

The mathematical electron: a particle and its mathematical representations

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Abstract

Since its “discovery” at the end of the nineteenth century, the mathematical description of the electron has been continuously refined. At the beginning, it was only a point in Newton mechanics, then a Schrödinger wave function in quantum mechanics, then a Dirac spinor in field theory, and at last a recombination of left-handed and right-handed Weyl spinors in the standard model of particles physics (after the Brout-Englert-Higgs mechanism). We will try to show how experimental constrains and theoretical ideas have motivated new and successful mathematical constructions, mainly in geometry, but also in algebra. This talk will be the opportunity to recall the deep, fruitful and intriguing entanglement of mathematics and theoretical physics through various examples taken from the twentieth century, as for instance the theory of spinor and gauge theories. It will also be the opportunity to ask about the notion of “elementary particle” at the beginning of the twenty-first century.

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Why the electron?

- First concept of “elementary particle”.
 - Related to the main achievements in theoretical physics in the XXth century:
 - Relativistic dynamics
 - Quantum mechanics
 - Field theory
 - Quantum Field theory
 - Standard Model of Particle Physics
 - It has been described using many mathematical structures:
 - Point particle
 - Wave function
 - Dirac spinor
 - Gauge field theories
 - Creation and annihilation operators in QFT
- ➔ Motivation for developments of modern mathematics.

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My objective

- Take a close look at the mathematical formulations of theories of the electron.
- Search for enduring properties in these successive descriptions.
- Give a “modern” view on the electron.
- What is now an “elementary particle”?
(after the discovery of the Higgs particle)

1 – Before the electron

Maxwell theory of electromagnetism

- One of the greatest achievement in the science of the nineteenth century.
- Explain all the known electric and magnetic phenomena.
- Unify all the established theories on electricity and magnetism.
- Written in terms of *field*, not in terms of *particles*.
- Propagation of electromagnetic waves at speed of light.
 - ➔ explains light.
 - ➔ need a “support” for the vibration of the waves: the *ether*...
- Theory of charges is not a part of the theory.
 - ➔ Charges are considered as “manifestation of the ether”.

Maxwell theory: the mathematics

“Original” description	Relativistic description
$\text{div } \vec{B} = 0$	$\partial_\xi F_{\mu\nu} + \partial_\mu F_{\nu\xi} + \partial_\nu F_{\xi\mu} = 0$
$\frac{\partial \vec{B}}{\partial t} + \text{rot } \vec{E} = 0$	
$\text{div } \vec{E} = \frac{\rho}{\epsilon_0}$	$\partial_\nu F^{\mu\nu} = \mu_0 j^\mu$
$\text{rot } \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{j}$	
$\vec{B} = \text{rot } \vec{A}$	$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \text{grad } V$	
\vec{E} and \vec{B} don't change through:	$F_{\mu\nu}$ doesn't change through:
$\vec{A} \mapsto \vec{A} + \text{grad } \chi$	$A_\mu \mapsto A_\mu + \partial_\mu \chi$
$V \mapsto V - \frac{\partial \chi}{\partial t}$	
\vec{E} = electric field,	ρ = charge density,
\vec{B} = magnetic field,	\vec{j} = charged current density,
$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$ = speed of light,	$\text{div } \vec{j} + \frac{\partial \rho}{\partial t} = 0$ conservation of charge.

$$\mu = 0, 1, 2, 3, \quad x^\mu = (ct, x, y, z), \quad \partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c} \frac{\partial}{\partial t}, \text{grad} \right)$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix} = -F_{\nu\mu}, \quad j^\mu = (c\rho, \vec{j})$$

Properties of grad , rot , and $\text{div} \Rightarrow \vec{A}$ and V exist locally.

\vec{A} = magnetic vector potential associated to \vec{B} ,

V = (scalar) electrostatic potential associated to \vec{E} .

$$A_\mu = (-V/c, \vec{A})$$

$$\partial_\xi F_{\mu\nu} + \partial_\mu F_{\nu\xi} + \partial_\nu F_{\xi\mu} = 0 \Rightarrow A_\mu \text{ exists locally.}$$

χ is a function on space-time

This is a *local symmetry due to the parametrization* $A_\mu = (-V/c, \vec{A})$

2 – An elementary particle

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The emergence of the electron

- Between 1890 and 1910 (approx.), the electron has been
 - conceived as a theoretical structure,
 - characterized as an experimental object,
 - discovered as an elementary particle,
 - accepted as an existing entity.
- Works by
 - Lorentz (~ 1892), Larmor (~ 1894) on the theory
 - Zeeman, Thomson, Millikan for main experimental evidences and characterization.
- *electron*: G. J. Stoney, 1891, to denote an *elementary quantity of electricity*.
Motivation: phenomenon of electrolysis, Faraday’s “chemical equivalents”.
- Characterization: *mass* m_e and *charge* q_e .
 - Measure of q_e/m : Zeeman effect (~ 1892), Thomson cathode rays (1897).
 - Measure of q_e : Millikan (1909, oil drop experiment).

