

# Gauge field theories: a comparison of various mathematical approaches

(Toward a pattern...)

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Paris, February 12, 2014

## **Abstract**

Gauge field theories are the cornerstone of the standard model of elementary particles. Since the discovery of the deep relation between the physical ideas of Yang and Mills and the mathematical structures of connections on principal fiber bundle, various mathematical approaches have been proposed to model gauge field theories. I will make a comparative review of some of these approaches, from ordinary differential geometry, noncommutative geometry, to transitive Lie algebroids.

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**How to construct a gauge field theory?**

The basic ingredients are:

1. A space of local symmetries (space-time dependence): a **gauge group**.
2. An implementation of the symmetry on matter fields: a **representation theory**.
3. A notion of derivation: some **differential structures**.
4. A (gauge compatible) replacement of ordinary derivations: **covariant derivative**.
5. A way to write a gauge invariant Lagrangian density: **action functional**.

Presentation of three mathematical schemes to construct gauge field theories:

- Ordinary differential geometry of principal fiber bundles.
- Noncommutative geometry.
- Transitive Lie algebroids.

Point 5 will not be covered. . .

# 1 – Ordinary differential geometry

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**Connections in differential geometry**

Let  $\mathcal{P} \xrightarrow{\pi} \mathcal{M}$  be a  $G$ -principal fiber bundle and  $\mathfrak{g}$  the Lie algebra of  $G$ .  
 $R_g$  the right action of  $g \in G$  on  $\mathcal{P}$

**Connection on  $\mathcal{P}$ :** 1-form  $\omega \in \Omega^1(\mathcal{P}) \otimes \mathfrak{g}$  such that:

$$\omega(\xi^{\mathcal{P}}) = \xi, \quad \forall \xi \in \mathfrak{g}, \quad R_g^* \omega = \text{Ad}_{g^{-1}} \omega, \quad \forall g \in G$$

**Curvature:**  $\Omega = d\omega + \frac{1}{2}[\omega, \omega] \in \Omega^2(\mathcal{P}) \otimes \mathfrak{g}$ .

Transfer this connection on any associated vector bundle  $\mathcal{E}$ :

**Covariant derivative:**  $X \in \Gamma(T\mathcal{M}) \rightarrow \nabla_X : \Gamma(\mathcal{E}) \rightarrow \Gamma(\mathcal{E})$  (smooth sections of  $\mathcal{E}$ ).

$$\nabla_X(f\varepsilon) = (X \cdot f)\varepsilon + f\nabla_X \varepsilon, \quad \text{and other relations...}$$

→ This is the minimal coupling in physics.

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**Three levels of description**

**Globally on  $\mathcal{P}$ :**  $\omega \in \Omega^1(\mathcal{P}) \otimes \mathfrak{g}$  connection 1-form on  $\mathcal{P}$ ,  $\Omega$  its curvature.

$\omega \in \Omega^1(\mathcal{P}) \otimes \mathfrak{g}$ , equivariant + vertical condition.

$\Omega \in \Omega^2(\mathcal{P}) \otimes \mathfrak{g}$ , equivariant + horizontal.

→ Preferred description for mathematicians...

**Locally on  $\mathcal{M}$ :**  $(U, \phi)$  local trivialisation of  $\mathcal{P}$  with  $\phi : U \times G \rightarrow \mathcal{P}|_U$ ,

$s$  its local section,  $s(x) = \phi(x, e)$ .

Local expressions of the connection and the curvature:

$$A = s^* \omega \in \Omega^1(U) \otimes \mathfrak{g} \quad F = s^* \Omega \in \Omega^2(U) \otimes \mathfrak{g}$$

$(U_i, \phi_i)$  and  $(U_j, \phi_j)$  two local trivialisations.  $g_{ij} : U_i \cap U_j \neq \emptyset \rightarrow G$  transition functions:

$$A_j = g_{ij}^{-1} A_i g_{ij} + g_{ij}^{-1} dg_{ij} \quad F_j = g_{ij}^{-1} F_i g_{ij}$$

→ Preferred description for physicists...

**Globally on  $\mathcal{M}$ :** The family  $\{F_i\}_i$  with its homogeneous gluing relations, or  $\Omega$  equivariant and horizontal, defines a section  $\mathbb{F}$  of  $\wedge^2 T^* \mathcal{M} \otimes \text{Ad} \mathcal{P}$ .

The family  $\{A_i\}_i$  is not associated to an algebraic object globally defined on  $\mathcal{M}$ .

→ Preferred description for me...

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**Covariant derivatives and algebraic structures**

$\mathcal{M}$  smooth manifold,  $(\Omega^\bullet(\mathcal{M}), d)$  de Rham differential calculus on  $\mathcal{M}$ .  
 $\mathcal{E}$  a vector bundle on  $\mathcal{M}$ .  
 $\mathbf{M} = \Gamma(\mathcal{E})$  the space of smooth sections of  $\mathcal{E}$ .

**Covariant derivative on  $\mathcal{E}$ :** linear map

$$\nabla : \mathbf{M} \rightarrow \Omega^1(\mathcal{M}) \otimes_{C^\infty(\mathcal{M})} \mathbf{M} \quad \text{such that} \quad \nabla(f\varepsilon) = df \otimes \varepsilon + f\nabla\varepsilon$$

for any  $\varepsilon \in \mathbf{M}$  and  $f \in C^\infty(\mathcal{M})$ .

**Natural extension:**

$$\nabla : \Omega^\bullet(\mathcal{M}) \otimes_{C^\infty(\mathcal{M})} \mathbf{M} \rightarrow \Omega^{\bullet+1}(\mathcal{M}) \otimes_{C^\infty(\mathcal{M})} \mathbf{M}$$

defined by

$$\nabla(\eta \otimes \varepsilon) = d\eta \otimes \varepsilon + (-1)^r \eta \wedge \nabla\varepsilon \quad \text{for any } \eta \in \Omega^r(\mathcal{M}).$$

**Curvature:** the  $C^\infty(\mathcal{M})$ -linear map

$$\nabla^2 : \mathbf{M} \rightarrow \Omega^2(\mathcal{M}) \otimes_{C^\infty(\mathcal{M})} \mathbf{M}$$

→ 2-form  $\mathbb{F}$  on  $\mathcal{M}$  with values in  $\text{Ad}\mathcal{P} \subset \text{End}(\mathcal{E}) = \mathcal{E}^* \otimes \mathcal{E}$  (modulo rep.).

→ This can be generalized...

.....  
**Assets and problems of ordinary gauge field theories**

- They are the prototype of theories used in the standard model of particle physics.
- The mathematical structures are now accustomed and popular.
- The gauge theories obtained are massless gauge theories.
  - Require the symmetry breaking mechanism to generate masses in the SM.
  - Higgs boson discovered: not a bad theoretical idea after all...

