The mathematical electron: a particle and its mathematical representations

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Thierry Masson

Centre de Physique Théorique
Campus de Luminy, Marseille

Why the electron?

- First concept of “elementary particle”.
- Related to the main achievements in theoretical physics in the XX$^{\text{th}}$ century:
  - Relativistic dynamics
  - Quantum mechanics
  - Field theory
  - Quantum Field theory
  - Standard Model of Particle Physics
- It has been described using many mathematical structures:
  - Point particle
  - Wave function
  - Dirac spinor
  - Gauge field theories
  - Creation and annihilation operators in QFT

➞ Motivation for developments of modern mathematics.

My objective

- Take a close look at the mathematical formulations of theories of the electron.
- Search for enduring properties in these successive descriptions.
- Give a “modern” view on the electron.
- What is now an “elementary particle”?
  (after the discovery of the Higgs particle)

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2. An elementary particle
3. Quantum description
4. Classical field theory
5. Quantum Electrodynamics
6. The Standard Model of Particle Physics
7. A composite particle
8. Conclusions

Before the electron

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Maxwell theory of electromagnetism

- One of the greatest achievement in the science of the nineteenth century.
- Explain all the known electric and magnetic phenomena.
- Unify all the established theories on electricity and magnetism.
- Written in terms of *field*, not in terms of *particles*.
- Propagation of electromagnetic waves at speed of light.
  ➙ explains light.
  ➙ need a “support” for the vibration of the waves: the *ether*…
- Theory of charges is not a part of the theory.
  ➙ Charges are considered as “manifestation of the ether”.

Maxwell theory: the mathematics

Maxwell theory: the mathematics

"Original" description

$$\operatorname{div} \vec{B} = 0 \qquad \frac{\partial \vec{B}}{\partial t} + \overrightarrow{\operatorname{rot}}\, \vec{E} = 0$$

$$\operatorname{div} \vec{E} = \frac{\rho}{\epsilon_0} \qquad \overrightarrow{\operatorname{rot}}\, \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{j}$$

$\vec{E}$ = electric field, ρ = charge density,
$\vec{B}$ = magnetic field, $\vec{j}$ = charged current density,
$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$ = speed of light, $\operatorname{div} \vec{j} + \frac{\partial \rho}{\partial t} = 0$ conservation of charge.

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Relativistic description

$$\partial_\xi F_{\mu\nu} + \partial_\mu F_{\nu\xi} + \partial_\nu F_{\xi\mu} = 0$$

$$\partial_\nu F^{\mu\nu} = \mu_0 j^\mu$$

$$\mu = 0, 1, 2, 3, \quad x^\mu = (ct, x, y, z), \quad \partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \overrightarrow{\operatorname{grad}} \right)$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix} = -F_{\nu\mu}, \quad j^\mu = (c\rho, \vec{j})$$

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$$\vec{B} = \overrightarrow{\operatorname{rot}}\, \vec{A} \qquad \vec{E} = -\frac{\partial \vec{A}}{\partial t} - \overrightarrow{\operatorname{grad}}\, V$$

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Properties of $\overrightarrow{\operatorname{grad}}$, $\overrightarrow{\operatorname{rot}}$, and div ➡ $\vec{A}$ and V exist locally.

$\vec{A}$ = magnetic vector potential associated to $\vec{B}$,
V = (scalar) electrostatic potential associated to $\vec{E}$.

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$$A_\mu = (-V/c, \vec{A})$$

$\partial_\xi F_{\mu\nu} + \partial_\mu F_{\nu\xi} + \partial_\nu F_{\xi\mu} = 0$ ➡ A_μ exists locally.

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$\vec{E}$ and $\vec{B}$ don't change through:

$$\vec{A} \mapsto \vec{A} + \overrightarrow{\operatorname{grad}}\, \chi$$

$$V \mapsto V - \frac{\partial \chi}{\partial t}$$

Relativistic description

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χ is a function on space-time

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$F_{\mu\nu}$ doesn't change through:

$$A_\mu \mapsto A_\mu + \partial_\mu \chi$$

χ is a function on space-time

This is a *local symmetry due to the parametrization* $A_\mu = (-V/c, \vec{A})$

An elementary particle

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The emergence of the electron

- Between 1890 and 1910 (approx.), the electron has been
  - conceived as a theoretical structure,
  - characterized as an experimental object,
  - discovered as an elementary particle,
  - accepted as an existing entity.
- Works by
  - Lorentz ($\sim$ 1892), Larmor ($\sim$ 1894) on the theory
  - Zeeman, Thomson, Millikan for main experimental evidences and characterization.
- *electron*: G. J. Stoney, 1891, to denote an *elementary quantity of electricity*. Motivation: phenomenon of electrolysis, Faraday's "chemical equivalents".
- Characterization: *mass* m_e and *charge* q_e.
  - Measure of q_e/m: Zeeman effect ($\sim$ 1892), Thomson cathode rays (1897).
  - Measure of q_e: Millikan (1909, oil drop experiment).