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Lorentz theory

- Larmor theory of charges relies only on ether (Maxwell theory).
- Lorentz theory adds “substance” to charges and currents → particles.
- Main elements of the theory:
 - The electron is a particle with a mass m_e and a charge q_e .
 - Point-like particle: no evidence for substructure and size.
 - It obeys the laws of classical mechanics (Newton).
 - The electromagnetic field (\vec{E}, \vec{B}) induces a force

$$\vec{F} = q_e(\vec{E} + \vec{v} \wedge \vec{B}) \quad \rightarrow \quad m\vec{a} = \vec{F}$$

- Hamilton’s equations with

$$H = \frac{1}{2m}(\vec{p} - q_e\vec{A})^2 + q_eV$$

- Successes of the theory:
 - Laplace force,
 - Faraday law of induction,
 - Conductance properties of metals...

→ The electron is part of Newton mechanics in a field theory.

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Relativistic from the beginning

- Theory of relativity motivated by the symmetries of Maxwell equations.
- The Lorentz theory is relativistic from the very beginning.
- Use the Faraday tensor

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix},$$

the four-velocity $v^\mu = \frac{dx^\mu}{d\tau} = \gamma(c, \vec{v})$ with $\vec{v} = \frac{d\vec{x}}{dt}$ and $\gamma = (1 - v^2/c^2)^{-1/2}$,
and the four-momentum $p^\mu = mv^\mu$:

$$\frac{dp^\mu}{d\tau} = q_e F^{\mu\nu} v_\nu$$

- These equations derive from the action

$$S = \int (-m \sqrt{v^\mu v_\mu} - q_e A_\mu v^\mu) d\tau$$

3 – Quantum description

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From particle to wave and vice-versa

- Long debate about nature of light:
waves (Hooke, Huygens...) or *particle* (Gassendi, Newton...)?
- 1817, Fresnel gives a description of light as a wave theory.
 ➔ This explains the polarization effect (transverse “vibrations”).
- Requires a medium to support the “vibrations” ➔ ether.
- 1905, Einstein explains the photoelectric effect.
 ➔ quantum of light of energy $E = h\nu$ ➔ particle description.
- 1923, Compton confirms the “corpuscle” behavior of light during “collisions”.
- 1924, de Broglie postulates the wave nature of the electron.
 - He uses relativistic arguments to generalize $E = h\nu$.
 - He associates to the electron a wavelength

$$\lambda = \frac{h}{p} = \frac{h}{m_e v} \sqrt{1 - \frac{v^2}{c^2}}$$

- 1927, Davisson and Germer confirm experimentally this hypothesis.
Scattering of slow moving electrons through a piece of nickel crystal.

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The Schrödinger equation

- A wave necessitates a... wave equation!
- 1926, Schrödinger proposes an equation:

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi$$

Ψ takes values in \mathbb{C} , H is a differential operator \simeq Hamiltonian.

- Explains the spectrum of the Hydrogen atom (Coulomb potential in H).
- Equation for an electron in a potential vector (V, \vec{A}) :

$$i\hbar \frac{\partial \Psi}{\partial t} = \frac{1}{2m} \left(-i\hbar \vec{\nabla} - q_e \vec{A} \right)^2 \Psi + q_e V \Psi$$

- If Ψ solution for (V, \vec{A}) and χ real function,
then $e^{\frac{iq_e}{\hbar} \chi} \Psi$ solution for $\left(V - \frac{\partial \chi}{\partial t}, \vec{A} + \vec{\text{grad}} \chi \right)$.
 ➔ \vec{E} and \vec{B} are left unchanged.
- This is a *gauge symmetry* (Weyl 1918, Fock 1926, London 1927).
 ➔ It is a (local) change of phase: $U(1) = \{z \in \mathbb{C} / |z| = 1\}$ gauge group.
- Mathematics of all the fundamental interactions...
Key ingredient: $\vec{p} - q_e \vec{A}$ with $\vec{p} \mapsto -i\hbar \vec{\nabla}$...