## 2 – Noncommutative geometry

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### The main noncommutative concepts

- Study geometric structures using equivalent algebraic structures.
- Replace commutative algebras of functions by noncommutative algebras.
- There is not a unique way to generalize key geometrical structures.
  - ➔ Several approaches to NCG:
    - Spectral triple (Connes' approach),
    - Quantum groups and covariant differential calculi,
    - Derivation-based noncommutative differential calculus (Dubois-Violette)...
      - ➔ The one used in the following.

## 2.1 Derivation-based differential calculus

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**Derivations of an associative algebra**

$\mathbf{A}$  associative algebra with unit.

$\mathcal{Z}(\mathbf{A})$  center of  $\mathbf{A}$ : commutative subalgebra.

**Vector space of derivations of  $\mathbf{A}$ :**

$$\text{Der}(\mathbf{A}) = \{ \mathfrak{X} : \mathbf{A} \rightarrow \mathbf{A} / \mathfrak{X} \text{ linear, } \mathfrak{X}(ab) = \mathfrak{X}(a)b + a\mathfrak{X}(b), \forall a, b \in \mathbf{A} \}$$

**Structure of  $\text{Der}(\mathbf{A})$ :** One has

- $\text{Der}(\mathbf{A})$  is a Lie algebra for the bracket  $[\mathfrak{X}, \mathfrak{Y}]a = \mathfrak{X}\mathfrak{Y}a - \mathfrak{Y}\mathfrak{X}a$ ,
- $\text{Der}(\mathbf{A})$ :  $\mathcal{Z}(\mathbf{A})$ -module for the product  $(f\mathfrak{X})a = f(\mathfrak{X}a)$ ,
- $\text{Int}(\mathbf{A}) = \{ \text{ad}_a : b \mapsto [a, b] / a \in \mathbf{A} \} \subset \text{Der}(\mathbf{A})$ , inner derivations:  
Lie ideal and  $\mathcal{Z}(\mathbf{A})$ -submodule,
- $\text{Out}(\mathbf{A}) = \text{Der}(\mathbf{A})/\text{Int}(\mathbf{A}) \twoheadrightarrow$  s.e.s. of Lie algebras and  $\mathcal{Z}(\mathbf{A})$ -modules

$$0 \rightarrow \text{Int}(\mathbf{A}) \rightarrow \text{Der}(\mathbf{A}) \rightarrow \text{Out}(\mathbf{A}) \rightarrow 0$$

.....  
**Derivation-based differential calculus**

$\mathbf{A}$  associative algebra with unit.

- $\underline{\Omega}_{\text{Der}}^\bullet(\mathbf{A})$  the space of  $\mathcal{Z}(\mathbf{A})$ -multilinear antisymmetric maps from  $\text{Der}(\mathbf{A})^n$  to  $\mathbf{A}$ ,  
with  $\underline{\Omega}_{\text{Der}}^0(\mathbf{A}) = \mathbf{A}$ .

$$\underline{\Omega}_{\text{Der}}^\bullet(\mathbf{A}) = \bigoplus_{n \geq 0} \underline{\Omega}_{\text{Der}}^n(\mathbf{A})$$

- $\mathbb{N}$ -graded algebra for the product

$$(\omega\eta)(\mathfrak{X}_1, \dots, \mathfrak{X}_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in \mathfrak{S}_{p+q}} (-1)^{\text{sign}(\sigma)} \omega(\mathfrak{X}_{\sigma(1)}, \dots, \mathfrak{X}_{\sigma(p)}) \eta(\mathfrak{X}_{\sigma(p+1)}, \dots, \mathfrak{X}_{\sigma(p+q)})$$

- **differential graded algebra** for the differential  $d$  defined by

$$\begin{aligned} d\omega(\mathfrak{X}_1, \dots, \mathfrak{X}_{n+1}) &= \sum_{i=1}^{n+1} (-1)^{i+1} \mathfrak{X}_i \omega(\mathfrak{X}_1, \dots, \overset{i}{\dot{\vee}} \dots, \mathfrak{X}_{n+1}) \\ &\quad + \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} \omega([\mathfrak{X}_i, \mathfrak{X}_j], \dots, \overset{i}{\dot{\vee}} \dots, \overset{j}{\dot{\vee}} \dots, \mathfrak{X}_{n+1}) \end{aligned}$$

## 2.2 Noncommutative connections and their properties

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**Noncommutative connections**

$M$  a right  $\mathbf{A}$ -module (representation space).

**Noncommutative connection:** linear map  $\widehat{\nabla}_{\mathfrak{X}} : M \rightarrow M$ , defined for any  $\mathfrak{X} \in \text{Der}(\mathbf{A})$ , such that  $\forall \mathfrak{X}, \mathfrak{Y} \in \text{Der}(\mathbf{A}), \forall a \in \mathbf{A}, \forall m \in M, \forall f \in \mathcal{Z}(\mathbf{A})$ :

$$\begin{aligned}\widehat{\nabla}_{\mathfrak{X}}(ma) &= m(\mathfrak{X}a) + (\widehat{\nabla}_{\mathfrak{X}}m)a, \\ \widehat{\nabla}_{\mathfrak{X}+\mathfrak{Y}}m &= \widehat{\nabla}_{\mathfrak{X}}m + \widehat{\nabla}_{\mathfrak{Y}}m, \\ \widehat{\nabla}_{f\mathfrak{X}}m &= (\widehat{\nabla}_{\mathfrak{X}}m)f\end{aligned}$$

**Curvature:** linear map  $\widehat{R}(\mathfrak{X}, \mathfrak{Y}) : M \rightarrow M$  defined for any  $\mathfrak{X}, \mathfrak{Y} \in \text{Der}(\mathbf{A})$  by

$$\widehat{R}(\mathfrak{X}, \mathfrak{Y})m = [\widehat{\nabla}_{\mathfrak{X}}, \widehat{\nabla}_{\mathfrak{Y}}]m - \widehat{\nabla}_{[\mathfrak{X}, \mathfrak{Y}]}m$$

.....  
**Gauge transformations, Hermitean structure**

$M$  a right  $\mathbf{A}$ -module.

**Gauge group of  $M$ :**

$\mathcal{G} = \text{Aut}(M)$ , the group of automorphisms of  $M$  as a right  $\mathbf{A}$ -module.

**Gauge transformations:** For any  $\Phi \in \mathcal{G}$  and any n.c. connection  $\widehat{\nabla}$ ,

$$\widehat{\nabla}_{\mathfrak{X}}^{\Phi} = \Phi^{-1} \circ \widehat{\nabla}_{\mathfrak{X}} \circ \Phi : M \rightarrow M$$

is a noncommutative connection.

➔ action of  $\mathcal{G}$  on the space of noncommutative connections.

When  $\mathbf{A}$  is an involutive algebra, notion of **Hermitean structure**:

- $\langle -, - \rangle : M \times M \rightarrow \mathbf{A}$ ,
- Hermitean n.c. connections,
- Unitary gauge transformations.



