (The (NC) Chern character). *For a large class of Fréchet algebras, the Chern character realizes an isomorphism*

$$\text{ch} : K_v(\mathbf{A}) \otimes \mathbb{C} \xrightarrow{\cong} HP_v(\mathbf{A})$$

➔ The Fréchet algebras $C^\infty(M)$ for locally compact manifolds M are in this class.

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Pairing and Fredholm modules

\mathbf{A} a topological algebra (Fréchet, pre- C^* , Banach).

- We have introduced the Chern character as a map $\text{ch} : K_v(\mathbf{A}) \rightarrow HP_v(\mathbf{A})$.
- It is also a pairing $K_v(\mathbf{A}) \times HP^v(\mathbf{A}) \rightarrow \mathbb{C}$.
- It is also a map $K^v(\mathbf{A}) \rightarrow HP^v(\mathbf{A})$, where $K^v(\mathbf{A})$ is the K -homology of \mathbf{A} .
 - Elements in $K^1(\mathbf{A})$ are classes of (odd) Fredholm modules $(\mathbf{A}, \mathcal{H}, F)$;
 - \mathcal{H} is a Hilbert space which supports an involutive representation π of \mathbf{A} ;
 - F is a bounded operator on \mathcal{H} such that $F = F^*$, $F^2 = 1$;
 - $[F, \pi(a)]$ is compact for any $a \in \mathbf{A}$;
 - then (Connes, 1985)
$$\tau(a^0, \dots, a^n) = \text{Tr}(a^0[F, a^1] \cdots [F, a^n])$$

defines an element in $HP^1(\mathbf{A})$.

- Need trace-class operators: “summability” of Fredholm modules.
 - ➔ Schatten ideals $\mathcal{L}^p(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) \mid \text{Tr}(|T|^p) < \infty\}$.
 - ➔ “Infinitesimals of order p in this quantum calculus” (Connes).

Remark 14. K -theory is a theory for NC topological spaces.

Periodic cyclic homology is a theory for algebras with “differentiable structures”.

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Spectral triples

Spectral triples are “unbounded Fredholm modules”.

\mathbf{A} an involutive unital associative algebra.

Definition 15 (Spectral triple). A spectral triple on \mathbf{A} is a triple $(\mathbf{A}, \mathcal{H}, \mathcal{D})$ where

- \mathcal{H} is a Hilbert space on which an involutive representation ρ of \mathbf{A} is given;
- \mathcal{D} is a (unbounded) self-adjoint operator on \mathcal{H} ;
- the resolvent of \mathcal{D} is compact;
- $[\mathcal{D}, \rho(a)]$ is bounded for any $a \in \mathbf{A}$.

Many more axioms for complete description:

- Grading \Rightarrow charge conjugation in physics.
- Reality operator \Rightarrow Tomita-Takesaki theory.
- Regularity condition
 \Rightarrow defines the “smooth” algebra \mathbf{A} as a dense subalgebra of a C^* -algebra

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Spectral triples (cont’d)

- $(\mathbf{A}, \mathcal{H}, \mathcal{D}(1 + \mathcal{D}^2)^{-1/2})$ defines a Fredholm modules (except for $F^2 = 1$):
 - class in $K^v(\mathbf{A})$;
 - element in $HP^v(\mathbf{A})$ by the Chern character;
 - Summability conditions on $(\mathbf{A}, \mathcal{H}, \mathcal{D})$.
- The operator \mathcal{D} is called a **Dirac operator**.
 - \Rightarrow This comes from $(\mathbf{A}, \mathcal{H}, \mathcal{D}) = (C^\infty(M), L^2(\mathcal{S}), \mathcal{D})$ where
 - M is a smooth compact Riemannian spin manifold,
 - \mathcal{S} is a spin bundle,
 - \mathcal{D} is the usual Dirac operator.
 - \Rightarrow This is the commutative prototype of spectral triples.
- **Reconstruction theorem by Connes (2008):**
 Commutative spectral triples (with additional axioms) are of the form $(C^\infty(M), L^2(\mathcal{S}), \mathcal{D})$.
- A spectral triple $(\mathbf{A}, \mathcal{H}, \mathcal{D})$ encodes some metric properties of the “NC spaces”.
 - \Rightarrow distance on the space of states of \mathbf{A} .
- Behavior of the eigenvalues of $|\mathcal{D}| \Rightarrow$ dimension of the spectral triple.