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The spin

- 1924, Pauli introduces new quantum numbers (spectrum of alkali metals).
- He adds a new principle: the *exclusion principle*.
- Kronig, Uhlenbeck, Goudsmit propose to associate this number to a “self-rotation” of the electron.
 - ➔ *Physical* idea based on the interaction with magnetic field.
- 1927, Pauli proposes a theory based on the Schrödinger equation:
 - 2-components wave function, $\Psi \in \mathbb{C}^2$,
 - 2×2 matrices, $\sigma_x = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,
 - interacting term with magnetic field

$$-\frac{q_e \hbar}{2m_e} \vec{\sigma} \cdot \vec{B}.$$

- Spin related to the Lie group $SU(2)$, itself related to rotation group $SO(3)$.
 - ➔ *Mathematical* interpretation of spin as “internal self-rotation”.
- Strong relation between spin and statistic of identical particles:
 - Bosons:** spin integer, predilection for identical states (ex. laser).
 - Fermions:** spin half-integer, exclusion principle.

➔ The electron is now a massive charged fermion with spin $\frac{1}{2}$.

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The electron as a current

- In quantum mechanics, what’s remains of the electron as a particle?
- “Classical” experiments don’t see electrons, but *currents*.
- The density of charge and density of current are

$$\rho = q_e \Psi^* \Psi \qquad \vec{j} = q_e \frac{i\hbar}{2m} \left(\Psi \vec{\nabla} \Psi^* - \Psi^* \vec{\nabla} \Psi + \frac{2iq_e}{\hbar} \vec{A} \Psi^* \Psi \right)$$

- For spin $\frac{1}{2}$ particles, add *spin* current

$$\rho = q_e \Psi^\dagger \Psi$$

$$\vec{j} = q_e \frac{i\hbar}{2m} \left((\vec{\nabla} \Psi^\dagger) \Psi - \Psi^\dagger (\vec{\nabla} \Psi) + \frac{2iq_e}{\hbar} \vec{A} \Psi^\dagger \Psi \right) + q_e \frac{\hbar}{2m} \vec{\nabla} \times (\Psi^\dagger \vec{\sigma} \Psi)$$

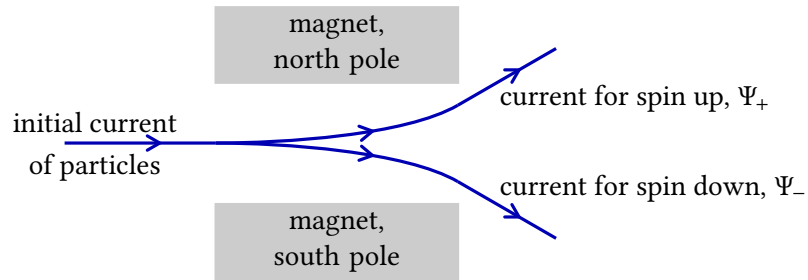
with $\Psi = \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix}$, $\Psi^\dagger = (\Psi_+^* \ \Psi_-^*)$.

- All these quantities are $U(1)$ -gauge invariant.
 - ➔ Only gauge invariant objects can be observed.

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The Stern-Gerlach experiment

- The spin was conceived to explain the spectrum of atoms.
- 1922, Stern and Gerlach show the interaction of spin with magnetic field (using silver atoms).
 - ➔ before the spin hypothesis!
- The (non homogeneous) magnetic field separates the currents for spin down and spin up particles.
- One of the most fundamental experiments in quantum mechanics.



4 – Classical field theory

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Field theories in a nutshell...

- Here, *field theory* \simeq *theory of relativistic “wave” functions*.
- Description in terms of Lagrangians (instead of Hamiltonians).
- Maxwell theory: the first field theory...

$$\mathcal{L}_{EM} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(\vec{E}^2 - \vec{B}^2)$$

- Schrödinger and Pauli theories are not relativistic.
- Relativistic versions of Schrödinger were proposed by Schrödinger, Fock, Kudar, de Donder, van der Dungen, Klein and Gordon...
They were almost all the same.
- Klein-Gordon equation based on the relation $E^2 = \vec{p}^2c^2 + m^2c^4$:

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \phi - \nabla^2 \phi + \frac{m^2c^2}{\hbar^2} \phi = 0, \quad \text{or} \quad (\partial_\mu \partial^\mu + m^2) \phi = 0 \quad \text{with } c = \hbar = 1.$$

- Lagrangian:

$$\mathcal{L}_{KG} = \frac{1}{2} \left((\partial_\mu \phi)^* \partial^\mu \phi - m^2 \phi^* \phi \right)$$

- Problems:
 - Solutions with negative energy.
 - Describe spinless particles (bosons).