.....  
**Noncommutative connections for  $M = \mathbf{A}$**

Special case: the right  $\mathbf{A}$ -module  $M = \mathbf{A}$  and  $\mathbf{A}$  has a unit  $\mathbf{1}$ .

$\widehat{\nabla}_{\mathfrak{X}} : \mathbf{A} \rightarrow \mathbf{A}$  be a n.c. connection.

- $\widehat{\nabla}$  completely given by  $\widehat{\nabla}_{\mathfrak{X}}\mathbf{1} = \omega(\mathfrak{X})$ , with  $\omega \in \underline{\Omega}_{\text{Der}}^1(\mathbf{A})$ :

$$\widehat{\nabla}_{\mathfrak{X}}a = \mathfrak{X}a + \omega(\mathfrak{X})a$$

- Curvature of  $\widehat{\nabla}$ : multiplication on the left on  $\mathbf{A}$  by the n.c. 2-form

$$\Omega(\mathfrak{X}, \mathfrak{Y}) = d\omega(\mathfrak{X}, \mathfrak{Y}) + [\omega(\mathfrak{X}), \omega(\mathfrak{Y})]$$

- Gauge group: identified with invertible elements  $g \in \mathbf{A}$  by  $\Phi_g(a) = ga$ .

- Gauge transformations on  $\widehat{\nabla}$ :

$$\omega \mapsto \omega^g = g^{-1}\omega g + g^{-1}dg \qquad \Omega \mapsto \Omega^g = g^{-1}\Omega g$$

- $\widehat{\nabla}_{\mathfrak{X}}^0 = \mathfrak{X}a$  is a n.c. connection on  $\mathbf{A}$  with vanishing curvature,  $\omega = 0$ .

➔ Replace  $\widehat{\nabla}$  by an algebraic object: **connection 1-form**.

If  $\mathbf{A}$  is involutive, hermitean structure:  $\langle a, b \rangle = a^*b$ .

$U(\mathbf{A}) = \{u \in \mathbf{A} / u^*u = uu^* = \mathbf{1}\}$  is the unitary gauge group.

## 2.3 The endomorphism algebra of a vector bundle

$$\dots\dots\dots$$

$$\mathbf{A} = C^\infty(\mathcal{M})$$

$\mathbf{A} = C^\infty(\mathcal{M})$  (complex valued), for a smooth compact manifold  $\mathcal{M}$ .  
 Involutive algebra for the complex conjugation.

- $\mathcal{Z}(\mathbf{A}) = C^\infty(\mathcal{M})$ .
- $\text{Der}(\mathbf{A}) = \Gamma(T\mathcal{M})$  (complex vector fields on  $\mathcal{M}$ ).
- $\text{Int}(\mathbf{A}) = 0$ .
- $\text{Out}(\mathbf{A}) = \Gamma(T\mathcal{M})$ .
- $\underline{\Omega}_{\text{Der}}^\bullet(\mathbf{A}) = \Omega^\bullet(\mathcal{M})$ , de Rham forms on  $\mathcal{M}$ .
- N.C. connections are ordinary connections.

$$\dots\dots\dots$$

$$\mathbf{A} = M_n(\mathbb{C})$$

$\mathbf{A} = M_n(\mathbb{C}) = M_n$ , finite dimensional algebra of  $n \times n$  complex matrices.  
 Involutive algebra for the adjointness of matrices.

- $\mathcal{Z}(M_n) = \mathbb{C}$ .
- $\text{Der}(M_n) = \text{Int}(M_n) \simeq \mathfrak{sl}_n = \mathfrak{sl}(n, \mathbb{C})$  (traceless matrices)  
 $\mathfrak{sl}_n(\mathbb{C}) \ni \gamma \mapsto \text{ad}_\gamma \in \text{Int}(M_n)$ .
- $\text{Out}(M_n) = 0$ .
- $\underline{\Omega}_{\text{Der}}^\bullet(M_n) \simeq M_n \otimes \wedge^\bullet \mathfrak{sl}_n^*$ ,  
 differential  $d' = \text{Chevalley-Eilenberg}$  for  $\mathfrak{sl}_n$  represented on  $M_n$  by  $\text{ad}$ .
- Canonical 1-form  $i\theta \in \underline{\Omega}_{\text{Der}}^1(M_n)$  such that for any  $\gamma \in M_n(\mathbb{C})$

$$i\theta(\text{ad}_\gamma) = \gamma - \frac{1}{n} \text{Tr}(\gamma) \mathbf{1}$$

- $\widehat{\nabla}_{\mathfrak{X}}^0 a = \mathfrak{X}a - i\theta(\mathfrak{X})a$  is a gauge invariant connection on  $\mathbf{M} = \mathbf{A}$  with zero curvature.  
 $\rightarrow$  Important role in gauge field theories...

$$\dots\dots\dots$$

$$\mathbf{A} = C^\infty(\mathcal{M}) \otimes M_n(\mathbb{C})$$

Mix of previous examples:  $\mathbf{A} = C^\infty(\mathcal{M}) \otimes M_n(\mathbb{C})$ , matrix valued functions on  $\mathcal{M}$ .

- $\mathcal{Z}(\mathbf{A}) = C^\infty(\mathcal{M})$ .
- $\text{Der}(\mathbf{A}) = [\text{Der}(C^\infty(\mathcal{M})) \otimes \mathbf{1}] \oplus [C^\infty(\mathcal{M}) \otimes \text{Der}(M_n)]$   
 $= \Gamma(T\mathcal{M}) \oplus [C^\infty(\mathcal{M}) \otimes \mathfrak{sl}_n]$  as Lie algebras and  $C^\infty(\mathcal{M})$ -modules.  
 $\mathfrak{X} = X + \text{ad}_\gamma$  for  $X \in \Gamma(T\mathcal{M})$  and  $\gamma \in C^\infty(\mathcal{M}) \otimes \mathfrak{sl}_n = \mathbf{A}_0$  (traceless elements in  $\mathbf{A}$ ).
- $\text{Int}(\mathbf{A}) = \mathbf{A}_0$  and  $\text{Out}(\mathbf{A}) = \Gamma(T\mathcal{M})$ .
- $\underline{\Omega}_{\text{Der}}^\bullet(\mathbf{A}) = \Omega^\bullet(\mathcal{M}) \otimes \underline{\Omega}_{\text{Der}}^\bullet(M_n)$  with  $\widehat{d} = d + d'$ .
- N.C. 1-form  $i\theta$  defined as  $i\theta(X + \text{ad}_\gamma) = \gamma \in \mathbf{A}_0 \subset \mathbf{A}$ .  
 It splits the short exact sequence

$$0 \rightarrow \mathbf{A}_0 \xrightarrow{i\theta} \text{Der}(\mathbf{A}) \rightarrow \Gamma(T\mathcal{M}) \rightarrow 0$$

- Gauge fields models of Yang-Mills-Higgs type:  
 Yang-Mills in the geometric direction, Higgs (scalar fields) in the algebraic direction.