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Differential structures

\mathbf{A} an associative algebra.

Definition 16 (Differential calculus on an algebra). A differential calculus on \mathbf{A} is a graded differential algebra (Ω^\bullet, d) such that $\Omega^0 = \mathbf{A}$.

→ Many differential calculi can be constructed on a given algebra.

Example 17 (Universal unital differential calculus). \mathbf{A} a unital associative algebra.

$(\Omega_U^\bullet(\mathbf{A}), d_U)$ is the free unital graded diff. alg. generated by \mathbf{A} in degree 0.

Elements are finite sum of $ad_U b_1 \cdots d_U b_n$ for $a, b_1, \dots, b_n \in \mathbf{A}$.

Universal property: for any unital diff. calc. (Ω^\bullet, d) on \mathbf{A} , there exists a unique morphism of unital diff. calc. $\phi : \Omega_U^\bullet(\mathbf{A}) \rightarrow \Omega^\bullet$ (of degree 0) such that $\phi(a) = a$ for any $a \in \mathbf{A} = \Omega_U^0(\mathbf{A}) = \Omega^0$.

→ Many diff. calc. are quotients of $(\Omega_U^\bullet(\mathbf{A}), d_U)$.

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Derivation based differential calculus

\mathbf{A} an associative algebra with unit $\mathbb{1}$.

- $\mathcal{Z}(\mathbf{A}) = \{a \in \mathbf{A} / ab = ba, \forall b \in \mathbf{A}\}$ its center.
- Space of derivations of \mathbf{A} :

$$\text{Der}(\mathbf{A}) = \{\mathfrak{X} : \mathbf{A} \rightarrow \mathbf{A} / \mathfrak{X} \text{ linear, } \mathfrak{X} \cdot (ab) = (\mathfrak{X} \cdot a)b + a(\mathfrak{X} \cdot b), \forall a, b \in \mathbf{A}\}.$$

→ Lie algebra: $[\mathfrak{X}, \mathfrak{Y}]a = \mathfrak{X}\mathfrak{Y}a - \mathfrak{Y}\mathfrak{X}a$ for all $\mathfrak{X}, \mathfrak{Y} \in \text{Der}(\mathbf{A})$,

→ $\mathcal{Z}(\mathbf{A})$ -module: $(f\mathfrak{X}) \cdot a = f(\mathfrak{X} \cdot a)$ for all $f \in \mathcal{Z}(\mathbf{A})$ and $\mathfrak{X} \in \text{Der}(\mathbf{A})$.

- $\underline{\Omega}_{\text{Der}}^0(\mathbf{A}) = \mathbf{A}$.
- $\underline{\Omega}_{\text{Der}}^n(\mathbf{A})$ space of $\mathcal{Z}(\mathbf{A})$ -mult. antisym. maps from $\text{Der}(\mathbf{A})^n$ to \mathbf{A} , $n \geq 1$.
- $\underline{\Omega}_{\text{Der}}^\bullet(\mathbf{A}) = \bigoplus_{n \geq 0} \underline{\Omega}_{\text{Der}}^n(\mathbf{A})$.
- \mathbb{N} -graded differential algebra (product by antisymmetrization):

$$\begin{aligned} \widehat{d}\omega(\mathfrak{X}_1, \dots, \mathfrak{X}_{n+1}) &= \sum_{i=1}^{n+1} (-1)^{i+1} \mathfrak{X}_i \cdot \omega(\mathfrak{X}_1, \dots, \overset{i}{\checkmark} \dots, \mathfrak{X}_{n+1}) \\ &\quad + \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} \omega([\mathfrak{X}_i, \mathfrak{X}_j], \dots, \overset{i}{\checkmark} \dots, \overset{j}{\checkmark} \dots, \mathfrak{X}_{n+1}). \end{aligned}$$

- $\mathbf{A} = C^\infty(M) \rightarrow (\underline{\Omega}_{\text{Der}}^\bullet(\mathbf{A}), \widehat{d})$ is the de Rham calculus.

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Spectral triples and forms

$(\mathbf{A}, \mathcal{H}, \mathcal{D})$ a spectral triple.

- Any form $\omega = \sum a d_U b_1 \cdots d_U b_n \in \Omega_U^\bullet(\mathbf{A})$ gives a bounded operator

$$\pi_{\mathcal{D}}(\omega) = \sum \pi(a)[\mathcal{D}, \pi(b_1)] \cdots [\mathcal{D}, \pi(b_n)]$$

- This is not a representation of the $(\Omega_U^\bullet(\mathbf{A}), d_U)$ as a graded diff. alg.
- Perturbation of \mathcal{D} by bounded operators does not change the K -homology class.
 - $\rightarrow \mathcal{D} + \pi_{\mathcal{D}}(\omega)$ in the same class.
- Case $(C^\infty(M), L^2(\mathcal{S}), \mathcal{D})$:
 - $E \rightarrow M$ vector bundle, ω connection 1-form on E
 - $\rightarrow \mathcal{D} + \omega$ is the twisted Dirac operator defined on $\mathcal{S} \otimes E$.

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NC connections

\mathbf{A} an associative algebra with unit $\mathbb{1}$. (Ω^\bullet, d) a diff. calc. on \mathbf{A} .

\mathbf{M} a finite projective right module over \mathbf{A} .

Definition 18 (NC connection). A NC connection on \mathbf{M} is a linear map $\widehat{\nabla} : \mathbf{M} \rightarrow \mathbf{M} \otimes_{\mathbf{A}} \Omega^1$ such that $\widehat{\nabla}(ma) = (\widehat{\nabla}m)a + m \otimes da$ for any $m \in \mathbf{M}$ and $a \in \mathbf{A}$.

$\widehat{\nabla}$ can be extended as $\widehat{\nabla} : \mathbf{M} \otimes_{\mathbf{A}} \Omega^p \rightarrow \mathbf{M} \otimes_{\mathbf{A}} \Omega^{p+1}$, for any $p \geq 0$, using the rule

$$\widehat{\nabla}(m \otimes \omega_p) = (\widehat{\nabla}m) \otimes \omega_p + m \otimes d\omega_p \quad \text{for any } \omega_p \in \Omega^p.$$

Definition 19 (Curvature). The curvature of $\widehat{\nabla}$ is defined as $\widehat{R} = \widehat{\nabla}^2 = \widehat{\nabla} \circ \widehat{\nabla} : \mathbf{M} \rightarrow \mathbf{M} \otimes_{\mathbf{A}} \Omega^2$.

It satisfies $\widehat{R}(ma) = (\widehat{R}m)a$ for any $m \in \mathbf{M}$ and $a \in \mathbf{A}$.

Definition 20 (Gauge transformations). \mathcal{G} the group of automorphisms of \mathbf{M} as a right \mathbf{A} -module.

For any $\Phi \in \mathcal{G}$, $\widehat{\nabla}^\Phi = \Phi^{-1} \circ \widehat{\nabla} \circ \Phi$ is also a NC connection on \mathbf{M} .

$\mathbf{A} = C^\infty(M) \rightarrow$ usual theory of connections on vector bundles $\mathbf{M} = \Gamma(E)$.

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NC connections: special case $M = \mathbf{A}$

Special example of right module: $M = \mathbf{A}$.

- Since \mathbf{A} is unital: $\widehat{\nabla}(a) = \widehat{\nabla}(\mathbf{1}a) = \widehat{\nabla}(\mathbf{1})a + \mathbf{1} \otimes da = \widehat{\nabla}(\mathbf{1})a + da$.
- $\widehat{\nabla}(\mathbf{1}) = \omega \in \Omega^1$ characterizes completely $\widehat{\nabla}$.
- ω is the NC connection 1-form.
- The curvature of $\widehat{\nabla}$ is the left multiplication by the 2-form $\Omega = d\omega + \omega\omega \in \Omega^2$.
- $\Phi \in \mathcal{G}$ is completely determined by $\Phi(\mathbf{1}) = g \in \mathbf{A}$ (invertible element).
- Gauge transformation of $\widehat{\nabla}$:

$$\omega \mapsto g^{-1}\omega g + g^{-1}dg, \quad \Omega \mapsto g^{-1}\Omega g.$$

→ This permits to construct NC gauge field theories:

- Need an appropriate Lagrangian → different approaches.
- For $\mathbf{A} = C^\infty(M) \otimes \mathbf{A}_F$ where $\mathbf{A}_F = \text{finite dim. alg. } \simeq M_n(\mathbb{C}), \mathbb{C}^N \dots$
 - NC connections split in two parts:
 - geometric along $C^\infty(M)$ → “ordinary” gauge field (Yang-Mills).
 - algebraic along \mathbf{A}_F → new fields \simeq scalar fields of the Higgs mechanism.

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NC connections and spectral triples

Definition 21. Two spectral triples $(\mathbf{A}, \mathcal{H}, \mathcal{D})$ and $(\mathbf{A}', \mathcal{H}', \mathcal{D}')$ are unitary equivalent if there exists a unitary operator $U : \mathcal{H} \rightarrow \mathcal{H}'$ and an algebra isomorphism $\phi : \mathbf{A} \rightarrow \mathbf{A}'$ such that $\pi' \circ \phi = U\pi U^{-1}$, $\mathcal{D}' = U\mathcal{D}U^{-1}$.

Inner fluctuations:

- Case $\mathcal{H}' = \mathcal{H}$, $\mathbf{A}' = \mathbf{A}$, $\pi' = \pi$ and $\phi_u(a) = u^*au$ for a unitary $u \in U(\mathcal{A})$.
 - Then $U = \pi(u)^*$.
- \mathcal{D} is transformed as $\mathcal{D}^u = U\mathcal{D}U^{-1} = \mathcal{D} + \pi(u)^*[\mathcal{D}, \pi(u)]$.
- Let $\omega \in \Omega_U^1(\mathbf{A})$ be a NC connection 1-form for the module $M = \mathbf{A}$.
 - Interpret $u \in U(\mathcal{A})$ as a (unitary) gauge transformation.
 - $\omega^u = u^*\omega u + u^*du$
 - $\pi_{\mathcal{D}}(\omega^u) = \pi(u)^*\pi_{\mathcal{D}}(\omega)\pi(u) + \pi(u)^*[\mathcal{D}, \pi(u)]$.
- Define $\mathcal{D}_\omega = \mathcal{D} + \pi_{\mathcal{D}}(\omega)$, then

$$(\mathcal{D}_\omega)^u = \mathcal{D}_{\omega^u} \quad (= \mathcal{D} + \pi(u)^*\pi_{\mathcal{D}}(\omega)\pi(u) + \pi(u)^*[\mathcal{D}, \pi(u)])$$

**To compensate the action of inner symmetries on \mathcal{D} ,
a NC connection is necessary.**

→ Implementation of the gauge principle.

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NC torus

θ a real number.

- On the Hilbert space $L^2(\mathbb{S}^1)$, consider the two unitary operators

$$(Uf)(t) = e^{2\pi it} f(t) \qquad (Vf)(t) = f(t - \theta)$$

where $f : \mathbb{S}^1 \rightarrow \mathbb{C}$ is a periodic function in the variable $t \in \mathbb{R}$.

- Then $UV = e^{2\pi i\theta} VU \in \mathcal{B}(L^2(\mathbb{S}^1))$.
- \mathcal{A}_θ the C^* -algebra generated by U and V in $\mathcal{B}(L^2(\mathbb{S}^1)) \rightarrow$ **NC torus**.

Suppose θ is irrational:

- \mathcal{A}_θ is the **irrational rotation algebra**.
It is simple and universal for the relation $UV = e^{2\pi i\theta} VU$.
- $\mathcal{A}_\theta = \mathcal{A}_{1-\theta}$.
- $\alpha : \mathbb{S}^1 \rightarrow \mathbb{S}^1, z \mapsto e^{2i\pi\theta} z \rightarrow$ action of \mathbb{Z} on $C(\mathbb{S}^1)$
 $\rightarrow \mathcal{A}_\theta = C(\mathbb{S}^1) \rtimes_\alpha \mathbb{Z}$ (cross product C^* -algebra).
- \mathcal{A}_θ is also associated to the Kronecker foliation $dx = \theta dy$ of the 2-torus.
- Finite projective modules are classified by two integers (p, q) s.t. $p + q\theta \geq 0$.

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NC torus: smooth structures

- $\mathcal{S}(\mathbb{Z}^2)$ the Schwartz space of sequences $(a_{m,n})_{m,n \in \mathbb{Z}}$ of rapid decay:
 $(|m| + |n|)^q |a_{m,n}|$ is bounded for any $q \in \mathbb{N}$.
- $\mathcal{A}_\theta^\infty =$ elements in \mathcal{A}_θ of the form $\sum_{m,n \in \mathbb{Z}} a_{m,n} U^m V^n$ for $(a_{m,n})_{m,n \in \mathbb{Z}} \in \mathcal{S}(\mathbb{Z}^2)$.
- By Fourier analysis $\mathcal{S}(\mathbb{Z}^2) \simeq C^\infty(\mathbb{T}^2)$.
 $\rightarrow \mathcal{A}_\theta^\infty$ is the space of “smooth functions on the NC torus”.
- $p_q(a) = \sup_{m,n \in \mathbb{Z}} \{(1 + |m| + |n|)^q |a_{m,n}|\}$ is a family of semi-norms on $\mathcal{A}_\theta^\infty$.
 \rightarrow Fréchet algebra.
- $\mathcal{A}_\theta^\infty$ has two derivations:

$$\begin{aligned} \delta_1(U^m) &= 2\pi im U^m, & \delta_1(V^n) &= V^n, \\ \delta_2(U^m) &= U^m, & \delta_2(V^n) &= 2\pi in V^n \end{aligned}$$

- $\tau \left(\sum_{m,n \in \mathbb{Z}} a_{m,n} U^m V^n \right) = a_{0,0}$ is a trace (unique, extends to \mathcal{A}_θ).
- Spectral triple:
 - $\mathbf{A} = \mathcal{A}_\theta^\infty$,
 - $\mathcal{H} =$ (Hilbert space of the GNS representation associated to τ) $\otimes \mathbb{C}^2$,
 - $\mathcal{D} = \begin{pmatrix} 0 & \delta_1 + i\delta_2 \\ \delta_1 - i\delta_2 & 0 \end{pmatrix}$.

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NC torus: homologies

Let $\lambda = \exp(2\pi i\theta)$.

Diophantine condition: there exists an integer k such that $|1 - \lambda^n|^{-1}$ is $O(n^k)$.

Theorem 22 (Connes, 1985). *If λ satisfies some diophantine condition:*

$$HH_0^{Cont}(\mathcal{A}_\theta^\infty) = \mathbb{C} \qquad HH_1^{Cont}(\mathcal{A}_\theta^\infty) = \mathbb{C}^2$$

For any λ :

$$HH_2^{Cont}(\mathcal{A}_\theta^\infty) = \mathbb{C} \qquad HH_n^{Cont}(\mathcal{A}_\theta^\infty) = 0 \text{ for } n \geq 3$$

If λ does not satisfy some diophantine condition:

$HH_0^{Cont}(\mathcal{A}_\theta^\infty)$ and $HH_1^{Cont}(\mathcal{A}_\theta^\infty)$ are infinite dimensional.

For any λ :

$$HP_0^{cont}(\mathcal{A}_\theta^\infty) = \mathbb{C}^2 \qquad HP_1^{cont}(\mathcal{A}_\theta^\infty) = \mathbb{C}^2$$

Theorem 23. *For any θ irrational,*

$$K_0(\mathcal{A}_\theta^\infty) = K_0(\mathcal{A}_\theta) = \mathbb{Z}^2 \simeq \theta\mathbb{Z} + \mathbb{Z}, \qquad K_1(\mathcal{A}_\theta^\infty) = K_1(\mathcal{A}_\theta) = \mathbb{Z}^2$$

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