Lorentz theory

- Larmor theory of charges relies only on ether (Maxwell theory).
- Lorentz theory adds "substance" to charges and currents ➡ particles.
- Main elements of the theory:
  - The electron is a particle with a mass m_e and a charge q_e.
  - Point-like particle: no evidence for substructure and size.
  - It obeys the laws of classical mechanics (Newton).
  - The electromagnetic field $(\vec{E}, \vec{B})$ induces a force

$$\vec{F} = q_e(\vec{E} + \vec{v} \wedge \vec{B}) \quad \longrightarrow \quad m\vec{a} = \vec{F}$$

  - Hamilton's equations with

$$H = \frac{1}{2m}(\vec{p} - q_e\vec{A})^2 + q_e V$$

- Successes of the theory:
  - Laplace force,
  - Faraday law of induction,
  - Conductance properties of metals…

➡ The electron is part of Newton mechanics in a field theory.

Relativistic from the beginning

- Theory of relativity motivated by the symmetries of Maxwell equations.
- The Lorentz theory is relativistic from the very beginning.
- Use the Faraday tensor

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix},$$

the four-velocity $v^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \gamma(c, \vec{v})$ with $\vec{v} = \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}$ and $\gamma = (1 - v^2/c^2)^{-1/2}$, and the four-momentum $p^\mu = mv^\mu$:

$$\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = q_e\, F^{\mu\nu}\, v_\nu$$

- These equations derive from the action

$$S = \int \left(-m\sqrt{v^\mu v_\mu} \; - \; q_e\, A_\mu v^\mu\right) \mathrm{d}\tau$$

Quantum description

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From particle to wave and vice-versa

- Long debate about nature of light:
  waves (Hooke, Huygens...) or *particle* (Gassendi, Newton...)?
- 1817, Fresnel gives a description of light as a wave theory.
  ➡ This explains the polarization effect (transverse "vibrations").
- Requires a medium to support the "vibrations" ➡ ether.
- 1905, Einstein explains the photoelectric effect.
  ➡ quantum of light of energy $E = h\nu$ ➡ particle description.
- 1923, Compton confirms the "corpuscle" behavior of light during "collisions".
- 1924, de Broglie postulates the wave nature of the electron.
  - He uses relativistic arguments to generalize $E = h\nu$.
  - He associates to the electron a wavelength

$$\lambda = \frac{h}{p} = \frac{h}{m_e\, v}\sqrt{1 - \frac{v^2}{c^2}}$$

- 1927, Davisson and Germer confirm experimentally this hypothesis.
  Scattering of slow moving electrons through a piece of nickel crystal.

The Schrödinger equation

- A wave necessitates a… wave equation!
- 1926, Schrödinger proposes an equation:

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi$$

  Ψ takes values in $\mathbb{C}$, H is a differential operator $\simeq$ Hamiltonian.
- Explains the spectrum of the Hydrogen atom (Coulomb potential in H).
- Equation for an electron in a potential vector $(V, \vec{A})$:

$$i\hbar \frac{\partial \Psi}{\partial t} = \frac{1}{2m}\left(-i\hbar\vec{\nabla} - q_e\vec{A}\right)^2 \Psi + q_e V \Psi$$

- If Ψ solution for $(V, \vec{A})$ and χ real function,
  then $e^{\frac{iq_e}{\hbar}\chi}\Psi$ solution for $\left(V - \frac{\partial \chi}{\partial t}, \vec{A} + \overrightarrow{\mathrm{grad}}\,\chi\right)$.
  ➞ $\vec{E}$ and $\vec{B}$ are left unchanged.
- This is a *gauge symmetry* (Weyl 1918, Fock 1926, London 1927).
  ➞ It is a (local) change of phase: $U(1) = \{z \in \mathbb{C} \,/\, |z| = 1\}$ gauge group.
- Mathematics of all the fundamental interactions…
  Key ingredient: $\vec{p} - q_e\vec{A}$ with $\vec{p} \mapsto -i\hbar\vec{\nabla}$…

The spin

- 1924, Pauli introduces new quantum numbers (spectrum of alkali metals).
- He adds a new principle: the *exclusion principle*.
- Kronig, Uhlenbeck, Goudsmit propose to associate this number to a "self-rotation" of the electron.