\dots\dots\dots

**The endomorphism algebra**

$\mathcal{M}$  smooth locally compact manifold.

$SU(n)$ -principal fiber bundle  $\mathcal{P}$ .

$\mathcal{E}$  associated vector bundle with fiber  $\mathbb{C}^n$ .

$\text{End}(\mathcal{E})$  fiber bundle of endomorphisms of  $\mathcal{E}$ .

$\mathbf{A}$  the algebra of smooth sections of  $\text{End}(\mathcal{E})$ .

“Trivial case”:  $\mathcal{E} = \mathcal{M} \times \mathbb{C}^n \rightarrow \mathbf{A} = C^\infty(\mathcal{M}) \otimes M_n$ .

In general,  $\mathbf{A}$  is (globally) more complicated.

Using trivialisations of  $\mathcal{E}$ , the algebra  $\mathbf{A}$  looks locally as  $C^\infty(U) \otimes M_n$ .

- $\mathcal{Z}(\mathbf{A}) = C^\infty(\mathcal{M})$ .
- Involution, trace map and determinant ( $\text{Tr}, \det : \mathbf{A} \rightarrow C^\infty(\mathcal{M})$ ) well defined.
- $SU(\mathbf{A})$  the unitaries in  $\mathbf{A}$  of determinant 1,  
 $\mathfrak{su}(\mathbf{A})$  the traceless antihermitean elements:

$\mathcal{G} = SU(\mathbf{A})$  is the gauge group of  $\mathcal{P}$

$\text{Lie}\mathcal{G} = \mathfrak{su}(\mathbf{A})$  is the Lie algebra of the gauge group

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**Derivations and differential calculus**

$\rho : \text{Der}(\mathbf{A}) \rightarrow \text{Der}(\mathbf{A})/\text{Int}(\mathbf{A}) = \text{Out}(\mathbf{A})$  the projection.

- $\text{Out}(\mathbf{A}) \simeq \text{Der}(C^\infty(\mathcal{M})) = \Gamma(T\mathcal{M})$ .  
 $\rho$  is the restriction of  $\mathfrak{X} \in \text{Der}(\mathbf{A})$  to  $\mathcal{Z}(\mathbf{A}) = C^\infty(\mathcal{M})$ .
- $\text{Int}(\mathbf{A})$  is isomorphic to  $\mathbf{A}_0$ , the traceless elements in  $\mathbf{A}$ .
- The s.e.s. of Lie algebras and  $C^\infty(\mathcal{M})$ -modules of derivations looks like

$$0 \longrightarrow \mathbf{A}_0 \xrightarrow{\text{ad}} \text{Der}(\mathbf{A}) \xrightarrow{\rho} \Gamma(T\mathcal{M}) \longrightarrow 0$$

- There is no *a priori* canonical splitting in the non trivial case.
- The “n.c. 1-form”  $i\theta$  cannot be defined here.  
 But one can define a map of  $C^\infty(\mathcal{M})$ -modules:

$$i\theta : \text{Int}(\mathbf{A}) \rightarrow \mathbf{A}_0 \qquad \text{ad}_\gamma \mapsto \gamma - \frac{1}{n} \text{Tr}(\gamma)\mathbb{1}$$

.....  
**Connections on  $\mathcal{E}$**

$\nabla^\mathcal{E}$  any (ordinary)  $SU(n)$ -connection on  $\mathcal{E}$ .

Associated connections:  $\nabla^{\mathcal{E}^*}$  on  $\mathcal{E}^*$  and  $\nabla$  on  $\text{End}(\mathcal{E}) = \mathcal{E}^* \otimes \mathcal{E}$ .

Notation  $X = \rho(\mathfrak{X}) \in \Gamma(T\mathcal{M})$  for any  $\mathfrak{X} \in \text{Der}(\mathbf{A})$ .

- $\Gamma(T\mathcal{M}) \ni X \mapsto \nabla_X \in \text{Der}(\mathbf{A})$ .  
 $\rightarrow X \mapsto \nabla_X$  is a splitting as  $C^\infty(\mathcal{M})$ -modules of the s.e.s.

$$0 \longrightarrow \mathbf{A}_0 \xrightarrow{\alpha} \text{Der}(\mathbf{A}) \xrightarrow{\nabla} \Gamma(T\mathcal{M}) \longrightarrow 0$$

- The obstruction to be a splitting of Lie algebras is the curvature of  $\nabla$ :

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}.$$

- $\mathfrak{X} - \nabla_X \in \text{Int}(\mathbf{A})$  for any  $\mathfrak{X} \in \text{Der}(\mathbf{A})$ .
- $\mathfrak{X} \mapsto \alpha(\mathfrak{X}) = -i\theta(\mathfrak{X} - \nabla_X)$  is a n.c. 1-form  $\alpha \in \underline{\Omega}_{\text{Der}}^1(\mathbf{A})$ .

$$\mathfrak{X} = \nabla_X - \text{ad}_{\alpha(x)}, \quad \forall \mathfrak{X} \in \text{Der}(\mathbf{A}) \qquad \alpha(\text{ad}_\gamma) = -\gamma, \quad \forall \gamma \in \mathbf{A}_0$$

$SU(n)$ -connection  $\nabla^\mathcal{E} \rightarrow$  n.c. 1-form  $\alpha$  s.t.  $\alpha(\text{ad}_\gamma) = -\gamma$ .

Identification of curvature and gauge transformations.

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**Noncommutative connections on  $M = \mathbf{A}$**

$M = \mathbf{A}$  with Hermitean structure  $(a, b) \mapsto a^*b$ .  
 $\omega \in \underline{\Omega}_{\text{Der}}^1(\mathbf{A}) \mapsto \widehat{\nabla}^\omega$  n.c. connection defined by  $\widehat{\nabla}_{\mathfrak{X}}^\omega a = \mathfrak{X}a + \omega(\mathfrak{X})a$ .  
 $\nabla^\mathcal{E}$  on  $\mathcal{E} \mapsto \nabla$  on  $\text{End}(\mathcal{E}) \mapsto \alpha \in \underline{\Omega}_{\text{Der}}^1(\mathbf{A}) \mapsto \widehat{\nabla}^\alpha$  n.c. connection.

**Theorem 1** (Ordinary connections as noncommutative connections). *One has:*

- *The space of n.c. connections on the right module  $\mathbf{A}$  compatible with the Hermitean structure  $(a, b) \mapsto a^*b$  contains the space of ordinary  $SU(n)$ -connections on  $\mathcal{E}$ .*
- *This inclusion is compatible with the corresponding definitions of curvature and gauge transformations.*
- $\alpha$  is a globally defined algebraic object on  $\mathcal{M}$  corresponding to the family  $\{A_i\}_i$ .
- What are n.c. connections from a physical point of view?  
 Their n.c. part can be interpreted as scalar Higgs fields in (natural) models.  
 $\mapsto$  Spontaneous symmetry breaking mechanism, mass generation...