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Dirac equation

- 1928, Dirac proposes an relativistic equation for spin $\frac{1}{2}$ particles.
- First order differential operator $\sqrt{\partial_\mu \partial^\mu} = \sqrt{\partial_t^2 - \vec{\nabla}^2}$.
- Dirac equation, $\psi \in \mathbb{C}^4$:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

with

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}, \quad k = 1, 2, 3$$

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} = \pm 2\delta^{\mu\nu}$$

- Lagrangian:

$$\mathcal{L}_{Dirac} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \quad \text{with } \bar{\psi} = \psi^\dagger \gamma^0$$

- Associated probabilistic (quadri-)current: $j^\mu = \bar{\psi} \gamma^\mu \psi$.
 ➔ associated density: $j^0 = \bar{\psi} \gamma^0 \psi = \psi^\dagger \psi \geq 0$.
- Non relativistic limit ➔ Pauli equation.
- The first order operator $\not{D} = i\gamma^\mu \partial_\mu$ has many applications in mathematics.
 ➔ Clifford algebras, spin geometry, noncommutative geometry...

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The positron

- Problem of the Dirac equation: solutions with negative energy persists.
- The *Dirac sea*: the vacuum is filled of electrons with negative energies.
 - ➔ Pauli exclusion principle: “usual” electrons can’t go there...
- A hole in the Dirac sea is a state of *positive* energy, with *opposite* charge.
 - ➔ interpretation as a new particle:
 - mass of the electron,
 - opposite charge,
 - annihilates when it meets an electron.
- 1932, Anderson detects such a particle with these properties: the *positron*.
- Notation: e^- for electron, e^+ for positron.
- Every particle has an *antiparticle*.
 - ⚠ a particle can be its antiparticle, ex. the photon.

➔ The (relativistic) electron is a member of a couple particle/antiparticle.

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Gauge field theories

- Matter fields are described by Dirac spinors or scalar Klein-Gordon fields.
- Electromagnetic interaction is described by gauge fields A_μ .
- Gauge principle: promote a “rigid/global” symmetry to a “local” symmetry.
 - ➔ Lagrangian invariant by $\phi \mapsto U \cdot \phi$ for $U \in G$ (Lie group).
 - ➔ Add new fields A_μ to the Lagrangian to get invariance for $U : \mathbb{R}^4 \rightarrow G$.
 - ➔ Electromagnetism is obtained from gauge principle with $G = U(1)$.
- 1954, Yang-Mills generalizes gauge principle to the group $SU(2)$.
 - ➔ $A_\mu = A_\mu^k T_k$ in the Lie algebra of $G = SU(2)$.
- Gauge principle produces natural coupling matter/gauge fields:

$$\partial_\mu \phi \mapsto (\partial_\mu - iqA_\mu)\phi, \quad q \text{ is the “charge” (minimal coupling)}$$

- The dynamics of A_μ is given by a term

$$\mathcal{L}_{\text{YM}} = \frac{1}{2} \text{tr}(F_{\mu\nu}F^{\mu\nu}) \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - iq[A_\mu, A_\nu] \text{ (field strength)}$$

- New Lagrangians:

$$\begin{aligned} \mathcal{L}_{\text{KG+YM}} &= \frac{1}{2} [(\partial_\mu - iqA_\mu)\phi]^\dagger [(\partial^\mu - iqA^\mu)\phi] - m^2 \phi^\dagger \phi + \frac{1}{2} \text{tr}(F_{\mu\nu}F^{\mu\nu}) \\ \mathcal{L}_{\text{Dirac+YM}} &= \bar{\psi}(i\gamma^\mu \partial_\mu + q\gamma^\mu A_\mu - m)\psi + \frac{1}{2} \text{tr}(F_{\mu\nu}F^{\mu\nu}) \end{aligned}$$

- *No mass terms for the gauge fields A_μ .*

5 – Quantum Electrodynamics

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Particles are quanta of fields

- Quantization of fields (“Second quantization”) = relativistic quantum theory.
- Main point: $E = mc^2$ implies creation and annihilation of particles.
 ➔ Quantum mechanics with indefinite number of particles.
- Main ideas for the quantization of a complex scalar field ϕ .