  ➞ *Physical* idea based on the interaction with magnetic field.
- 1927, Pauli proposes a theory based on the Schrödinger equation:
  - 2-components wave function, $\Psi \in \mathbb{C}^2$,
  - 2×2 matrices, $\sigma_x = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,
  - interacting term with magnetic field

$$-\frac{q_e \hbar}{2m_e} \vec{\sigma} \cdot \vec{B}.$$

- Spin related to the Lie goup $SU(2)$, itself related to rotation group $SO(3)$.

  ➞ *Mathematical* interpretation of spin as "internal self-rotation".
- Strong relation between spin and statistic of identical particles:

  Bosons: spin integer, predilection for identical states (ex. laser).

  Fermions: spin half-integer, exclusion principle.

➞ The electron is now a massive charged fermion with spin $\frac{1}{2}$.

The electron as a current

- In quantum mechanics, what's remains of the electron as a particle?
- "Classical" experiments don't see electrons, but *currents.*
- The density of charge and density of current are

$$\rho = q_e \Psi^* \Psi \qquad \vec{j} = q_e \frac{i\hbar}{2m} \left(\Psi \vec{\nabla} \Psi^* - \Psi^* \vec{\nabla} \Psi + \frac{2iq_e}{\hbar} \vec{A} \Psi^* \Psi \right)$$

- For spin $\frac{1}{2}$ particles, add *spin* current

$$\rho = q_e \Psi^\dagger \Psi$$

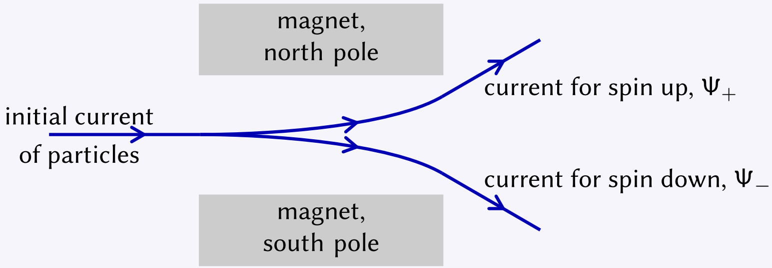
$$\vec{j} = q_e \frac{i\hbar}{2m} \left((\vec{\nabla} \Psi^\dagger) \Psi - \Psi^\dagger (\vec{\nabla} \Psi) + \frac{2iq_e}{\hbar} \vec{A} \Psi^\dagger \Psi \right) + q_e \frac{\hbar}{2m} \vec{\nabla} \times (\Psi^\dagger \vec{\sigma} \Psi)$$

with $\Psi = \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix}$, $\Psi^\dagger = \begin{pmatrix} \Psi_+^* & \Psi_-^* \end{pmatrix}$.

- All these quantities are $U(1)$*-gauge invariant.*
  ➞ Only gauge invariant objects can be observed.

The Stern-Gerlach experiment

- The spin was conceived to explain the spectrum of atoms.
- 1922, Stern and Gerlach show the interaction of spin with magnetic field (using silver atoms).
  ➞ before the spin hypothesis!
- The (non homogeneous) magnetic field separates the currents for spin down and spin up particles.
- One of the most fundamental experiments in quantum mechanics.



Classical field theory

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Field theories in a nutshell…

- Here, *field theory* $\simeq$ *theory of relativistic "wave" functions.*
- Description in terms of Lagrangians (instead of Hamiltonians).
- Maxwell theory: the first field theory…

$$\mathcal{L}_{\text{EM}} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} = \tfrac{1}{2}(\vec{E}^2 - \vec{B}^2)$$

- Schrödinger and Pauli theories are not relativistic.
- Relativistic versions of Schrödinger were proposed by Schrödinger, Fock, Kudar, de Donder, van der Dungen, Klein and Gordon…
  They were almost all the same.
- Klein-Gordon equation based on the relation $E^2 = \vec{p}^2 c^2 + m^2 c^4$:

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi - \nabla^2\phi + \frac{m^2 c^2}{\hbar^2}\phi = 0, \quad \text{or} \quad (\partial_\mu \partial^\mu + m^2)\phi = 0 \quad \text{with } c = \hbar = 1.$$

- Lagrangian:

$$\mathcal{L}_{\text{KG}} = \tfrac{1}{2}\left((\partial_\mu \phi)^* \partial^\mu \phi - m^2 \phi^* \phi\right)$$

- Problems:
  - Solutions with negative energy.
  - Describe spinless particles (bosons).