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**Assets and problems of n.c. gauge field theories**

- The gauge theories obtained in NCG are very diverse:  
 Many approaches to NCG...  
 Many algebras can be considered...  
 $\mapsto$  Gauge field theories can be realistic or exotic.
- Many of these theories are of Yang-Mills-Higgs type.  
 $\mapsto$  The n.c. standard model by Chamseddine, Connes and Marcolli.
- The gauge group is the group of automorphisms of a right module.  
 $\mapsto$  Not all Lie groups are accessible, for instance  $U(1)$ ...
- The mathematical structures can be very involved.  
 $\mapsto$  Spectral triples require a lot of mathematical skill.  
 $\mapsto$  Not convenient to explore new models related to particle physics...

### 3 – Transitive Lie algebroids

#### 3.1 Generalities on Lie algebroids

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**Lie algebroids: the algebraic point of view**

- Lie algebroids are usually considered as generalizations of the tangent bundle.
- Here the geometric structure is ignored in favor of the algebraic structure, as in NCG.

$\mathcal{M}$  a smooth manifold.

$\Gamma(T\mathcal{M})$  the Lie algebra and  $C^\infty(\mathcal{M})$ -module of vector fields.

Definition given in the “language” of algebras and modules:

A **Lie algebroid**  $A$  is:

- a finite projective module over  $C^\infty(\mathcal{M})$ ,
- equipped with a Lie bracket  $[-, -]$ ,
- equipped with a  $C^\infty(\mathcal{M})$ -linear Lie morphism  $\rho : A \rightarrow \Gamma(T\mathcal{M})$  such that

$$[\mathfrak{X}, f\mathfrak{Y}] = f[\mathfrak{X}, \mathfrak{Y}] + (\rho(\mathfrak{X}) \cdot f) \mathfrak{Y}$$

for any  $\mathfrak{X}, \mathfrak{Y} \in A$  and  $f \in C^\infty(\mathcal{M})$ .

$\rho$  is the **anchor** map.

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**Transitive Lie algebroids**

A Lie algebroid  $A \xrightarrow{\rho} \Gamma(T\mathcal{M})$  is **transitive** if  $\rho$  is surjective.

- $L = \text{Ker } \rho$  is a Lie algebroid with null anchor on  $\mathcal{M}$ .
- The vector bundle  $\mathcal{L}$  such that  $L = \Gamma(\mathcal{L})$  is a locally trivial bundle in Lie algebras.  
 $\rightarrow$  gives the Lie structure on  $L$ .
- One has the s.e.s. of Lie algebras and  $C^\infty(\mathcal{M})$ -modules

$$0 \longrightarrow L \xrightarrow{\iota} A \xrightarrow{\rho} \Gamma(T\mathcal{M}) \longrightarrow 0$$

- $L$  is called the **kernel** of  $A$ .

This s.e.s. is the key structure for various considerations.

Think about it as an infinitesimal version of a principal fiber bundle

$$G \longrightarrow \mathcal{P} \xrightarrow{\pi} \mathcal{M}$$

### 3.2 Examples, representation theory

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**Example 1: Derivations of a vector bundle**

$\mathcal{E}$  a vector bundle over  $\mathcal{M}$ .

$\text{End}(\mathcal{E})$  the fiber bundle of endomorphisms of  $\mathcal{E}$ ,  $\mathbf{A}(\mathcal{E}) = \Gamma(\text{End}(\mathcal{E}))$ .

$\mathfrak{D}(\mathcal{E})$  the space of first-order differential operators on  $\mathcal{E}$  with scalar symbols.

$\sigma : \mathfrak{D}(\mathcal{E}) \rightarrow \Gamma(T\mathcal{M})$  the symbol map.

$$0 \longrightarrow \mathbf{A}(\mathcal{E}) \xrightarrow{\iota} \mathfrak{D}(\mathcal{E}) \xrightarrow{\sigma} \Gamma(T\mathcal{M}) \longrightarrow 0$$

is the **transitive Lie algebroid of derivations** of  $\mathcal{E}$ .

Remark:  $\mathbf{A}(\mathcal{E})$  is the endomorphism algebra of  $\mathcal{E}$  defined before...

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**Representation of a Lie algebroid**

$A \xrightarrow{\rho} \Gamma(T\mathcal{M})$  a Lie algebroid and  $\mathcal{E} \rightarrow \mathcal{M}$  a vector bundle.

A **representation** of  $A$  on  $\mathcal{E}$  is a morphism of Lie algebroids  $\phi : A \rightarrow \mathfrak{D}(\mathcal{E})$ .

When  $A$  is transitive, one has the commutative diagram of exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & L & \xrightarrow{\iota} & A & \xrightarrow{\rho} & \Gamma(T\mathcal{M}) \longrightarrow 0 \\ & & \downarrow \phi_L & & \downarrow \phi & & \parallel \\ 0 & \longrightarrow & \mathbf{A}(\mathcal{E}) & \xrightarrow{\iota} & \mathfrak{D}(\mathcal{E}) & \xrightarrow{\sigma} & \Gamma(T\mathcal{M}) \longrightarrow 0 \end{array}$$

$\phi_L : L \rightarrow \mathbf{A}(\mathcal{E})$  is a morphism of Lie algebras.

Reminder:

- Principal fiber bundles: representation theory is played by associated vector bundles.
- Noncommutative geometry: representation theory is played by modules.

.....  
**Example 2: Atiyah Lie algebroids**

$\mathcal{P} \xrightarrow{\pi} \mathcal{M}$  a  $G$ -principal fiber bundle,  $\mathfrak{g}$  the Lie algebra of  $G$ .  
 $R_g : \mathcal{P} \rightarrow \mathcal{P}$ ,  $R_g(p) = p \cdot g$ , the right action of  $G$  on  $\mathcal{P}$ .

$$\begin{aligned} \Gamma_G(T\mathcal{P}) &= \{\mathfrak{X} \in \Gamma(T\mathcal{P}) / R_{g*}\mathfrak{X} = \mathfrak{X} \text{ for all } g \in G\} \\ \Gamma_G(\mathcal{P}, \mathfrak{g}) &= \{v : \mathcal{P} \rightarrow \mathfrak{g} / v(p \cdot g) = \text{Ad}_{g^{-1}}v(p) \text{ for all } g \in G\} \end{aligned}$$

Both are Lie algebras and  $C^\infty(\mathcal{M})$ -modules.

$\Gamma_G(T\mathcal{P})$  is the space of  $\pi_*$ -projectable vector fields in  $\Gamma(T\mathcal{P}) \xrightarrow{\pi_*} \Gamma(TM) \rightarrow \Gamma(TM)$ .

$$\iota : \Gamma_G(\mathcal{P}, \mathfrak{g}) \rightarrow \Gamma_G(T\mathcal{P}) \quad \text{defined by} \quad \iota(v)|_p = v(p)|_p = \left( \frac{d}{dt} p \cdot \exp(tv(p)) \right)_{|t=0}$$

$\mathfrak{g} \ni v \mapsto v^{\mathcal{P}}$  the fundamental vector field on  $\mathcal{P}$ .