1. Decompose ϕ in Fourier modes, with $d\tilde{k} = \frac{d^3k}{(2\pi)^3 2\omega_{\mathbf{k}}}$ (Lorentz invariant measure),

$$\phi(x) = \int d\tilde{k} \left(a(\mathbf{k})e^{-ik \cdot x} + b^\dagger(\mathbf{k})e^{ik \cdot x} \right), \quad \phi^*(x) = \int d\tilde{k} \left(b(\mathbf{k})e^{-ik \cdot x} + a^\dagger(\mathbf{k})e^{ik \cdot x} \right)$$

2. Promote $a^\dagger(\mathbf{k})$, $a(\mathbf{k})$, $b^\dagger(\mathbf{k})$, and $b(\mathbf{k})$ to operators

$$[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = (2\pi)^3 2\omega_{\mathbf{k}} \delta^3(\mathbf{k} - \mathbf{k}'), \quad [b(\mathbf{k}), b^\dagger(\mathbf{k}')] = (2\pi)^3 2\omega_{\mathbf{k}} \delta^3(\mathbf{k} - \mathbf{k}').$$

$a^\dagger(\mathbf{k})$ and $b^\dagger(\mathbf{k})$ are creation operators, $a(\mathbf{k})$ and $b(\mathbf{k})$ are annihilation operators.

3. $N = \int d\tilde{k} a^\dagger(\mathbf{k})a(\mathbf{k})$ = number of particles,
 $\bar{N} = \int d\tilde{k} b^\dagger(\mathbf{k})b(\mathbf{k})$ = number of antiparticles.

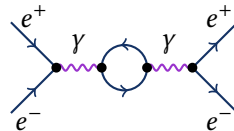
4. Particles identify as quanta created by $a^\dagger(\mathbf{k})$.
 Antiparticles identify as quanta created by $b^\dagger(\mathbf{k})$.

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QED

Quantum ElectroDynamics = Quantization of the Lagrangian

$$\mathcal{L}_{\text{Dirac+YM}} = \bar{\psi}(i\gamma^\mu \partial_\mu + q_e \gamma^\mu A_\mu - m)\psi + \frac{1}{2} \text{tr}(F_{\mu\nu} F^{\mu\nu})$$

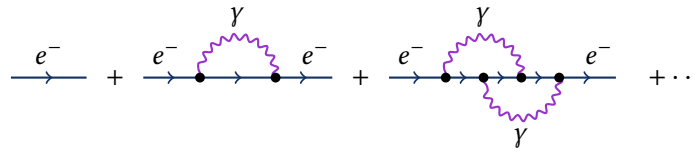
- The electron and positron are the quanta of the Dirac spinor ψ .
 ➔ A lot of subtleties: spin, anticommutations relations...
- The photon is the quantum of the gauge field A_μ .
 ➔ A lot of (other) subtleties: vector field, gauge symmetry...
- Interaction = exchange of (virtual) particles.
 ➔ Successive annihilations and creations of particles.
 ➔ Computation using Feynman diagrams.



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The self-energy of the electron

- During its propagation, an electron interacts with the electromagnetic field:



- This interaction produces an “effective mass”: *self-energy* of the electron.
 - Direct computation: this energy is infinite.
 - 1940’s, *renormalization procedure* by Schwinger, Feynman, Tomonaga.
 - Idea: the parameter m_e inserted in the Lagrangian *is not* the measured mass.
 - ➔ Insert a “bare” value \bar{m}_e for the mass in the initial computation.
 - Same problem for the electric charge q_e of the electron.
 - ➔ The electron has a “bare” electric charge \bar{q}_e .
 - The bare mass \bar{m}_e and the bare electric charge \bar{q}_e are infinite...
- ➔ In QED, the electron has no definite mass and electric charge...

6 – The Standard Model of Particle Physics

Phenomenology: a lot of particles

- Known particles around 1930: electron, proton, photon.
- 1930, Pauli predicts a light neutral particle: the neutrino ν_e .
 - ➔ Forced by conservation of energy,
 - ➔ discovered in 1956.
- 1940's–1960's: cosmic radiation, bubble chamber (1952), linear particle accelerators (1928), cyclotron (1929)... ➔ A lot of new particles are discovered.
 - 1932, *neutron*, “neutral proton”.
 - 1933, *positron*, anti-electron.
 - 1936, *muon* μ^- , “heavy electron”, $m_\mu \simeq 200 \times m_e$.
 - ➔ associated neutrino ν_μ discovered in 1962.
 - 1940–1960, lot of baryons, mesons (sensible to strong interaction).
 - 1974–1977, *tau*, τ^- , another “heavy electron”, $m_\tau \simeq 3500 \times m_e$ (➔ ν_τ).
- 1964, Gell-Mann, Zweig: hypothesis of quarks to classify baryons and mesons.
 - Three elementary particle (quarks): *u* (up), *d* (down), *s* (strange).
 - 1970's: hypothesis of three more quarks: *c* (charm), *b* (bottom), *t* (top).
 - $p^+ = uud$ and $n = ddu$.

Phenomenology: the elementary particles

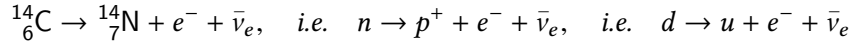
Leptons spin = 1/2			Quarks spin = 1/2		
Flavor	Mass GeV/c ²	Electric charge	Flavor	Approx. Mass GeV/c ²	Electric charge
ν_L lightest neutrino*	$(0-0.13) \times 10^{-9}$	0	u up	0.002	2/3
e electron	0.000511	-1	d down	0.005	-1/3
ν_M middle neutrino*	$(0.009-0.13) \times 10^{-9}$	0	c charm	1.3	2/3
μ muon	0.106	-1	s strange	0.1	-1/3
ν_H heaviest neutrino*	$(0.04-0.14) \times 10^{-9}$	0	t top	173	2/3
τ tau	1.777	-1	b bottom	4.2	-1/3

➔ The electron e^- is now the lightest member of a family (e^-, μ^-, τ^-).

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Phenomenology: some new interactions

- Electromagnetism is a gauge interaction based on the group $U(1)$.
- The *weak* interaction is responsible for radioactive decay and nuclear fusion.



This is an example of β^- -decay, where nowadays β^- means e^- .

- 1933, Fermi proposes a theory of β -decay in terms of a constant G_F .
 - ➔ This theory does not work at high energy.
 - ➔ It is a “contact” theory (no mediating bosons).
- 1960’s, search for a theory of weak interaction:
 - with mediating vector bosons,
 - with short range interaction ➔ massive vector bosons.
- The *strong* interaction is responsible for the gluing of quarks in hadrons.
- Success of QED and renormalization
 - ➔ construct weak and strong interactions on gauge principles.
 - The strong interaction is a gauge field based on the group $SU(3)$.
 - ➔ 8 gauge bosons, the *gluons*.
 - What about the weak interaction?
 - ➔ Phenomenology suggests gauge group $SU(2)$...

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Phenomenology: parity violation

- Parity is a global symmetry: $(x^0, x^1, x^2, x^3) \mapsto (x^0, -x^1, -x^2, -x^3)$.
- Electromagnetism and strong interactions preserve parity.
- Weak interaction violates parity (1957, Wu & *al.*).

Implications in field theory:

- Gamma matrices (chiral presentation), $k = 1, 2, 3$:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}, \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}$$

➔ $(\gamma^5)^2 = \mathbb{1}_4$ ➔ $\frac{1 \pm \gamma^5}{2}$ are projections.

- Chiral decomposition of Dirac spinor: $\psi = \psi_L + \psi_R$ with

$$\psi_L = \frac{1 - \gamma^5}{2} \psi \qquad \psi_R = \frac{1 + \gamma^5}{2} \psi$$

➔ Left-handed and right-handed spinors, valued in \mathbb{C}^2 (half of ψ).

- *Parity symmetry exchanges left-handed and right-handed spinors.*
- ψ_L and ψ_R are Weyl spinors (or chiral spinors) ➔ irred. rep. of Lorentz group.
- Each ψ_L and ψ_R have antiparticles.
 - ➔ This decomposition is not related to particle/antiparticle.

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The weak interaction: a massive problem

- Mathematical facts:
 - If a gauge symmetry acts differently on ψ_L and ψ_R , then there is no possible Dirac mass term for ψ .
Dirac mass term = $m\bar{\psi}\psi = m(\psi_L^\dagger\psi_R + \psi_R^\dagger\psi_L)$
 - There is no mass term for Yang-Mills gauge bosons.
- Phenomenological constrains:
 - Weak interaction is based on the gauge group $SU(2)$.
 - Weak interaction is mediated by *massive* bosons.
 - Weak interaction violates parity conservation.
 ➔ ψ_L and ψ_R are in different representation of $SU(2)$.
- Gauge principle + parity violation:
 1. Weak gauge bosons should be massless.
 2. The electron should be massless.

➔ A gauge theory of weak interaction can only have massless particles!

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The electroweak model

An elegant solution (1968, Glashow, Salam, Weinberg):

- Based on a gauge field theory.
- Provide mass terms by a *Spontaneous Symmetry Breaking Mechanism* (SSBM).
 ➔ Also known as the Brout-Englert-Higgs mechanism (Nobel Prize 2014).
- Unify electromagnetism and weak interaction: *electroweak interaction*.
- It is a two steps theory:
 1. The Lagrangian for a gauge theory with group $U(1) \times SU(2)$.
 2. The “broken” Lagrangian after the SSBM.

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The electroweak model: step 1

$$\begin{aligned} \mathcal{L}_{\text{Before SSBM}} = & \bar{\psi}_L i\gamma^\mu D_\mu^L \psi_L + \bar{\psi}_R i\gamma^\mu D_\mu^R \psi_R + q''(\bar{\psi}_L \phi \psi_R + \bar{\psi}_R \phi^\dagger \psi_L) \\ & - \frac{1}{4} f_{\mu\nu} f^{\mu\nu} - \frac{1}{4} \sum_{k=1,2,3} g_{\mu\nu}^k g^{k\mu\nu} \\ & + \frac{1}{2} (D_\mu \phi)^\dagger (D^\mu \phi) + \frac{\mu^2}{2} \phi^\dagger \phi - \frac{\lambda}{4!} (\phi^\dagger \phi)^2 \end{aligned}$$

- A left-handed $SU(2)$ -doublet $\psi_L = \begin{pmatrix} N_L \\ E_L \end{pmatrix}$ of Weyl (massless) spinors.
- A right-handed $SU(2)$ -singlet $\psi_R = E_R$ of Weyl (massless) spinor.
- A (massless) $U(1)$ -gauge boson a_μ , with field strength $f_{\mu\nu}$.
- Three (massless) $SU(2)$ -gauge bosons b_μ^k , $k = 1, 2, 3$, with field strength $g_{\mu\nu}$.
- A $SU(2)$ -doublet $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ of complex scalar fields.
- Minimal couplings

$$\begin{aligned} D_\mu \phi &= (\partial_\mu - i\frac{q}{2} b_\mu - i\frac{q'}{2} a_\mu) \phi, \\ D_\mu^L \psi_L &= (\partial_\mu - i\frac{q}{2} b_\mu + i\frac{q'}{2} a_\mu) \psi_L, \quad D_\mu^R \psi_R = (\partial_\mu + iq' a_\mu) \psi_R \end{aligned}$$

.....
The electroweak model: step 2

$$\begin{aligned} \mathcal{L}_{\text{After SSBM}} = & \bar{\nu}_L i\gamma^\mu \partial_\mu \nu_L + \bar{\psi}_e i\gamma^\mu D_\mu^Y \psi_e - m_e \bar{\psi}_e \psi_e \\ & - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m_Z^2 Z_\mu^0 Z^{0\mu} + m_W^2 W_\mu^+ W^{-\mu} \\ & + (\partial_\mu H)(\partial^\mu H) - \mu^2 H^2 \\ & + \text{dynamical terms for } W_\mu^+, W_\mu^- \text{ and } Z_\mu^0 \\ & + \text{a lot of interacting terms...} \end{aligned}$$

with $v = \sqrt{\frac{6}{\lambda}} \mu$, $m_W^2 = v^2 \frac{q^2}{2}$, $m_Z^2 = v^2 \frac{q^2 + q'^2}{2}$, $m_e = vq''$.

- Minimal coupling $D_\mu^Y \psi_e = (\partial_\mu + iq_e A_\mu) \psi_e$ with $q_e = \frac{qq'}{\sqrt{q^2 + q'^2}}$.
- A massless neutrino ν_L (left-handed).
- A massive electron ψ_e (Dirac spinor).
- A massless $U(1)$ -gauge boson A_μ (the photon), with field strength $F_{\mu\nu}$.
- Three massive vector bosons W_μ^+ , W_μ^- and Z_μ^0 .
- A “residual” massive particle H , the Higgs boson.

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What happens to the electron?

1. Start with three Weyl spinors N_L , E_L and E_R :

- Grouped in *different* $SU(2)$ representations: $\psi_L = \begin{pmatrix} N_L \\ E_L \end{pmatrix}$ and $\psi_R = E_R$.
- They have different coupling constants with the $U(1)$ -gauge bosons a_μ .
- ψ_L and ψ_R are independent fields, they describe independent massless particles.

2. Transformation during the SSBM:

$$N_L \mapsto \nu_L \qquad E_L \mapsto e_L \qquad E_R \mapsto e_R$$

The *emerging Dirac electron* is

$$\psi_e = e_L + e_R = \begin{pmatrix} e_L \\ e_R \end{pmatrix}$$

- The symmetry $SU(2)$ is “broken” \rightarrow no restriction on the mass.
- ψ_e is a Dirac spinor with a mass term.
- A_μ, W_μ^+, W_μ^- and Z_μ^0 are obtained from $a_\mu, b_\mu^k, k = 1, 2, 3$.
- ψ_e is minimally coupled to the electromagnetic field $A_\mu \rightarrow$ QED theory.

\rightarrow In the SM, the electron (as we know it) can only be identified *after* the SSBM.