Dirac equation

- 1928, Dirac proposes an relativistic equation for spin $\frac{1}{2}$ particles.
- First order differential operator $\sqrt{\partial_\mu \partial^\mu} = \sqrt{\partial_t^2 - \vec{\nabla}^2}$.
- Dirac equation, $\psi \in \mathbb{C}^4$:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

  with

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}, \; k = 1, 2, 3$$

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} = \pm 2\delta^{\mu\nu}$$

- Lagrangian:

$$\mathcal{L}_{\text{Dirac}} = \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi \quad \text{with } \overline{\psi} = \psi^\dagger \gamma^0$$

- Associated probabilistic (quadri-)current: $j^\mu = \overline{\psi}\gamma^\mu\psi$.
  ➞ associated density: $j^0 = \overline{\psi}\gamma^0\psi = \psi^\dagger\psi \geq 0$.
- Non relativistic limit ➞ Pauli equation.
- The first order operator $\not{D} = i\gamma^\mu \partial_\mu$ has many applications in mathematics.
  ➞ Clifford algebras, spin geometry, noncommutative geometry…

The positron

- Problem of the Dirac equation: solutions with negative energy persists.
- The *Dirac sea*: the vacuum is filled of electrons with negative energies.
  ➞ Pauli exclusion principle: "usual" electrons can't go there...
- A hole in the Dirac sea is a state of *positive* energy, with *opposite* charge.
  ➞ interpretation as a new particle:
  - mass of the electron,
  - opposite charge,
  - annihilates when it meets an electron.
- 1932, Anderson detects such a particle with these properties: the *positron*.
- Notation: e^- for electron, e^+ for positron.
- Every particle has an *antiparticle*.
  ⚠ a particle can be its antiparticle, ex. the photon.

➞ The (relativistic) electron is a member of a couple particle/antiparticle.

Gauge field theories

- Matter fields are described by Dirac spinors or scalar Klein-Gordon fields.
- Electromagnetic interaction is described by gauge fields A_μ.
- Gauge principle: promote a “rigid/global” symmetry to a “local” symmetry.
  ➞ Lagrangian invariant by $\phi \mapsto U \cdot \phi$ for $U \in G$ (Lie group).
  ➞ Add new fields A_μ to the Lagrangian to get invariance for $U : \mathbb{R}^4 \to G$.
  ➞ Electromagnetism is obtained from gauge principle with $G = U(1)$.
- 1954, Yang-Mills generalizes gauge principle to the group $SU(2)$.
  ➞ $A_\mu = A_\mu^k T_k$ in the Lie algebra of $G = SU(2)$.
- Gauge principle produces natural coupling matter/gauge fields:
  $$\partial_\mu \phi \mapsto (\partial_\mu - iqA_\mu)\phi, \quad q \text{ is the “charge” (minimal coupling)}$$
- The dynamics of A_μ is given by a term
  $\mathcal{L}_{\text{YM}} = \frac{1}{2}\operatorname{tr}(F_{\mu\nu}F^{\mu\nu})$ with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - iq[A_\mu, A_\nu]$ (field strength)
- New Lagrangians:
  $$\mathcal{L}_{\text{KG+YM}} = \tfrac{1}{2}\left[(\partial_\mu - iqA_\mu)\phi\right]^\dagger \left[(\partial^\mu - iqA^\mu)\phi\right] - m^2\phi^\dagger\phi + \tfrac{1}{2}\operatorname{tr}(F_{\mu\nu}F^{\mu\nu})$$
  $$\mathcal{L}_{\text{Dirac+YM}} = \overline{\psi}(i\gamma^\mu\partial_\mu + q\gamma^\mu A_\mu - m)\psi + \tfrac{1}{2}\operatorname{tr}(F_{\mu\nu}F^{\mu\nu})$$
- *No mass terms for the gauge fields A_μ.*

Quantum Electrodynamics

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Particles are quanta of fields

- Quantization of fields ("Second quantization") = relativistic quantum theory.
- Main point: $E = mc^2$ implies creation and annihilation of particles.
  ➞ Quantum mechanics with indefinite number of particles.
- Main ideas for the quantization of a complex scalar field ϕ.

1. Decompose ϕ in Fourier modes, with $\mathrm{d}\tilde{k} = \frac{\mathrm{d}^3 k}{(2\pi)^3 2\omega_{\mathbf{k}}}$ (Lorentz invariant measure),

$$\phi(x) = \int \mathrm{d}\tilde{k} \left(a(\mathbf{k}) e^{-ik\cdot x} + b^\dagger(\mathbf{k}) e^{ik\cdot x} \right), \quad \phi^*(x) = \int \mathrm{d}\tilde{k} \left(b(\mathbf{k}) e^{-ik\cdot x} + a^\dagger(\mathbf{k}) e^{ik\cdot x} \right)$$

2. Promote $a^\dagger(\mathbf{k})$, $a(\mathbf{k})$, $b^\dagger(\mathbf{k})$, and $b(\mathbf{k})$ to operators

$$[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = (2\pi)^3 2\omega_{\mathbf{k}} \delta^3(k - k'), \quad [b(\mathbf{k}), b^\dagger(\mathbf{k}')] = (2\pi)^3 2\omega_{\mathbf{k}} \delta^3(k - k').$$

   $a^\dagger(\mathbf{k})$ and $b^\dagger(\mathbf{k})$ are creation operators, $a(\mathbf{k})$ and $b(\mathbf{k})$ are annihilation operators.

3. $N = \int \mathrm{d}\tilde{k} a^\dagger(\mathbf{k}) a(\mathbf{k})$ = number of particles,
   $\overline{N} = \int \mathrm{d}\tilde{k} b^\dagger(\mathbf{k}) b(\mathbf{k})$ = number of antiparticles.

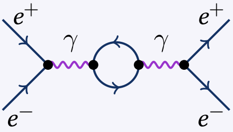
4. Particles identify as quanta created by $a^\dagger(\mathbf{k})$.
   Antiparticles identify as quanta created by $b^\dagger(\mathbf{k})$.

QED

Quantum ElectroDynamics = Quantization of the Lagrangian

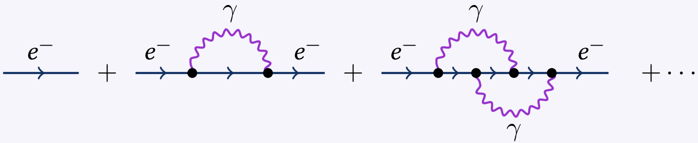
$$\mathcal{L}_{\text{Dirac+YM}} = \overline{\psi}(i\gamma^\mu \partial_\mu + q_e \gamma^\mu A_\mu - m)\psi + \frac{1}{2}\operatorname{tr}(F_{\mu\nu}F^{\mu\nu})$$

- The electron and positron are the quanta of the Dirac spinor ψ.
  ➡ A lot of subtleties: spin, anticommutations relations…
- The photon is the quantum of the gauge field A_μ.
  ➡ A lot of (other) subtleties: vector field, gauge symmetry…
- Interaction = exchange of (virtual) particles.
  ➡ Successive annihilations and creations of particles.
  ➡ Computation using Feynman diagrams.



The self-energy of the electron

- During its propagation, an electron interacts with the electromagnetic field:



- This interaction produces an "effective mass": *self-energy* of the electron.
- Direct computation: this energy is infinite.
- 1940's, *renormalization procedure* by Schwinger, Feynman, Tomonaga.
- Idea: the parameter m_e inserted in the Lagrangian *is not* the measured mass.
  ➞ Insert a "bare" value $\bar{m}_e$ for the mass in the initial computation.
- Same problem for the electric charge q_e of the electron.
  ➞ The electron has a "bare" electric charge $\bar{q}_e$.

- The bare mass $\bar{m}_e$ and the bare electric charge $\bar{q}_e$ are infinite...

➞ In QED, the electron has no definite mass and electric charge...

The Standard Model of Particle Physics

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6. **The Standard Model of Particle Physics**
7. A composite particle
8. Conclusions

Phenomenology: a lot of particles

- Known particles around 1930: electron, proton, photon.
- 1930, Pauli predicts a light neutral particle: the neutrino ν_e.
  - ➞ Forced by conservation of energy,
  - ➞ discovered in 1956.
- 1940's–1960's: cosmic radiation, bubble chamber (1952), linear particle accelerators (1928), cyclotron (1929)... ➞ A lot of new particles are discovered.
  - 1932, *neutron*, "neutral proton".
  - 1933, *positron*, anti-electron.
  - 1936, *muon* μ^-, "heavy electron", $m_\mu \simeq 200 \times m_e$.
    - ➞ associated neutrino ν_μ discovered in 1962.
  - 1940–1960, lot of baryons, mesons (sensible to strong interaction).
  - 1974–1977, *tau*, τ^-, another "heavy electron", $m_\tau \simeq 3500 \times m_e$ (➞ ν_τ).
- 1964, Gell-Mann, Zweig: hypothesis of quarks to classify baryons and mesons.
  - Three elementary particle (quarks): u (up), d (down), s (strange).
  - 1970's: hypothesis of three more quarks: c (charm), b (bottom), t (top).
  - $p^+ = uud$ and $n = ddu$.

Phenomenology: the elementary particles

FERMIONS matter constituents
spin = 1/2, 3/2, 5/2, …

Leptons spin =1/2

Flavor	Mass GeV/c²	Electric charge
ν_L lightest neutrino*	$(0-0.13)\times10^{-9}$	0
e electron	0.000511	–1
ν_M middle neutrino*	$(0.009-0.13)\times10^{-9}$	0
μ muon	0.106	–1
ν_H heaviest neutrino*	$(0.04-0.14)\times10^{-9}$	0
τ tau	1.777	–1

Quarks spin =1/2

Flavor	Approx. Mass GeV/c²	Electric charge
u up	0.002	2/3
d down	0.005	–1/3
c charm	1.3	2/3
s strange	0.1	–1/3
t top	173	2/3
b bottom	4.2	–1/3

➛ The electron e^- is now the lightest member of a family (e^-, μ^-, τ^-).

Phenomenology: some new interactions

- Electromagnetism is a gauge interaction based on the group $U(1)$.
- The *weak* interaction is responsible for radioactive decay and nuclear fusion.
  $${}^{14}_{6}\mathrm{C} \to {}^{14}_{7}\mathrm{N} + e^- + \bar{\nu}_e, \quad \textit{i.e.} \quad n \to p^+ + e^- + \bar{\nu}_e, \quad \textit{i.e.} \quad d \to u + e^- + \bar{\nu}_e$$
  This is an example of β^--decay, where nowadays β^- means e^-.
- 1933, Fermi proposes a theory of β-decay in terms of a constant G_F.
  - ➞ This theory does not work at high energy.
  - ➞ It is a "contact" theory (no mediating bosons).
- 1960's, search for a theory of weak interaction:
  - with mediating vector bosons,
  - with short range interaction ➞ massive vector bosons.
- The *strong* interaction is responsible for the gluing of quarks in hadrons.
- Success of QED and renormalization
  - ➞ construct weak and strong interactions on gauge principles.
  - The strong interaction is a gauge field based on the group $SU(3)$.
    - ➞ 8 gauge bosons, the *gluons*.
  - What about the weak interaction?
    - ➞ Phenomenology suggests gauge group $SU(2)$…

Phenomenology: parity violation

- Parity is a global symmetry: $(x^0, x^1, x^2, x^3) \mapsto (x^0, -x^1, -x^2, -x^3)$.
- Electromagnetism and strong interactions preserve parity.
- Weak interaction violates parity (1957, Wu & *al.*).

Implications in field theory:

- Gamma matrices (chiral presentation), $k = 1, 2, 3$:
$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}, \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}$$
➞ $(\gamma^5)^2 = \mathbb{1}_4$ ➞ $\frac{1\pm\gamma^5}{2}$ are projections.
- Chiral decomposition of Dirac spinor: $\psi = \psi_L + \psi_R$ with
$$\psi_L = \frac{1-\gamma^5}{2}\psi \qquad \psi_R = \frac{1+\gamma^5}{2}\psi$$
➞ Left-handed and right-handed spinors, valued in $\mathbb{C}^2$ (half of ψ).
- *Parity symmetry exchanges left-handed and right-handed spinors.*
- ψ_L and ψ_R are Weyl spinors (or chiral spinors) ➞ irred. rep. of Lorentz group.
- Each ψ_L and ψ_R have antiparticles.
➞ This decomposition is not related to particle/antiparticle.

The weak interaction: a massive problem

- Mathematical facts:
  - If a gauge symmetry acts differently on ψ_L and ψ_R, then there is no possible Dirac mass term for ψ.

    $$\text{Dirac mass term} = m\overline{\psi}\psi = m(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L)$$
  - There is no mass term for Yang-Mills gauge bosons.
- Phenomenological constrains:
  - Weak interaction is based on the gauge group $SU(2)$.
  - Weak interaction is mediated by *massive* bosons.
  - Weak interaction violates parity conservation.

    ➞ ψ_L and ψ_R are in different representation of $SU(2)$.
- Gauge principle + parity violation:
  1. Weak gauge bosons should be massless.
  2. The electron should be massless.

➞ A gauge theory of weak interaction can only have massless particles!

The electroweak model

An elegant solution (1968, Glashow, Salam, Weinberg):

- Based on a gauge field theory.
- Provide mass terms by a *Spontaneous Symmetry Breaking Mechanism* (SSBM).
  ➞ Also known as the Brout-Englert-Higgs mechanism (Nobel Prize 2014).
- Unify electromagnetism and weak interaction: *electroweak interaction*.
- It is a two steps theory:
  1. The Lagrangian for a gauge theory with group $U(1) \times SU(2)$.
  2. The“broken” Lagrangian after the SSBM.

The electroweak model: step 1

$$\begin{aligned}\mathcal{L}_{\text{Before SSBM}} = {} & \overline{\psi_L} i\gamma^\mu D^L_\mu \psi_L + \overline{\psi_R} i\gamma^\mu D^R_\mu \psi_R + q''(\overline{\psi_L}\phi\psi_R + \overline{\psi_R}\phi^\dagger\psi_L) \\ & - \frac{1}{4} f_{\mu\nu} f^{\mu\nu} - \frac{1}{4}\sum_{k=1,2,3} g^k_{\mu\nu} g^{k\;\mu\nu} \\ & + \tfrac{1}{2}(D_\mu\phi)^\dagger(D^\mu\phi) + \tfrac{\mu^2}{2}\phi^\dagger\phi - \tfrac{\lambda}{4!}(\phi^\dagger\phi)^2\end{aligned}$$

- A left-handed $SU(2)$-doublet $\psi_L = \begin{pmatrix} N_L \\ E_L \end{pmatrix}$ of Weyl (massless) spinors.
- A right-handed $SU(2)$-singlet $\psi_R = E_R$ of Weyl (massless) spinor.
- A (massless) $U(1)$-gauge boson a_μ, with field strength $f_{\mu\nu}$.
- Three (massless) $SU(2)$-gauge bosons b^k_μ, $k = 1, 2, 3$, with field strength $g_{\mu\nu}$.
- A $SU(2)$-doublet $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ of complex scalar fields.
- Minimal couplings

$$D_\mu\phi = (\partial_\mu - i\tfrac{q}{2} b_\mu - i\tfrac{q'}{2} a_\mu)\phi,$$

$$D^L_\mu\psi_L = (\partial_\mu - i\tfrac{q}{2} b_\mu + i\tfrac{q'}{2} a_\mu)\psi_L, \quad D^R_\mu\psi_R = (\partial_\mu + iq' a_\mu)\psi_R$$

The electroweak model: step 2

$$\begin{aligned}\mathcal{L}_{\text{After SSBM}} = {} & \overline{\nu_L} i\gamma^\mu \partial_\mu \nu_L + \overline{\psi}_e i\gamma^\mu D^\gamma_\mu \psi_e - m_e \overline{\psi}_e \psi_e \\ & - \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \tfrac{1}{2} m_Z^2 Z^0_\mu Z^{0\,\mu} + m_W^2 W^+_\mu W^{-\,\mu} \\ & + (\partial_\mu H)(\partial^\mu H) - \mu^2 H^2 \\ & + \text{dynamical terms for } W^+_\mu,\ W^-_\mu \text{ and } Z^0_\mu \\ & + \text{a lot of interacting terms...}\end{aligned}$$

with $v = \sqrt{\frac{6}{\lambda}}\mu$, $m_W^2 = v^2 \frac{q^2}{2}$, $m_Z^2 = v^2 \frac{q^2 + q'^2}{2}$, $m_e = vq''$.

- Minimal coupling $D^\gamma_\mu \psi_e = (\partial_\mu + iq_e A_\mu)\psi_e$ with $q_e = \frac{qq'}{\sqrt{q^2+q'^2}}$.
- A massless neutrino ν_L (left-handed).
- A massive electron ψ_e (Dirac spinor).
- A massless $U(1)$-gauge boson A_μ (the photon), with field strength $F_{\mu\nu}$.
- Three massive vector bosons W^+_μ, W^-_μ and Z^0_μ.
- A “residual” massive particle H, the Higgs boson.

What happens to the electron?

1. Start with three Weyl spinors N_L, E_L and E_R:
   - Grouped in *different* $SU(2)$ representations: $\psi_L = \begin{pmatrix} N_L \\ E_L \end{pmatrix}$ and $\psi_R = E_R$.
   - They have different coupling constants with the $U(1)$-gauge bosons a_μ.
   - ψ_L and ψ_R are independent fields, they describe independent massless particles.

2. Transformation during the SSBM:

$$N_L \mapsto \nu_L \qquad E_L \mapsto e_L \qquad E_R \mapsto e_R$$

   The *emerging Dirac electron* is

$$\psi_e = e_L + e_R = \begin{pmatrix} e_L \\ e_R \end{pmatrix}$$

   - The symmetry $SU(2)$ is "broken" ➸ no restriction on the mass.
   - ψ_e is a Dirac spinor with a mass term.
   - A_μ, W^+_μ, W^-_μ and Z^0_μ are obtained from a_μ, b^k_μ, $k = 1, 2, 3$.
   - ψ_e is minimally coupled to the electromagnetic field A_μ ➸ QED theory.

➸ In the SM, the electron (as we know it) can only be identified *after* the SSBM.

The bosons of interaction

BOSONS force carriers
spin = 0, 1, 2, …

Unified Electroweak spin = 1

Name	Mass GeV/c^2	Electric charge
γ photon	0	0
W^-	80.39	–1
W^+ W bosons	80.39	+1
Z^0 Z boson	91.188	0

Strong (color) spin =1

Name	Mass GeV/c^2	Electric charge
g gluon	0	0

➛ The electron interacts only with bosons in the left tabular.

➛ True for all leptons, while quarks interact with all bosons.

A composite particle

1 Before the electron
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The electroweak model revisited

- The original electroweak model relies on the SSBM.
- There is a more elegant and comprehensible way to look at it:

The transformation from step 1 to step 2
is a mere change of variables in the space of gauge fields.

  - Detach the $SU(2)$-dependence of ϕ as: $\phi = U\eta \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$.
    ➞ U is a $SU(2)$-valued field, η is a (positive) real field.
  - Create $SU(2)$-*gauge invariant composite fields*:

$$\begin{pmatrix} \nu_L \\ e_L \end{pmatrix} = U^{-1}\psi_L = U^{-1}\begin{pmatrix} N_L \\ E_L \end{pmatrix}, \qquad e_R = E_R,$$

$$B_\mu = U^{-1} b_\mu U + \tfrac{2i}{q} U^{-1}(\partial_\mu U) \qquad (a_\mu, B_\mu) \mapsto (A_\mu, Z^0_\mu, W^+_\mu, W^-_\mu).$$

  - Rewrite the Lagrangian in terms of these new variables.
    ➞ It is only written in terms of $SU(2)$-gauge invariant fields.
  - The Lagrangian makes apparent the *observed* particles:
    the electron, the photon, the W^+_μ, W^-_μ and Z^0_μ vector bosons…
- In the electroweak theory, the electron has two equivalent presentations:
  - In terms of representations of $U(1) \times SU(2)$: $\psi_L = \left(\begin{smallmatrix} N_L \\ E_L \end{smallmatrix}\right)$ and $\psi_R = E_R$.
  - In terms of an EM-charged particle ψ_e which is invariant under $SU(2)$.

➞ The Dirac spinor of the electron is a composite field of more fundamental fields.

Conclusions

1. Before the electron
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A (historical) trajectory that leaves traces

The elaboration of successive mathematical descriptions of the electron has motivated new mathematics:

- Quantum mechanics, QFT
  ➞ Hilbert spaces, Operator algebras, Functionnal analysis…
- Lorentz invariance, Quantum mechanics
  ➞ Group representation theory (Spin, Poincaré group)…
- Dirac spinor
  ➞ Spin geometry, Clifford algebras, Elliptic operators, K-homology…
- Field theories, Gauge field theories
  ➞ Lie groups, Fibers and connections, Gauge invariants…
- Renormalization
  ➞ Graph theory, Combinatorial methods…

Great strides on a firm ground

The electron has gone through various successive mathematical descriptions.

Some enduring properties can be identified:

- Mass: $m_e = 9.109\,382\,91(40) \times 10^{-31}$ kg
- Charge: $q_e = -1.602\,176\,565(35) \times 10^{-19}$ C
- Spin: $\frac{1}{2}$
- No evidence for size ("electromagnetic size") ➡ point-like particle.
  $R_e < 2 \times 10^{-20}$ m (2001, LEP, contact interaction)
- "Minimal coupling" with electromagnetic field: "$p - q_e A$"
  ➡ true in CM, QM, QFT (QED, SM)...

But there are also missteps in this journey:

- Does it have an "intrinsic" mass and charge in QFT?
- Does it remain "elementary" in weak interaction processes?

Life and death of an elementary particle

- The electron "started his life" as the first *elementary particle*.
- The notion of "elementarity" is relative to interactions.
  ➞ Ex.: the nucleus of atoms with respect to electromagnetism.
- The electron is an elementary particle with respect to electromagnetism.
  ➞ Lorentz theory, Quantum mechanics, QED.
- The electron is not an elementary particle with respect to the weak interaction.
  - Parity violation forces us to break the electron into left and right components.
  - *Left-handed and right-handed components are independent.*
  - The same is true for quarks…
- Nevertheless: *The electron is not a bounded state of subparticles.*
  - This notion of "non elementarity" is not "spatial".
  - This is not a "physical" gluing of particles (like proton composed of quarks).

- The geometric interpretation of the Standard Model of Particles Physics forces us to conceive a new kind of non-elementarity.
  ➞ This non elementarity takes place in *inner* spaces.

➞ Today, the electron is the quantum of a composite field.

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Thanks you for your attention