This induces a s.e.s. of Lie algebras and  $C^\infty(\mathcal{M})$ -modules

$$0 \longrightarrow \Gamma_G(\mathcal{P}, \mathfrak{g}) \xrightarrow{\iota} \Gamma_G(T\mathcal{P}) \xrightarrow{\pi_*} \Gamma(TM) \longrightarrow 0.$$

$\Gamma_G(T\mathcal{P})$  is the **Atiyah (transitive) Lie algebroid** associated to  $\mathcal{P}$ .

The representations of  $\Gamma_G(T\mathcal{P})$  are given by the associated vector bundles to  $\mathcal{P}$ .

**Atiyah Lie algebroids permit to embed ordinary gauge theories in this framework.**

.....  
**Example 3: Trivial Lie algebroids**

Trivial Lie algebroid = Atiyah Lie algebroid of a trivial principal bundle  $\mathcal{P} = \mathcal{M} \times G$ .

Compact notation:  $\text{TLA}(\mathcal{M}, \mathfrak{g}) \equiv A = \Gamma(TM \oplus (\mathcal{M} \times \mathfrak{g}))$ .

**Proposition 2.** *Every transitive Lie algebroid  $A$  is locally of the form  $\text{TLA}(\mathcal{U}, \mathfrak{g})$  for  $\mathcal{U} \subset \mathcal{M}$  open subset.*

Trivialization of an Atiyah Lie algebroid  $\leftrightarrow$  Trivialization of the principal fiber bundle.



### 3.3 Differential structures

.....  
**Differential forms: general definition**

$0 \longrightarrow L \xrightarrow{\iota} A \xrightarrow{\rho} \Gamma(T\mathcal{M}) \longrightarrow 0$  a transitive Lie algebroid.

$\phi : A \rightarrow \mathfrak{D}(\mathcal{E})$  a representation of  $A$  on  $\mathcal{E}$ .

- For  $p = 0$ :  $\Omega^0(A, \mathcal{E}) = \Gamma(\mathcal{E})$ .
- For  $p \geq 1$ :  $\Omega^p(A, \mathcal{E})$  the linear space of  $C^\infty(\mathcal{M})$ -multilinear antisymmetric maps from  $A^p$  to  $\Gamma(\mathcal{E})$ .
- $\Omega^\bullet(A, \mathcal{E}) = \bigoplus_{p \geq 0} \Omega^p(A, \mathcal{E})$  is equipped with the differential

$$\begin{aligned} (\widehat{d}_\phi \widehat{\omega})(\mathfrak{X}_1, \dots, \mathfrak{X}_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i+1} \phi(\mathfrak{X}_i) \cdot \widehat{\omega}(\mathfrak{X}_1, \dots, \overset{i}{\checkmark} \dots, \mathfrak{X}_{p+1}) \\ &\quad + \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} \widehat{\omega}([\mathfrak{X}_i, \mathfrak{X}_j], \mathfrak{X}_1, \dots, \overset{i}{\checkmark} \dots \overset{j}{\checkmark} \dots, \mathfrak{X}_{p+1}) \end{aligned}$$

$\phi(\mathfrak{X}) \cdot \varphi$  is the action of the first order diff. op.  $\phi(\mathfrak{X})$  on  $\varphi \in \Gamma(\mathcal{E})$ .

- $\widehat{d}_\phi^2 = 0$  ( $\phi$  is a morphism of Lie algebras).

.....  
**Differential forms: two examples**

Let  $\mathcal{E} = \mathcal{M} \times \mathbb{C}$ , then  $\Gamma(\mathcal{E}) = C^\infty(\mathcal{M})$ .

The anchor map  $\rho$  is a representation of  $A$  on  $C^\infty(\mathcal{M})$ .

**Forms with values in  $C^\infty(\mathcal{M})$ :**  $(\Omega^\bullet(A), \widehat{d}_A)$  is the graded commutative differential algebra of forms on  $A$  with values in  $C^\infty(\mathcal{M})$  associated to the representation  $\rho$ .

$\mathcal{E} = \mathcal{L}$  the vector bundle such that  $L = \Gamma(\mathcal{L})$ .

For  $\mathfrak{X} \in A$  and  $\ell \in L$ , define  $\text{ad}_\mathfrak{X}(\ell) \in L$  such that  $\iota(\text{ad}_\mathfrak{X}(\ell)) = [\mathfrak{X}, \iota(\ell)]$ .

This is the adjoint representation of  $A$  on  $\mathcal{L}$ .

**Forms with values in the kernel:**  $(\Omega^\bullet(A, L), \widehat{d})$  is the graded differential Lie algebra of forms on  $A$  with values in the kernel  $L$  associated to the adjoint representation.

→ graded Lie algebra and graded differential module on  $\Omega^\bullet(A)$ .

.....  
**Differential forms on trivial Lie algebroids**

$A = \text{TLA}(\mathcal{M}, \mathfrak{g})$  a trivial Lie algebroid.

$\Omega^\bullet(A)$  is the total complex of the bigraded commutative algebra  $\Omega^\bullet(\mathcal{M}) \otimes \wedge^\bullet \mathfrak{g}^*$ .  
 $\widehat{d}_A = d + s$  with

$$\begin{aligned} d : \Omega^\bullet(\mathcal{M}) \otimes \wedge^\bullet \mathfrak{g}^* &\rightarrow \Omega^{\bullet+1}(\mathcal{M}) \otimes \wedge^\bullet \mathfrak{g}^* && \text{de Rham differential} \\ s : \Omega^\bullet(\mathcal{M}) \otimes \wedge^\bullet \mathfrak{g}^* &\rightarrow \Omega^\bullet(\mathcal{M}) \otimes \wedge^{\bullet+1} \mathfrak{g}^* && \text{Chevalley-Eilenberg differential} \end{aligned}$$

$\Omega^\bullet(A, L)$  is the total complex of the bigraded Lie algebra  $\Omega^\bullet(\mathcal{M}) \otimes \wedge^\bullet \mathfrak{g}^* \otimes \mathfrak{g}$ .  
 $\widehat{d} = d + s'$  with  $s'$  the Chevalley-Eilenberg differential on  $\wedge^\bullet \mathfrak{g}^* \otimes \mathfrak{g}$  (for the ad rep.).  
Compact notation:  $(\Omega_{\text{TLA}}^\bullet(\mathcal{M}, \mathfrak{g}), \widehat{d}_{\text{TLA}}) = (\Omega^\bullet(A, L), \widehat{d})$ .

They are models for trivializations of forms on any transitive Lie algebroid.