.....
The bosons of interaction

BOSONS			force carriers spin = 0, 1, 2, ...
Unified Electroweak spin = 1			Strong (color) spin = 1
Name	Mass GeV/c ²	Electric charge	
γ photon	0	0	g gluon
W^-	80.39	-1	
W^+	80.39	+1	
W bosons			
Z^0 Z boson	91.188	0	

\rightarrow The electron interacts only with bosons in the left tabular.

\rightarrow True for all leptons, while quarks interact with all bosons.

7 – A composite particle

..... The electroweak model revisited

- The original electroweak model relies on the SSBM.
- There is a more elegant and comprehensible way to look at it:

The transformation from step 1 to step 2
is a mere change of variables in the space of gauge fields.

- Detach the $SU(2)$ -dependence of ϕ as: $\phi = U\eta \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
 ➔ U is a $SU(2)$ -valued field, η is a (positive) real field.
- Create $SU(2)$ -gauge invariant composite fields:

$$\begin{aligned} \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} &= U^{-1}\psi_L = U^{-1} \begin{pmatrix} N_L \\ E_L \end{pmatrix}, & e_R &= E_R, \\ B_\mu &= U^{-1}b_\mu U + \frac{2i}{q}U^{-1}(\partial_\mu U) & (a_\mu, B_\mu) &\mapsto (A_\mu, Z_\mu^0, W_\mu^+, W_\mu^-). \end{aligned}$$

- Rewrite the Lagrangian in terms of these new variables.
 ➔ It is only written in terms of $SU(2)$ -gauge invariant fields.
 - The Lagrangian makes apparent the *observed* particles:
 the electron, the photon, the W_μ^+ , W_μ^- and Z_μ^0 vector bosons...
 - In the electroweak theory, the electron has two equivalent presentations:
 - In terms of representations of $U(1) \times SU(2)$: $\psi_L = \begin{pmatrix} N_L \\ E_L \end{pmatrix}$ and $\psi_R = E_R$.
 - In terms of an EM-charged particle ψ_e which is invariant under $SU(2)$.
- ➔ The Dirac spinor of the electron is a composite field of more fundamental fields.

8 – Conclusions

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A (historical) trajectory that leaves traces

The elaboration of successive mathematical descriptions of the electron has motivated new mathematics:

- Quantum mechanics, QFT
 - ➔ Hilbert spaces, Operator algebras, Functionnal analysis...
- Lorentz invariance, Quantum mechanics
 - ➔ Group representation theory (Spin, Poincaré group)...
- Dirac spinor
 - ➔ Spin geometry, Clifford algebras, Elliptic operators, K -homology...
- Field theories, Gauge field theories
 - ➔ Lie groups, Fibers and connections, Gauge invariants...
- Renormalization
 - ➔ Graph theory, Combinatorial methods...

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Great strides on a firm ground

The electron has gone through various successive mathematical descriptions.

Some enduring properties can be identified:

- Mass: $m_e = 9.109\,382\,91(40) \times 10^{-31}$ kg
- Charge: $q_e = -1.602\,176\,565(35) \times 10^{-19}$ C
- Spin: $\frac{1}{2}$
- No evidence for size (“electromagnetic size”) ➔ point-like particle.
 $R_e < 2 \times 10^{-20}$ m (2001, LEP, contact interaction)
- “Minimal coupling” with electromagnetic field: “ $p - q_e A$ ”
 - ➔ true in CM, QM, QFT (QED, SM)...

But there are also missteps in this journey:

- Does it have an “intrinsic” mass and charge in QFT?
- Does it remain “elementary” in weak interaction processes?

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Life and death of an elementary particle

- The electron “started his life” as the first *elementary particle*.
 - The notion of “elementarity” is relative to interactions.
 - ➔ Ex.: the nucleus of atoms with respect to electromagnetism.
 - The electron is an elementary particle with respect to electromagnetism.
 - ➔ Lorentz theory, Quantum mechanics, QED.
 - The electron is not an elementary particle with respect to the weak interaction.
 - Parity violation forces us to break the electron into left and right components.
 - *Left-handed and right-handed components are independent*.
 - The same is true for quarks...
 - Nevertheless: *The electron is not a bounded state of subparticles*.
 - This notion of “non elementarity” is not “spatial”.
 - This is not a “physical” gluing of particles (like proton composed of quarks).
 - The geometric interpretation of the Standard Model of Particles Physics forces us to conceive a new kind of non-elementarity.
 - ➔ This non elementarity takes place in *inner* spaces.
- ➔ Today, the electron is the quantum of a composite field.

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