.....  
**Differential forms on Atiyah Lie algebroids**

$A$  the Atiyah Lie algebroid of a  $G$ -principal fiber bundle  $\mathcal{P} \xrightarrow{\pi} \mathcal{M}$ .

Compact notation:  $\Omega_{\text{Lie}}^\bullet(\mathcal{P}, \mathfrak{g}) = \Omega^\bullet(A, L)$ .

$$\mathfrak{g}_{\text{equ}} = \{\xi^{\mathcal{P}} \oplus \xi \mid \xi \in \mathfrak{g}\} \subset \text{TLA}(\mathcal{P}, \mathfrak{g}) = \Gamma(T\mathcal{P} \oplus (\mathcal{P} \times \mathfrak{g}))$$

is a Lie algebra, which defines a Cartan operation on  $(\Omega_{\text{TLA}}^\bullet(\mathcal{P}, \mathfrak{g}), \widehat{d}_{\text{TLA}})$ .

$(\Omega_{\text{TLA}}^\bullet(\mathcal{P}, \mathfrak{g})_{\mathfrak{g}_{\text{equ}}}, \widehat{d}_{\text{TLA}})$  the differential graded subcomplex of basic elements.

**Theorem 3** (S. Lazzarini, T.M.). *If  $G$  is connected and simply connected then  $(\Omega_{\text{Lie}}^\bullet(\mathcal{P}, \mathfrak{g}), \widehat{d})$  is isomorphic to  $(\Omega_{\text{TLA}}^\bullet(\mathcal{P}, \mathfrak{g})_{\mathfrak{g}_{\text{equ}}}, \widehat{d}_{\text{TLA}})$*

$$\rightarrow \Omega_{\text{Lie}}^\bullet(\mathcal{P}, \mathfrak{g}) \subset \Omega_{\text{TLA}}^\bullet(\mathcal{P}, \mathfrak{g}) \simeq \Omega^\bullet(\mathcal{P}) \otimes \wedge^\bullet \mathfrak{g}^* \otimes \mathfrak{g}.$$

When  $G$  is connected and simply connected, a form can be described as:

- a  $\mathfrak{g}_{\text{equ}}$ -basic elements in  $\Omega_{\text{TLA}}^\bullet(\mathcal{P}, \mathfrak{g})$ ;
- a form in  $\Omega_{\text{Lie}}^\bullet(\mathcal{P}, \mathfrak{g})$ ;
- a family of local trivializations in  $\Omega_{\text{TLA}}^\bullet(\mathcal{U}, \mathfrak{g})$  with gluing relations.

### 3.4 Connections

.....  
**Ordinary connections on transitive Lie algebroids**

$0 \longrightarrow L \xrightarrow{\iota} A \xrightarrow{\rho} \Gamma(TM) \longrightarrow 0$  a transitive Lie algebroid.

**Connection on a transitive Lie algebroid:** splitting  $\nabla : \Gamma(TM) \rightarrow A$  as  $C^\infty(\mathcal{M})$ -modules of the s.e.s.

$$0 \longrightarrow L \xleftarrow{\omega^\nabla} A \xleftarrow{\rho} \Gamma(TM) \longrightarrow 0$$

**Curvature:** obstruction to be a morphism of Lie algebras:

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]} \in \iota(L)$$

$$\mathfrak{X} = \nabla_X - \iota \circ \omega^\nabla(\mathfrak{X})$$

- $\omega^\nabla \in \Omega^1(A, L)$  and  $\omega^\nabla \circ \iota(\ell) = -\ell$  for any  $\ell \in L$  (normalization on  $L$ ).
- $R^\nabla(\mathfrak{X}, \mathfrak{Y}) = (\widehat{d}\omega^\nabla)(\mathfrak{X}, \mathfrak{Y}) + [\omega^\nabla(\mathfrak{X}), \omega^\nabla(\mathfrak{Y})]$ ,  $R^\nabla \in \Omega^2(A, L)$ ,  
 $\iota \circ R^\nabla(\mathfrak{X}, \mathfrak{Y}) = R(X, Y)$  (vanishes when  $\mathfrak{X}$  or  $\mathfrak{Y}$  is in  $\iota(L)$ ).

$\omega^\nabla$  is the **connection 1-form** associated to  $\nabla$ .

.....  
**Ordinary connections on Atiyah Lie algebroid**

$$0 \longrightarrow \Gamma_G(\mathcal{P}, \mathfrak{g}) \xrightarrow{\iota} \Gamma_G(T\mathcal{P}) \xrightarrow{\pi^*} \Gamma(TM) \longrightarrow 0$$

**Proposition 4** (Connections). *Ordinary connection on the Atiyah Lie algebroid = connection on  $\mathcal{P}$ .*

*The notions of curvature coincide.*

This example explains the terminology “ordinary connection”.

**The geometric equivalence:** A connection on  $\mathcal{P}$  defines a horizontal lift  $\Gamma(TM) \rightarrow \Gamma_G(T\mathcal{P})$ ,  
 $X \mapsto X^h$ .

**The algebraic equivalence:** Suppose  $G$  is connected and simply connected.

$\omega^\mathcal{P} \in \Omega^1(\mathcal{P}) \otimes \mathfrak{g}$  a connection 1-form on  $\mathcal{P}$ .

$\theta \in \mathfrak{g}^* \otimes \mathfrak{g}$  the Maurer-Cartan 1-form on  $G$ .

$$\widehat{\omega}_{\mathfrak{g}_{\text{equ}}} = \omega^\mathcal{P} - \theta \in \Omega_{\text{TLA}}^1(\mathcal{P}, \mathfrak{g}) \subset \Omega^\bullet(\mathcal{P}) \otimes \wedge^\bullet \mathfrak{g}^* \otimes \mathfrak{g}$$

is  $\mathfrak{g}_{\text{equ}}$ -basic.

➔ It corresponds to the connection 1-form  $\omega^\nabla \in \Omega_{\text{Lie}}^1(\mathcal{P}, \mathfrak{g})$  associated to  $\omega^\mathcal{P}$ .

.....  
**Generalized connections on transitive Lie algebroids**

$A$  a transitive Lie algebroid.

**Generalized connection:** a 1-form  $\widehat{\omega} \in \Omega^1(A, L)$ .

**Curvature:** the 2-form  $\widehat{R} = d\widehat{\omega} + \frac{1}{2}[\widehat{\omega}, \widehat{\omega}] \in \Omega^2(A, L)$ .

An ordinary connection is a generalized connection for which  $\widehat{\omega} \circ \iota = -\text{Id}_L$ .

Consider a representation of  $A$  on  $\mathcal{E}$ :

$$\begin{array}{ccccccc} 0 & \longrightarrow & L & \xleftarrow{\widehat{\omega}} & A & \xrightarrow{\rho} & \Gamma(T\mathcal{M}) \longrightarrow 0 \\ & & \phi_L \downarrow & & \widehat{\nabla} \downarrow \phi & & \parallel \\ 0 & \longrightarrow & \mathbf{A}(\mathcal{E}) & \xrightarrow{\iota} & \mathfrak{D}(\mathcal{E}) & \xrightarrow{\sigma} & \Gamma(T\mathcal{M}) \longrightarrow 0 \end{array}$$

$\widehat{\omega}$  defines a covariant derivative on  $\mathcal{E}$ :

$$\widehat{\nabla}_{\mathfrak{X}}\varphi = \phi(\mathfrak{X}) \cdot \varphi + \phi_L(\widehat{\omega}(\mathfrak{X}))\varphi.$$

$[\widehat{\nabla}_{\mathfrak{X}}, \widehat{\nabla}_{\mathfrak{Y}}] - \widehat{\nabla}_{[\mathfrak{X}, \mathfrak{Y}]} = \iota \circ \phi_L \circ \widehat{R}(\mathfrak{X}, \mathfrak{Y}) \rightarrow \widehat{\nabla}$  is not a representation in general.

.....  
**Generalized connections on Atiyah Lie algebroids**

Suppose  $G$  is connected and simply connected.

$$0 \longrightarrow \Gamma_G(\mathcal{P}, \mathfrak{g}) \xrightarrow{\iota} \Gamma_G(T\mathcal{P}) \xrightarrow{\pi^*} \Gamma(T\mathcal{M}) \longrightarrow 0$$

A connection  $\widehat{\omega}$  on  $\Gamma_G(T\mathcal{P})$  is a  $\mathfrak{g}_{\text{equ}}$ -basic 1-form  $\widehat{\omega}_{\mathfrak{g}_{\text{equ}}} \in \Omega_{\text{TLA}}^1(\mathcal{P}, \mathfrak{g})$ .

$$\widehat{\omega}_{\mathfrak{g}_{\text{equ}}} = \omega + \varphi \in (\Omega^1(\mathcal{P}) \otimes \mathfrak{g}) \oplus (C^\infty(\mathcal{P}) \otimes \mathfrak{g}^* \otimes \mathfrak{g}).$$

If  $\varphi = -\theta$ , then  $\widehat{\omega}$  is an ordinary connection on  $\Gamma_G(T\mathcal{P})$ .

$\rightarrow \omega$  is an (ordinary) connection 1-form on  $\mathcal{P}$ .

Otherwise,  $\varphi + \theta$  measures the deviation of  $\widehat{\omega}$  from an ordinary connection.

$\rightarrow \varphi$  contains new degrees of freedom for gauge field theories.

### 3.5 Gauge transformations

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**Gauge group of a representation**

$\mathcal{E}$  a representation space of the transitive Lie algebroid  $A$ :

$$\begin{array}{ccccccc}
 0 & \longrightarrow & L & \xrightarrow{\iota} & A & \xrightarrow{\rho} & \Gamma(TM) \longrightarrow 0 \\
 & & \phi_L \downarrow & & \downarrow \phi & & \parallel \\
 0 & \longrightarrow & \mathbf{A}(\mathcal{E}) & \xrightarrow{\iota} & \mathfrak{D}(\mathcal{E}) & \xrightarrow{\sigma} & \Gamma(TM) \longrightarrow 0
 \end{array}$$

The **gauge group of  $\mathcal{E}$**  is the group  $\text{Aut}(\mathcal{E})$  (vertical automorphisms of  $\mathcal{E}$ ).

$\text{Aut}(\mathcal{E}) \subset \mathbf{A}(\mathcal{E})$ ,

$\mathbf{A}(\mathcal{E})$  are the infinitesimal gauge transformations on  $\mathcal{E}$ .

→  $\phi_L(\xi)$  in an infinitesimal gauge transformation for any  $\xi \in L$ .

.....  
**Infinitesimal gauge transformations**

**Infinitesimal gauge transformation:** any element  $\xi \in L$ .

→ No notion of finite gauge transformation at the level of  $A$  ( $\simeq$  NCG).

**Gauge transformation of connection:**  $\widehat{d}\xi + [\widehat{\omega}, \xi]$ .

**Gauge transformation of curvature:**  $[\widehat{R}, \xi]$ .

→ The (local) gauge principle is implemented at the infinitesimal level.

→ BRST-like.

**Gauge transformations on Atiyah Lie algebroids:** The infinitesimal gauge transformations on generalized connections on  $\Gamma_G(T\mathcal{P})$  are the (ordinary) infinitesimal gauge transformations on  $\mathcal{P}$ .

→ Here there is a notion of gauge group...

.....  
**Assets and problems**  
**of gauge field theories on Lie algebroids**

- The mathematical structures are (very) close to ordinary geometry.
  - ➔ Natural extension of the ordinary differential geometry of gauge fields.
- The gauge theories are of Yang-Mills-Higgs type.
  - ➔ We know where the Yang-Mills theories are:  
 $\omega - \theta$  as a special case of  $\omega + \varphi \dots$
- All Lie groups are accessible (Atiyah Lie algebroids).
- BRST-like differential structures.
  - ➔ Work in progress to understand if this is only a coincidence. . .
- New framework: require some work to construct realistic models.

# Conclusion

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## Before the conclusion: a unifying theorem

$SL(n)$ -principal bundle  $\mathcal{P}$  over  $\mathcal{M}$ .

$\mathcal{E}$  associated vector bundle with fiber  $\mathbb{C}^n$ .

$\mathbf{A} = \Gamma(\text{End}(\mathcal{E}))$  the algebra of endomorphisms of  $\mathcal{E}$ .

The short exact sequence

$$0 \longrightarrow \text{Int}(\mathbf{A}) \xrightarrow{\text{ad}} \text{Der}(\mathbf{A}) \xrightarrow{\rho} \Gamma(T\mathcal{M}) \longrightarrow 0$$

defines  $\text{Der}(\mathbf{A})$  as a transitive Lie algebroid over  $\mathcal{M}$ , with  $\iota = \text{ad}$ .

**Theorem 5** (S. Lazzarini, T.M.). *The following three spaces are isomorphic:*

1. *The space of generalized connections on  $\Gamma_G(T\mathcal{P})$ .*
2. *The space of generalized connections on  $\text{Der}(\mathbf{A})$ .*
3. *The space of traceless n.c. connections on the right  $\mathbf{A}$ -module  $\mathbf{M} = \mathbf{A}$*

*These isomorphisms are compatible with curvatures and gauge transformations. All these spaces contain ordinary (Yang-Mills) connections on  $\mathcal{P}$ .*

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## Conclusion

- Gauge field theories can be generalized in at least two directions:
  - noncommutative geometry,
  - transitive Lie algebroids.

➔ These generalizations can coincide in specific examples.
- Common feature: add some purely algebraic directions to space-time.
- We naturally get Yang-Mills-Higgs type gauge theories in both situations.
  - ➔ They contain ordinary Yang-Mills gauge theories used in physics.
- A pattern for (realistic) gauge field theories:

