Gauge field theories: a comparison of various mathematical approaches

(Toward a pattern…)

Paris, February 12, 2014

Thierry Masson

Centre de Physique Théorique
Campus de Luminy, Marseille

How to construct a gauge field theory?

The basic ingredients are:

1. A space of local symmetries (space-time dependence): a **gauge group**.
2. An implementation of the symmetry on matter fields: a **representation theory**.
3. A notion of derivation: some **differential structures**.
4. A (gauge compatible) replacement of ordinary derivations: **covariant derivative**.
5. A way to write a gauge invariant Lagrangian density: **action functional**.

How to construct a gauge field theory?

The basic ingredients are:

1. A space of local symmetries (space-time dependence): a **gauge group**.
2. An implementation of the symmetry on matter fields: a **representation theory**.
3. A notion of derivation: some **differential structures**.
4. A (gauge compatible) replacement of ordinary derivations: **covariant derivative**.
5. A way to write a gauge invariant Lagrangian density: **action functional**.

Presentation of three mathematical schemes to construct gauge field theories:

- Ordinary differential geometry of principal fiber bundles.
- Noncommutative geometry.
- Transitive Lie algebroids.

How to construct a gauge field theory?

The basic ingredients are:

1. A space of local symmetries (space-time dependence): a **gauge group**.
2. An implementation of the symmetry on matter fields: a **representation theory**.
3. A notion of derivation: some **differential structures**.
4. A (gauge compatible) replacement of ordinary derivations: **covariant derivative**.
5. A way to write a gauge invariant Lagrangian density: **action functional**.

Presentation of three mathematical schemes to construct gauge field theories:

- Ordinary differential geometry of principal fiber bundles.
- Noncommutative geometry.
- Transitive Lie algebroids.

Point 5 will not be covered…

Ordinary differential geometry

1. Ordinary differential geometry
2. Noncommutative geometry
3. Transitive Lie algebroids

Connections in differential geometry

Connections in differential geometry

Let $\mathcal{P} \xrightarrow{\pi} \mathcal{M}$ be a G-principal fiber bundle and $\mathfrak{g}$ the Lie algebra of G.
R_g the right action of $g \in G$ on $\mathcal{P}$

Connection on $\mathcal{P}$: 1-form $\omega \in \Omega^1(\mathcal{P}) \otimes \mathfrak{g}$ such that:

$$\omega(\xi^{\mathcal{P}}) = \xi, \quad \forall \xi \in \mathfrak{g}, \qquad\qquad R_g^* \omega = \mathrm{Ad}_{g^{-1}} \omega, \quad \forall g \in G$$

Curvature: $\Omega = d\omega + \frac{1}{2}[\omega, \omega] \in \Omega^2(\mathcal{P}) \otimes \mathfrak{g}$.

Transfer this connection on any associated vector bundle $\mathcal{E}$:

Covariant derivative: $X \in \Gamma(T\mathcal{M}) \longrightarrow \nabla_X : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ (smooth sections of $\mathcal{E}$).

$$\nabla_X(f\varepsilon) = (X{\cdot}f)\varepsilon + f\nabla_X \varepsilon, \quad \text{and other relations...}$$

➞ This is the minimal coupling in physics.

Three levels of description

Three levels of description

Globally on $\mathcal{P}$: $\omega \in \Omega^1(\mathcal{P}) \otimes \mathfrak{g}$ connection 1-form on $\mathcal{P}$, Ω its curvature.

$\omega \in \Omega^1(\mathcal{P}) \otimes \mathfrak{g}$, equivariant + vertical condition.

$\Omega \in \Omega^2(\mathcal{P}) \otimes \mathfrak{g}$, equivariant + horizontal.

Three levels of description

Globally on $\mathcal{P}$: $\omega \in \Omega^1(\mathcal{P}) \otimes \mathfrak{g}$ connection 1-form on $\mathcal{P}$, Ω its curvature.

$\omega \in \Omega^1(\mathcal{P}) \otimes \mathfrak{g}$, equivariant + vertical condition.

$\Omega \in \Omega^2(\mathcal{P}) \otimes \mathfrak{g}$, equivariant + horizontal.

➡ **Preferred description for mathematicians...**

Three levels of description

Globally on $\mathcal{P}$: $\omega \in \Omega^1(\mathcal{P}) \otimes \mathfrak{g}$ connection 1-form on $\mathcal{P}$, Ω its curvature.
$\omega \in \Omega^1(\mathcal{P}) \otimes \mathfrak{g}$, equivariant + vertical condition.
$\Omega \in \Omega^2(\mathcal{P}) \otimes \mathfrak{g}$, equivariant + horizontal.

➞ **Preferred description for mathematicians…**

Locally on $\mathcal{M}$: (U, ϕ) local trivialisation of $\mathcal{P}$ with $\phi : U \times G \to \mathcal{P}_{|U}$,
s its local section, $s(x) = \phi(x, e)$.
Local expressions of the connection and the curvature:

$$A = s^*\omega \in \Omega^1(U) \otimes \mathfrak{g} \qquad\qquad F = s^*\Omega \in \Omega^2(U) \otimes \mathfrak{g}$$

(U_i, ϕ_i) and (U_j, ϕ_j) two local trivialisations. $g_{ij} : U_i \cap U_j \neq \varnothing \to G$ transition functions:

$$A_j = g_{ij}^{-1} A_i g_{ij} + g_{ij}^{-1} dg_{ij} \qquad\qquad F_j = g_{ij}^{-1} F_i g_{ij}$$

Three levels of description

Globally on $\mathcal{P}$: $\omega \in \Omega^1(\mathcal{P}) \otimes \mathfrak{g}$ connection 1-form on $\mathcal{P}$, Ω its curvature.

$\omega \in \Omega^1(\mathcal{P}) \otimes \mathfrak{g}$, equivariant + vertical condition.

$\Omega \in \Omega^2(\mathcal{P}) \otimes \mathfrak{g}$, equivariant + horizontal.

➞ **Preferred description for mathematicians...**

Locally on $\mathcal{M}$: (U, ϕ) local trivialisation of $\mathcal{P}$ with $\phi : U \times G \to \mathcal{P}_{|U}$,

s its local section, $s(x) = \phi(x, e)$.

Local expressions of the connection and the curvature:

$$A = s^*\omega \in \Omega^1(U) \otimes \mathfrak{g} \qquad\qquad F = s^*\Omega \in \Omega^2(U) \otimes \mathfrak{g}$$

(U_i, ϕ_i) and (U_j, ϕ_j) two local trivialisations. $g_{ij} : U_i \cap U_j \neq \varnothing \to G$ transition functions:

$$A_j = g_{ij}^{-1} A_i g_{ij} + g_{ij}^{-1} dg_{ij} \qquad\qquad F_j = g_{ij}^{-1} F_i g_{ij}$$

➞ **Preferred description for physicists...**

Three levels of description

Globally on $\mathcal{P}$: $\omega \in \Omega^1(\mathcal{P}) \otimes \mathfrak{g}$ connection 1-form on $\mathcal{P}$, Ω its curvature.
$\omega \in \Omega^1(\mathcal{P}) \otimes \mathfrak{g}$, equivariant + vertical condition.
$\Omega \in \Omega^2(\mathcal{P}) \otimes \mathfrak{g}$, equivariant + horizontal.

➞ **Preferred description for mathematicians…**

Locally on $\mathcal{M}$: (U, ϕ) local trivialisation of $\mathcal{P}$ with $\phi : U \times G \to \mathcal{P}_{|U}$,
s its local section, $s(x) = \phi(x, e)$.
Local expressions of the connection and the curvature:

$$A = s^*\omega \in \Omega^1(U) \otimes \mathfrak{g} \qquad\qquad F = s^*\Omega \in \Omega^2(U) \otimes \mathfrak{g}$$

(U_i, ϕ_i) and (U_j, ϕ_j) two local trivialisations. $g_{ij} : U_i \cap U_j \neq \varnothing \to G$ transition functions:

$$A_j = g_{ij}^{-1} A_i g_{ij} + g_{ij}^{-1} dg_{ij} \qquad\qquad F_j = g_{ij}^{-1} F_i g_{ij}$$

➞ **Preferred description for physicists…**

Globally on $\mathcal{M}$: The family $\{F_i\}_i$ with its homogeneous gluing relations, or Ω equivariant and horizontal, defines a section $\mathbb{F}$ of $\bigwedge^2 T^*\mathcal{M} \otimes \mathrm{Ad}\mathcal{P}$.

Three levels of description

Globally on $\mathcal{P}$: $\omega \in \Omega^1(\mathcal{P}) \otimes \mathfrak{g}$ connection 1-form on $\mathcal{P}$, Ω its curvature.
$\omega \in \Omega^1(\mathcal{P}) \otimes \mathfrak{g}$, equivariant + vertical condition.
$\Omega \in \Omega^2(\mathcal{P}) \otimes \mathfrak{g}$, equivariant + horizontal.

➞ **Preferred description for mathematicians...**

Locally on $\mathcal{M}$: (U, ϕ) local trivialisation of $\mathcal{P}$ with $\phi : U \times G \to \mathcal{P}_{|U}$,
s its local section, $s(x) = \phi(x, e)$.
Local expressions of the connection and the curvature:

$$A = s^*\omega \in \Omega^1(U) \otimes \mathfrak{g} \qquad\qquad F = s^*\Omega \in \Omega^2(U) \otimes \mathfrak{g}$$

(U_i, ϕ_i) and (U_j, ϕ_j) two local trivialisations. $g_{ij} : U_i \cap U_j \neq \varnothing \to G$ transition functions:

$$A_j = g_{ij}^{-1} A_i g_{ij} + g_{ij}^{-1} \mathrm{d} g_{ij} \qquad\qquad F_j = g_{ij}^{-1} F_i g_{ij}$$

➞ **Preferred description for physicists...**

Globally on $\mathcal{M}$: The family $\{F_i\}_i$ with its homogeneous gluing relations, or Ω equivariant and horizontal, defines a section $\mathbb{F}$ of $\bigwedge^2 T^*\mathcal{M} \otimes \mathrm{Ad}\mathcal{P}$.
The family $\{A_i\}_i$ is not associated to an algebraic object globally defined on $\mathcal{M}$.

Three levels of description

Globally on $\mathcal{P}$: $\omega \in \Omega^1(\mathcal{P}) \otimes \mathfrak{g}$ connection 1-form on $\mathcal{P}$, Ω its curvature.
$\omega \in \Omega^1(\mathcal{P}) \otimes \mathfrak{g}$, equivariant + vertical condition.
$\Omega \in \Omega^2(\mathcal{P}) \otimes \mathfrak{g}$, equivariant + horizontal.

➙ **Preferred description for mathematicians…**

Locally on $\mathcal{M}$: (U, ϕ) local trivialisation of $\mathcal{P}$ with $\phi : U \times G \to \mathcal{P}_{|U}$,
s its local section, $s(x) = \phi(x, e)$.
Local expressions of the connection and the curvature:

$$A = s^*\omega \in \Omega^1(U) \otimes \mathfrak{g} \qquad\qquad F = s^*\Omega \in \Omega^2(U) \otimes \mathfrak{g}$$

(U_i, ϕ_i) and (U_j, ϕ_j) two local trivialisations. $g_{ij} : U_i \cap U_j \neq \varnothing \to G$ transition functions:

$$A_j = g_{ij}^{-1} A_i g_{ij} + g_{ij}^{-1} \mathrm{d} g_{ij} \qquad\qquad F_j = g_{ij}^{-1} F_i g_{ij}$$

➙ **Preferred description for physicists…**

Globally on $\mathcal{M}$: The family $\{F_i\}_i$ with its homogeneous gluing relations, or Ω equivariant and horizontal, defines a section $\mathbb{F}$ of $\bigwedge^2 T^*\mathcal{M} \otimes \mathrm{Ad}\mathcal{P}$.
The family $\{A_i\}_i$ is not associated to an algebraic object globally defined on $\mathcal{M}$.

➙ **Preferred description for me…**

Covariant derivatives and algebraic structures

$\mathcal{M}$ smooth manifold, $(\Omega^\bullet(\mathcal{M}), \mathrm{d})$ de Rham differential calculus on $\mathcal{M}$.
$\mathcal{E}$ a vector bundle on $\mathcal{M}$.
$\boldsymbol{M} = \Gamma(\mathcal{E})$ the space of smooth sections of $\mathcal{E}$.

Covariant derivative on $\mathcal{E}$: linear map

$$\nabla : \boldsymbol{M} \to \Omega^1(\mathcal{M}) \otimes_{C^\infty(\mathcal{M})} \boldsymbol{M} \qquad \text{such that} \qquad \nabla(f\varepsilon) = \mathrm{d}f \otimes \varepsilon + f\nabla\varepsilon$$

for any $\varepsilon \in \boldsymbol{M}$ and $f \in C^\infty(\mathcal{M})$.

Covariant derivatives and algebraic structures

$\mathcal{M}$ smooth manifold, $(\Omega^\bullet(\mathcal{M}), \mathrm{d})$ de Rham differential calculus on $\mathcal{M}$.
$\mathcal{E}$ a vector bundle on $\mathcal{M}$.
$\boldsymbol{M} = \Gamma(\mathcal{E})$ the space of smooth sections of $\mathcal{E}$.

Covariant derivative on $\mathcal{E}$: linear map

$$\nabla : \boldsymbol{M} \to \Omega^1(\mathcal{M}) \otimes_{C^\infty(\mathcal{M})} \boldsymbol{M} \qquad \text{such that} \qquad \nabla(f\varepsilon) = \mathrm{d}f \otimes \varepsilon + f\nabla\varepsilon$$

for any $\varepsilon \in \boldsymbol{M}$ and $f \in C^\infty(\mathcal{M})$.

Natural extension:

$$\nabla : \Omega^\bullet(\mathcal{M}) \otimes_{C^\infty(\mathcal{M})} \boldsymbol{M} \to \Omega^{\bullet+1}(\mathcal{M}) \otimes_{C^\infty(\mathcal{M})} \boldsymbol{M}$$

defined by

$$\nabla(\eta \otimes \varepsilon) = \mathrm{d}\eta \otimes \varepsilon + (-1)^r \eta \wedge \nabla\varepsilon \qquad \text{for any } \eta \in \Omega^r(\mathcal{M}).$$

Curvature: the $C^\infty(\mathcal{M})$-linear map

$$\nabla^2 : \boldsymbol{M} \to \Omega^2(\mathcal{M}) \otimes_{C^\infty(\mathcal{M})} \boldsymbol{M}$$

➞ 2-form $\mathbb{F}$ on $\mathcal{M}$ with values in $\mathrm{Ad}\mathcal{P} \subset \mathrm{End}(\mathcal{E}) = \mathcal{E}^* \otimes \mathcal{E}$ (modulo rep.).

Covariant derivatives and algebraic structures

$\mathcal{M}$ smooth manifold, $(\Omega^\bullet(\mathcal{M}), \mathrm{d})$ de Rham differential calculus on $\mathcal{M}$.
$\mathcal{E}$ a vector bundle on $\mathcal{M}$.
$\boldsymbol{M} = \Gamma(\mathcal{E})$ the space of smooth sections of $\mathcal{E}$.

Covariant derivative on $\mathcal{E}$: linear map

$$\nabla : \boldsymbol{M} \to \Omega^1(\mathcal{M}) \otimes_{C^\infty(\mathcal{M})} \boldsymbol{M} \qquad \text{such that} \qquad \nabla(f\varepsilon) = \mathrm{d}f \otimes \varepsilon + f\nabla\varepsilon$$

for any $\varepsilon \in \boldsymbol{M}$ and $f \in C^\infty(\mathcal{M})$.

Natural extension:

$$\nabla : \Omega^\bullet(\mathcal{M}) \otimes_{C^\infty(\mathcal{M})} \boldsymbol{M} \to \Omega^{\bullet+1}(\mathcal{M}) \otimes_{C^\infty(\mathcal{M})} \boldsymbol{M}$$

defined by

$$\nabla(\eta \otimes \varepsilon) = \mathrm{d}\eta \otimes \varepsilon + (-1)^r \eta \wedge \nabla\varepsilon \qquad \text{for any } \eta \in \Omega^r(\mathcal{M}).$$

Curvature: the $C^\infty(\mathcal{M})$-linear map

$$\nabla^2 : \boldsymbol{M} \to \Omega^2(\mathcal{M}) \otimes_{C^\infty(\mathcal{M})} \boldsymbol{M}$$

➞ 2-form $\mathbb{F}$ on $\mathcal{M}$ with values in $\mathrm{Ad}\mathcal{P} \subset \mathrm{End}(\mathcal{E}) = \mathcal{E}^* \otimes \mathcal{E}$ (modulo rep.).

➞ **This can be generalized. . .**

Assets and problems of ordinary gauge field theories

- They are the prototype of theories used in the standard model of particle physics.
- The mathematical structures are now accustomed and popular.
- The gauge theories obtained are massless gauge theories.
  - ➞ Require the symmetry breaking mechanism to generate masses in the SM.
  - ➞ Higgs boson discovered: not a bad theoretical idea after all…

Noncommutative geometry

1. Ordinary differential geometry
2. **Noncommutative geometry**
   - Derivation-based differential calculus
   - Noncommutative connections and their properties
   - The endomorphism algebra of a vector bundle
3. Transitive Lie algebroids

The main noncommutative concepts

- Study geometric structures using equivalent algebraic structures.
- Replace commutative algebras of functions by noncommutative algebras.
- There is not a unique way to generalize key geometrical structures.
  ➞ Several approaches to NCG:
  - Spectral triple (Connes' approach),
  - Quantum groups and covariant differential calculi,
  - Derivation-based noncommutative differential calculus (Dubois-Violette)…
    ➞ The one used in the following.

Derivations of an associative algebra

$\mathbf{A}$ associative algebra with unit.
$\mathcal{Z}(\mathbf{A})$ center of $\mathbf{A}$: commutative subalgebra.

Vector space of derivations of A:

$$\mathrm{Der}(\mathbf{A}) = \{\mathfrak{X} : \mathbf{A} \to \mathbf{A} \;/\; \mathfrak{X} \text{ linear}, \mathfrak{X}(ab) = \mathfrak{X}(a)b + a\mathfrak{X}(b), \forall a, b \in \mathbf{A}\}$$

Structure of Der(A): One has

- $\mathrm{Der}(\mathbf{A})$ is a Lie algebra for the bracket $[\mathfrak{X}, \mathfrak{Y}]a = \mathfrak{X}\mathfrak{Y}a - \mathfrak{Y}\mathfrak{X}a$,
- $\mathrm{Der}(\mathbf{A})$: $\mathcal{Z}(\mathbf{A})$-module for the product $(f\mathfrak{X})a = f(\mathfrak{X}a)$,
- $\mathrm{Int}(\mathbf{A}) = \{\mathrm{ad}_a : b \mapsto [a, b] \;/\; a \in \mathbf{A}\} \subset \mathrm{Der}(\mathbf{A})$, inner derivations:
  Lie ideal and $\mathcal{Z}(\mathbf{A})$-submodule,
- $\mathrm{Out}(\mathbf{A}) = \mathrm{Der}(\mathbf{A})/\mathrm{Int}(\mathbf{A})$ ➨ s.e.s. of Lie algebras and $\mathcal{Z}(\mathbf{A})$-modules

$$0 \longrightarrow \mathrm{Int}(\mathbf{A}) \longrightarrow \mathrm{Der}(\mathbf{A}) \longrightarrow \mathrm{Out}(\mathbf{A}) \longrightarrow 0$$

Derivation-based differential calculus

A associative algebra with unit.

- $\underline{\Omega}^\bullet_{\text{Der}}(\mathbf{A})$ the space of $\mathcal{Z}(\mathbf{A})$-multilinear antisymmetric maps from $\text{Der}(\mathbf{A})^n$ to **A**, with $\underline{\Omega}^0_{\text{Der}}(\mathbf{A}) = \mathbf{A}$.

$$\underline{\Omega}^\bullet_{\text{Der}}(\mathbf{A}) = \bigoplus_{n\geq 0} \underline{\Omega}^n_{\text{Der}}(\mathbf{A})$$

- $\mathbb{N}$**-graded algebra** for the product

$$(\omega\eta)(\mathfrak{X}_1, \dots, \mathfrak{X}_{p+q}) = \frac{1}{p!q!} \sum_{\sigma\in\mathfrak{S}_{p+q}} (-1)^{\text{sign}(\sigma)} \omega(\mathfrak{X}_{\sigma(1)}, \dots, \mathfrak{X}_{\sigma(p)}) \eta(\mathfrak{X}_{\sigma(p+1)}, \dots, \mathfrak{X}_{\sigma(p+q)})$$

- **differential graded algebra** for the differential d defined by

$$\text{d}\omega(\mathfrak{X}_1, \dots, \mathfrak{X}_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} \mathfrak{X}_i \omega(\mathfrak{X}_1, \dots \overset{i}{\vee} \dots, \mathfrak{X}_{n+1}) + \sum_{1\leq i<j\leq n+1} (-1)^{i+j} \omega([\mathfrak{X}_i, \mathfrak{X}_j], \dots \overset{i}{\vee} \dots \overset{j}{\vee} \dots, \mathfrak{X}_{n+1})$$

Noncommutative connections

M a right **A**-module (representation space).

Noncommutative connection: linear map $\widehat{\nabla}_{\mathfrak{X}} : \boldsymbol{M} \to \boldsymbol{M}$, defined for any $\mathfrak{X} \in \mathrm{Der}(\mathbf{A})$, such that $\forall \mathfrak{X}, \mathfrak{Y} \in \mathrm{Der}(\mathbf{A}), \forall a \in \mathbf{A}, \forall m \in \boldsymbol{M}, \forall f \in \mathcal{Z}(\mathbf{A})$:

$$
\begin{aligned}
\widehat{\nabla}_{\mathfrak{X}}(ma) &= m(\mathfrak{X}a) + (\widehat{\nabla}_{\mathfrak{X}} m)a, \\
\widehat{\nabla}_{\mathfrak{X}+\mathfrak{Y}} m &= \widehat{\nabla}_{\mathfrak{X}} m + \widehat{\nabla}_{\mathfrak{Y}} m, \\
\widehat{\nabla}_{f\mathfrak{X}} m &= (\widehat{\nabla}_{\mathfrak{X}} m) f
\end{aligned}
$$

Curvature: linear map $\widehat{R}(\mathfrak{X}, \mathfrak{Y}) : \boldsymbol{M} \to \boldsymbol{M}$ defined for any $\mathfrak{X}, \mathfrak{Y} \in \mathrm{Der}(\mathbf{A})$ by

$$\widehat{R}(\mathfrak{X}, \mathfrak{Y})m = [\widehat{\nabla}_{\mathfrak{X}}, \widehat{\nabla}_{\mathfrak{Y}}]m - \widehat{\nabla}_{[\mathfrak{X},\mathfrak{Y}]} m$$

Gauge transformations, Hermitean structure

$\boldsymbol{M}$ a right $\mathbf{A}$-module.

Gauge group of M:

$\mathcal{G} = \mathrm{Aut}(\boldsymbol{M})$, the group of automorphisms of $\boldsymbol{M}$ as a right $\mathbf{A}$-module.

Gauge transformations: For any $\Phi \in \mathcal{G}$ and any n.c. connection $\widehat{\nabla}$,

$$\widehat{\nabla}^{\Phi}_{\mathfrak{X}} = \Phi^{-1} \circ \widehat{\nabla}_{\mathfrak{X}} \circ \Phi : \boldsymbol{M} \to \boldsymbol{M}$$

is a noncommutative connection.

➞ action of $\mathcal{G}$ on the space of noncommutative connections.

Gauge transformations, Hermitean structure

$\boldsymbol{M}$ a right $\mathbf{A}$-module.

Gauge group of M:

$\mathcal{G} = \mathrm{Aut}(\boldsymbol{M})$, the group of automorphisms of $\boldsymbol{M}$ as a right $\mathbf{A}$-module.

Gauge transformations: For any $\Phi \in \mathcal{G}$ and any n.c. connection $\widehat{\nabla}$,

$$\widehat{\nabla}^{\Phi}_{\mathfrak{X}} = \Phi^{-1} \circ \widehat{\nabla}_{\mathfrak{X}} \circ \Phi : \boldsymbol{M} \to \boldsymbol{M}$$

is a noncommutative connection.

➡ action of $\mathcal{G}$ on the space of noncommutative connections.

When $\mathbf{A}$ is an involutive algebra, notion of **Hermitean structure**:

- $\langle -, - \rangle : \boldsymbol{M} \times \boldsymbol{M} \to \mathbf{A}$,
- Hermitean n.c. connections,
- Unitary gauge transformations.

Noncommutative connections for $M = \mathbf{A}$

Special case: the right $\mathbf{A}$-module $M = \mathbf{A}$ and $\mathbf{A}$ has a unit $\mathbb{1}$.

$\widehat{\nabla}_{\mathfrak{X}} : \mathbf{A} \to \mathbf{A}$ be a n.c. connection.

- $\widehat{\nabla}$ completely given by $\widehat{\nabla}_{\mathfrak{X}} \mathbb{1} = \omega(\mathfrak{X})$, with $\omega \in \underline{\Omega}^1_{\mathrm{Der}}(\mathbf{A})$:
$$\widehat{\nabla}_{\mathfrak{X}} a = \mathfrak{X} a + \omega(\mathfrak{X}) a$$
- Curvature of $\widehat{\nabla}$: multiplication on the left on $\mathbf{A}$ by the n.c. 2-form
$$\Omega(\mathfrak{X}, \mathfrak{Y}) = d\omega(\mathfrak{X}, \mathfrak{Y}) + [\omega(\mathfrak{X}), \omega(\mathfrak{Y})]$$
- Gauge group: identified with invertible elements $g \in \mathbf{A}$ by $\Phi_g(a) = ga$.
- Gauge transformations on $\widehat{\nabla}$:
$$\omega \mapsto \omega^g = g^{-1} \omega g + g^{-1} dg \qquad \Omega \mapsto \Omega^g = g^{-1} \Omega g$$
- $\widehat{\nabla}^0_{\mathfrak{X}} = \mathfrak{X} a$ is a n.c. connection on $\mathbf{A}$ with vanishing curvature, $\omega = 0$.

➡ Replace $\widehat{\nabla}$ by an algebraic object: **connection 1-form**.

Noncommutative connections for $M = \mathbf{A}$

Special case: the right $\mathbf{A}$-module $M = \mathbf{A}$ and $\mathbf{A}$ has a unit $\mathbb{1}$.

$\widehat{\nabla}_{\mathfrak{X}} : \mathbf{A} \to \mathbf{A}$ be a n.c. connection.

- $\widehat{\nabla}$ completely given by $\widehat{\nabla}_{\mathfrak{X}}\mathbb{1} = \omega(\mathfrak{X})$, with $\omega \in \underline{\Omega}^1_{\text{Der}}(\mathbf{A})$:
$$\widehat{\nabla}_{\mathfrak{X}} a = \mathfrak{X}a + \omega(\mathfrak{X})a$$
- Curvature of $\widehat{\nabla}$: multiplication on the left on $\mathbf{A}$ by the n.c. 2-form
$$\Omega(\mathfrak{X}, \mathfrak{Y}) = d\omega(\mathfrak{X}, \mathfrak{Y}) + [\omega(\mathfrak{X}), \omega(\mathfrak{Y})]$$
- Gauge group: identified with invertible elements $g \in \mathbf{A}$ by $\Phi_g(a) = ga$.
- Gauge transformations on $\widehat{\nabla}$:
$$\omega \mapsto \omega^g = g^{-1}\omega g + g^{-1}dg \qquad \Omega \mapsto \Omega^g = g^{-1}\Omega g$$
- $\widehat{\nabla}^0_{\mathfrak{X}} = \mathfrak{X}a$ is a n.c. connection on $\mathbf{A}$ with vanishing curvature, $\omega = 0$.

➡ Replace $\widehat{\nabla}$ by an algebraic object: **connection 1-form**.

If $\mathbf{A}$ is involutive, hermitean structure: $\langle a, b \rangle = a^* b$.
$U(\mathbf{A}) = \{u \in \mathbf{A} \,/\, u^*u = uu^* = \mathbb{1}\}$ is the unitary gauge group.

$\mathbf{A} = C^\infty(\mathcal{M})$

$\mathbf{A} = C^\infty(\mathcal{M})$ (complex valued), for a smooth compact manifold $\mathcal{M}$.
Involutive algebra for the complex conjugation.

- $\mathcal{Z}(\mathbf{A}) = C^\infty(\mathcal{M})$.
- $\mathrm{Der}(\mathbf{A}) = \Gamma(T\mathcal{M})$ (complex vector fields on $\mathcal{M}$).
- $\mathrm{Int}(\mathbf{A}) = 0$.
- $\mathrm{Out}(\mathbf{A}) = \Gamma(T\mathcal{M})$.
- $\underline{\Omega}^\bullet_{\mathrm{Der}}(\mathbf{A}) = \Omega^\bullet(\mathcal{M})$, de Rham forms on $\mathcal{M}$.
- N.C. connections are ordinary connections.

$\mathbf{A} = M_n(\mathbb{C})$

$\mathbf{A} = M_n(\mathbb{C}) = M_n$, finite dimensional algebra of $n \times n$ complex matrices.
Involutive algebra for the adjointness of matrices.

- $\mathcal{Z}(M_n) = \mathbb{C}$.
- $\text{Der}(M_n) = \text{Int}(M_n) \simeq \mathfrak{sl}_n = \mathfrak{sl}(n, \mathbb{C})$ (traceless matrices)
  $$\mathfrak{sl}_n(\mathbb{C}) \ni \gamma \mapsto \text{ad}_\gamma \in \text{Int}(M_n).$$
- $\text{Out}(M_n) = 0$.
- $\underline{\Omega}^\bullet_{\text{Der}}(M_n) \simeq M_n \otimes \bigwedge^\bullet \mathfrak{sl}_n^*$,
  differential $\text{d}' =$ Chevalley-Eilenberg for $\mathfrak{sl}_n$ represented on M_n by ad.
- Canonical 1-form $i\theta \in \underline{\Omega}^1_{\text{Der}}(M_n)$ such that for any $\gamma \in M_n(\mathbb{C})$
  $$i\theta(\text{ad}_\gamma) = \gamma - \tfrac{1}{n}\,\text{Tr}(\gamma)\mathbb{1}$$
- $\widehat{\nabla}^0_{\mathfrak{X}} a = \mathfrak{X}a - i\theta(\mathfrak{X})a$ is a gauge invariant connection on $\boldsymbol{M} = \mathbf{A}$ with zero curvature.
  ➡ Important role in gauge field theories…

$\mathbf{A} = C^\infty(\mathcal{M}) \otimes M_n(\mathbb{C})$

Mix of previous examples: $\mathbf{A} = C^\infty(\mathcal{M}) \otimes M_n(\mathbb{C})$, matrix valued functions on $\mathcal{M}$.

- $\mathcal{Z}(\mathbf{A}) = C^\infty(\mathcal{M})$.
- $\mathrm{Der}(\mathbf{A}) = [\mathrm{Der}(C^\infty(\mathcal{M})) \otimes \mathbb{1}] \oplus [C^\infty(\mathcal{M}) \otimes \mathrm{Der}(M_n)]$
  $= \Gamma(T\mathcal{M}) \oplus [C^\infty(\mathcal{M}) \otimes \mathfrak{sl}_n]$ as Lie algebras and $C^\infty(\mathcal{M})$-modules.
  $\mathfrak{X} = X + \mathrm{ad}_\gamma$ for $X \in \Gamma(T\mathcal{M})$ and $\gamma \in C^\infty(\mathcal{M}) \otimes \mathfrak{sl}_n = \mathbf{A}_0$ (traceless elements in $\mathbf{A}$).
- $\mathrm{Int}(\mathbf{A}) = \mathbf{A}_0$ and $\mathrm{Out}(\mathbf{A}) = \Gamma(T\mathcal{M})$.
- $\underline{\Omega}^\bullet_{\mathrm{Der}}(\mathbf{A}) = \Omega^\bullet(\mathcal{M}) \otimes \underline{\Omega}^\bullet_{\mathrm{Der}}(M_n)$ with $\widehat{\mathrm{d}} = \mathrm{d} + \mathrm{d}'$.
- N.C. 1-form $i\theta$ defined as $i\theta(X + \mathrm{ad}_\gamma) = \gamma \in \mathbf{A}_0 \subset \mathbf{A}$.
  It splits the short exact sequence

$$0 \longrightarrow \mathbf{A}_0 \underset{\overset{i\theta}{\longleftarrow}}{\longrightarrow} \mathrm{Der}(\mathbf{A}) \longrightarrow \Gamma(T\mathcal{M}) \longrightarrow 0$$

- Gauge fields models of Yang-Mills-Higgs type:
  Yang-Mills in the geometric direction, Higgs (scalar fields) in the algebraic direction.

The endomorphism algebra

$\mathcal{M}$ smooth locally compact manifold.
$SU(n)$-principal fiber bundle $\mathcal{P}$.
$\mathcal{E}$ associated vector bundle with fiber $\mathbb{C}^n$.
$\mathrm{End}(\mathcal{E})$ fiber bundle of endomorphisms of $\mathcal{E}$.

A the algebra of smooth sections of $\mathrm{End}(\mathcal{E})$.

The endomorphism algebra

$\mathcal{M}$ smooth locally compact manifold.
$SU(n)$-principal fiber bundle $\mathcal{P}$.
$\mathcal{E}$ associated vector bundle with fiber $\mathbb{C}^n$.
$\operatorname{End}(\mathcal{E})$ fiber bundle of endomorphisms of $\mathcal{E}$.

A the algebra of smooth sections of $\operatorname{End}(\mathcal{E})$.

“Trivial case”: $\mathcal{E} = \mathcal{M} \times \mathbb{C}^n$ ➡ $\mathbf{A} = C^\infty(\mathcal{M}) \otimes M_n$.
In general, **A** is (globally) more complicated.
Using trivialisations of $\mathcal{E}$, the algebra **A** looks locally as $C^\infty(U) \otimes M_n$.

The endomorphism algebra

$\mathcal{M}$ smooth locally compact manifold.
$SU(n)$-principal fiber bundle $\mathcal{P}$.
$\mathcal{E}$ associated vector bundle with fiber $\mathbb{C}^n$.
$\mathrm{End}(\mathcal{E})$ fiber bundle of endomorphisms of $\mathcal{E}$.

$\mathbf{A}$ the algebra of smooth sections of $\mathrm{End}(\mathcal{E})$.

"Trivial case": $\mathcal{E} = \mathcal{M} \times \mathbb{C}^n$ ➞ $\mathbf{A} = C^\infty(\mathcal{M}) \otimes M_n$.
In general, $\mathbf{A}$ is (globally) more complicated.
Using trivialisations of $\mathcal{E}$, the algebra $\mathbf{A}$ looks locally as $C^\infty(U) \otimes M_n$.

- $\mathcal{Z}(\mathbf{A}) = C^\infty(\mathcal{M})$.
- Involution, trace map and determinant ($\mathrm{Tr}, \det : \mathbf{A} \to C^\infty(\mathcal{M})$) well defined.
- $SU(\mathbf{A})$ the unitaries in $\mathbf{A}$ of determinant 1,
  $\mathfrak{su}(\mathbf{A})$ the traceless antihermitean elements:

$$\mathcal{G} = SU(\mathbf{A}) \text{ is the gauge group of } \mathcal{P}$$

$$\mathrm{Lie}\mathcal{G} = \mathfrak{su}(\mathbf{A}) \text{ is the Lie algebra of the gauge group}$$

Derivations and differential calculus

$\rho : \mathrm{Der}(\mathbf{A}) \to \mathrm{Der}(\mathbf{A})/\mathrm{Int}(\mathbf{A}) = \mathrm{Out}(\mathbf{A})$ the projection.

- $\mathrm{Out}(\mathbf{A}) \simeq \mathrm{Der}(C^\infty(\mathcal{M})) = \Gamma(T\mathcal{M})$.
  ρ is the restriction of $\mathfrak{X} \in \mathrm{Der}(\mathbf{A})$ to $\mathcal{Z}(\mathbf{A}) = C^\infty(\mathcal{M})$.
- $\mathrm{Int}(\mathbf{A})$ is isomorphic to $\mathbf{A}_0$, the traceless elements in $\mathbf{A}$.
- The s.e.s. of Lie algebras and $C^\infty(\mathcal{M})$-modules of derivations looks like

$$0 \longrightarrow \mathbf{A}_0 \xrightarrow{\mathrm{ad}} \mathrm{Der}(\mathbf{A}) \xrightarrow{\rho} \Gamma(T\mathcal{M}) \longrightarrow 0$$

- There is no *a priori* canonical splitting in the non trivial case.
- The "n.c. 1-form" $i\theta$ cannot be defined here.
  But one can define a map of $C^\infty(\mathcal{M})$-modules:

$$i\theta : \mathrm{Int}(\mathbf{A}) \to \mathbf{A}_0 \qquad \mathrm{ad}_\gamma \mapsto \gamma - \tfrac{1}{n}\,\mathrm{Tr}(\gamma)\mathbb{1}$$

Connections on $\mathcal{E}$

$\nabla^{\mathcal{E}}$ any (ordinary) $SU(n)$-connection on $\mathcal{E}$.
Associated connections: $\nabla^{\mathcal{E}^*}$ on $\mathcal{E}^*$ and ∇ on $\mathrm{End}(\mathcal{E}) = \mathcal{E}^* \otimes \mathcal{E}$.
Notation $X = \rho(\mathfrak{X}) \in \Gamma(T\mathcal{M})$ for any $\mathfrak{X} \in \mathrm{Der}(\mathbf{A})$.

- $\Gamma(T\mathcal{M}) \ni X \mapsto \nabla_X \in \mathrm{Der}(\mathbf{A})$.
  ➥ $X \mapsto \nabla_X$ is a splitting as $C^\infty(\mathcal{M})$-modules of the s.e.s.

$$0 \longrightarrow \mathbf{A}_0 \longrightarrow \mathrm{Der}(\mathbf{A}) \longrightarrow \Gamma(T\mathcal{M}) \longrightarrow 0 \quad \text{(with } \nabla: \Gamma(T\mathcal{M}) \to \mathrm{Der}(\mathbf{A})\text{)}$$

- The obstruction to be a splitting of Lie algebras is the curvature of ∇:

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

Connections on $\mathcal{E}$

$\nabla^{\mathcal{E}}$ any (ordinary) $SU(n)$-connection on $\mathcal{E}$.
Associated connections: $\nabla^{\mathcal{E}^*}$ on $\mathcal{E}^*$ and ∇ on $\mathrm{End}(\mathcal{E}) = \mathcal{E}^* \otimes \mathcal{E}$.
Notation $X = \rho(\mathfrak{X}) \in \Gamma(T\mathcal{M})$ for any $\mathfrak{X} \in \mathrm{Der}(\mathbf{A})$.

- $\Gamma(T\mathcal{M}) \ni X \mapsto \nabla_X \in \mathrm{Der}(\mathbf{A})$.
  ➡ $X \mapsto \nabla_X$ is a splitting as $C^\infty(\mathcal{M})$-modules of the s.e.s.

$$0 \longrightarrow \mathbf{A}_0 \underset{\overset{\alpha}{\longleftarrow}}{\longrightarrow} \mathrm{Der}(\mathbf{A}) \underset{\overset{\nabla}{\longleftarrow}}{\longrightarrow} \Gamma(T\mathcal{M}) \longrightarrow 0$$

- The obstruction to be a splitting of Lie algebras is the curvature of ∇:
$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

- $\mathfrak{X} - \nabla_X \in \mathrm{Int}(\mathbf{A})$ for any $\mathfrak{X} \in \mathrm{Der}(\mathbf{A})$.

- $\mathfrak{X} \mapsto \alpha(\mathfrak{X}) = -i\theta(\mathfrak{X} - \nabla_X)$ is a n.c. 1-form $\alpha \in \underline{\Omega}^1_{\mathrm{Der}}(\mathbf{A})$.
$$\mathfrak{X} = \nabla_X - \mathrm{ad}_{\alpha(\mathfrak{X})}, \ \forall \mathfrak{X} \in \mathrm{Der}(\mathbf{A}) \qquad \alpha(\mathrm{ad}_\gamma) = -\gamma, \ \forall \gamma \in \mathbf{A}_0$$

Connections on $\mathcal{E}$

$\nabla^{\mathcal{E}}$ any (ordinary) $SU(n)$-connection on $\mathcal{E}$.
Associated connections: $\nabla^{\mathcal{E}^*}$ on $\mathcal{E}^*$ and ∇ on $\mathrm{End}(\mathcal{E}) = \mathcal{E}^* \otimes \mathcal{E}$.
Notation $X = \rho(\mathfrak{X}) \in \Gamma(T\mathcal{M})$ for any $\mathfrak{X} \in \mathrm{Der}(\mathbf{A})$.

- $\Gamma(T\mathcal{M}) \ni X \mapsto \nabla_X \in \mathrm{Der}(\mathbf{A})$.
  ➡ $X \mapsto \nabla_X$ is a splitting as $C^\infty(\mathcal{M})$-modules of the s.e.s.

$$0 \longrightarrow \mathbf{A}_0 \underset{\alpha}{\overset{}{\rightleftarrows}} \mathrm{Der}(\mathbf{A}) \underset{\nabla}{\rightleftarrows} \Gamma(T\mathcal{M}) \longrightarrow 0$$

- The obstruction to be a splitting of Lie algebras is the curvature of ∇:
$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

- $\mathfrak{X} - \nabla_X \in \mathrm{Int}(\mathbf{A})$ for any $\mathfrak{X} \in \mathrm{Der}(\mathbf{A})$.

- $\mathfrak{X} \mapsto \alpha(\mathfrak{X}) = -i\theta(\mathfrak{X} - \nabla_X)$ is a n.c. 1-form $\alpha \in \underline{\Omega}^1_{\mathrm{Der}}(\mathbf{A})$.
$$\mathfrak{X} = \nabla_X - \mathrm{ad}_{\alpha(\mathfrak{X})}, \ \forall \mathfrak{X} \in \mathrm{Der}(\mathbf{A}) \qquad \alpha(\mathrm{ad}_\gamma) = -\gamma, \ \forall \gamma \in \mathbf{A}_0$$

$SU(n)$-connection $\nabla^{\mathcal{E}}$ ➡ n.c. 1-form α s.t. $\alpha(\mathrm{ad}_\gamma) = -\gamma$.
Identification of curvature and gauge transformations.

Noncommutative connections on $M = \mathbf{A}$

$\boldsymbol{M} = \mathbf{A}$ with Hermitean structure $(a, b) \mapsto a^* b$.

$\omega \in \underline{\Omega}^1_{\mathrm{Der}}(\mathbf{A})$ ➤ $\widehat{\nabla}^\omega$ n.c. connection defined by $\widehat{\nabla}^\omega_{\mathfrak{X}} a = \mathfrak{X} a + \omega(\mathfrak{X}) a$.

Noncommutative connections on $M = \mathbf{A}$

$\boldsymbol{M} = \mathbf{A}$ with Hermitean structure $(a, b) \mapsto a^* b$.
$\omega \in \underline{\Omega}^1_{\mathrm{Der}}(\mathbf{A}) \longrightarrow \widehat{\nabla}^\omega$ n.c. connection defined by $\widehat{\nabla}^\omega_{\mathfrak{X}} a = \mathfrak{X} a + \omega(\mathfrak{X}) a$.

$\nabla^{\mathcal{E}}$ on $\mathcal{E} \longrightarrow \nabla$ on $\mathrm{End}(\mathcal{E}) \longrightarrow \alpha \in \underline{\Omega}^1_{\mathrm{Der}}(\mathbf{A}) \longrightarrow \widehat{\nabla}^\alpha$ n.c. connection.

Theorem (Ordinary connections as noncommutative connections)

- *The space of n.c. connections on the right module* $\mathbf{A}$ *compatible with the Hermitean structure* $(a, b) \mapsto a^* b$ *contains the space of ordinary* $SU(n)$*-connections on* $\mathcal{E}$.
- *This inclusion is compatible with the corresponding definitions of curvature and gauge transformations.*

Noncommutative connections on $\boldsymbol{M} = \mathbf{A}$

$\boldsymbol{M} = \mathbf{A}$ with Hermitean structure $(a, b) \mapsto a^* b$.
$\omega \in \underline{\Omega}^1_{\mathrm{Der}}(\mathbf{A}) \twoheadrightarrow \widehat{\nabla}^\omega$ n.c. connection defined by $\widehat{\nabla}^\omega_{\mathfrak{X}} a = \mathfrak{X} a + \omega(\mathfrak{X}) a$.

$\nabla^{\mathcal{E}}$ on $\mathcal{E} \twoheadrightarrow \nabla$ on $\mathrm{End}(\mathcal{E}) \twoheadrightarrow \alpha \in \underline{\Omega}^1_{\mathrm{Der}}(\mathbf{A}) \twoheadrightarrow \widehat{\nabla}^\alpha$ n.c. connection.

Theorem (Ordinary connections as noncommutative connections)

- *The space of n.c. connections on the right module* **A** *compatible with the Hermitean structure $(a, b) \mapsto a^* b$ contains the space of ordinary $SU(n)$-connections on $\mathcal{E}$.*
- *This inclusion is compatible with the corresponding definitions of curvature and gauge transformations.*

- α is a globally defined algebraic object on $\mathcal{M}$ corresponding to the family $\{A_i\}_i$.
- What are n.c. connections from a physical point of view?
  Their n.c. part can be interpreted as scalar Higgs fields in (natural) models.
  ➞ Spontaneous symmetry breaking mechanism, mass generation…

Assets and problems of n.c. gauge field theories

- The gauge theories obtained in NCG are very diverse:
  Many approaches to NCG…
  Many algebras can be considered…
  ➞ Gauge field theories can be realistic or exotic.
- Many of these theories are of Yang-Mills-Higgs type.
  ➞ The n.c. standard model by Chamseddine, Connes and Marcolli.
- The gauge group is the group of automorphisms of a right module.
  ➞ Not all Lie groups are accessible, for instance $U(1)$…
- The mathematical structures can be very involved.
  ➞ Spectral triples require a lot of mathematical skill.
  ➞ Not convenient to explore new models related to particle physics…

Transitive Lie algebroids

1 Ordinary differential geometry

2 Noncommutative geometry

3 Transitive Lie algebroids
- Generalities on Lie algebroids
- Examples, representation theory
- Differential structures
- Connections
- Gauge transformations

Lie algebroids: the algebraic point of view

- Lie algebroids are usually considered as generalizations of the tangent bundle.
- Here the geometric structure is ignored in favor of the algebraic structure, as in NCG.

Lie algebroids: the algebraic point of view

- Lie algebroids are usually considered as generalizations of the tangent bundle.
- Here the geometric structure is ignored in favor of the algebraic structure, as in NCG.

$\mathcal{M}$ a smooth manifold.
$\Gamma(T\mathcal{M})$ the Lie algebra and $C^\infty(\mathcal{M})$-module of vector fields.

Definition given in the "language" of algebras and modules:

A **Lie algebroid** A is:

- a finite projective module over $C^\infty(\mathcal{M})$,
- equipped with a Lie bracket $[-,-]$,
- equipped with a $C^\infty(\mathcal{M})$-linear Lie morphism $\rho : A \to \Gamma(T\mathcal{M})$ such that
$$[\mathfrak{X}, f\mathfrak{Y}] = f[\mathfrak{X}, \mathfrak{Y}] + (\rho(\mathfrak{X})\cdot f)\, \mathfrak{Y}$$
for any $\mathfrak{X}, \mathfrak{Y} \in A$ and $f \in C^\infty(\mathcal{M})$.
ρ is the **anchor** map.

Transitive Lie algebroids

A Lie algebroid $A \xrightarrow{\rho} \Gamma(T\mathcal{M})$ is **transitive** if ρ is surjective.

- $L = \operatorname{Ker} \rho$ is a Lie algebroid with null anchor on $\mathcal{M}$.
- The vector bundle $\mathcal{L}$ such that $L = \Gamma(\mathcal{L})$ is a locally trivial bundle in Lie algebras.
  ➡ gives the Lie structure on L.
- One has the s.e.s. of Lie algebras and $C^\infty(\mathcal{M})$-modules

$$0 \longrightarrow L \xrightarrow{\iota} A \xrightarrow{\rho} \Gamma(T\mathcal{M}) \longrightarrow 0$$

- L is called the **kernel** of A.

This s.e.s. is the key structure for various considerations.

Think about it as an infinitesimal version of a principal fiber bundle

$$G \longrightarrow \mathcal{P} \xrightarrow{\pi} \mathcal{M}$$

Example 1: Derivations of a vector bundle

$\mathcal{E}$ a vector bundle over $\mathcal{M}$.

$\text{End}(\mathcal{E})$ the fiber bundle of endomorphisms of $\mathcal{E}$, $\mathbf{A}(\mathcal{E}) = \Gamma(\text{End}(\mathcal{E}))$.

$\mathfrak{D}(\mathcal{E})$ the space of first-order differential operators on $\mathcal{E}$ with scalar symbols.

$\sigma : \mathfrak{D}(\mathcal{E}) \to \Gamma(T\mathcal{M})$ the symbol map.

$$0 \longrightarrow \mathbf{A}(\mathcal{E}) \xrightarrow{\iota} \mathfrak{D}(\mathcal{E}) \xrightarrow{\sigma} \Gamma(T\mathcal{M}) \longrightarrow 0$$

is the **transitive Lie algebroid of derivations** of $\mathcal{E}$.

Remark: $\mathbf{A}(\mathcal{E})$ is the endomorphism algebra of $\mathcal{E}$ defined before…

Representation of a Lie algebroid

$A \xrightarrow{\rho} \Gamma(T\mathcal{M})$ a Lie algebroid and $\mathcal{E} \to \mathcal{M}$ a vector bundle.

A **representation** of A on $\mathcal{E}$ is a morphism of Lie algebroids $\phi : A \to \mathfrak{D}(\mathcal{E})$.

Representation of a Lie algebroid

$A \xrightarrow{\rho} \Gamma(T\mathcal{M})$ a Lie algebroid and $\mathcal{E} \to \mathcal{M}$ a vector bundle.

A **representation** of A on $\mathcal{E}$ is a morphism of Lie algebroids $\phi : A \to \mathfrak{D}(\mathcal{E})$.

When A is transitive, one has the commutative diagram of exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \xrightarrow{\iota} & A & \xrightarrow{\rho} & \Gamma(T\mathcal{M}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi_L} & & \downarrow{\scriptstyle \phi} & & \| & & \\
0 & \longrightarrow & \mathbf{A}(\mathcal{E}) & \xrightarrow{\iota} & \mathfrak{D}(\mathcal{E}) & \xrightarrow{\sigma} & \Gamma(T\mathcal{M}) & \longrightarrow & 0
\end{array}$$

$\phi_L : L \to \mathbf{A}(\mathcal{E})$ is a morphism of Lie algebras.

Representation of a Lie algebroid

$A \xrightarrow{\rho} \Gamma(T\mathcal{M})$ a Lie algebroid and $\mathcal{E} \to \mathcal{M}$ a vector bundle.

A **representation** of A on $\mathcal{E}$ is a morphism of Lie algebroids $\phi : A \to \mathfrak{D}(\mathcal{E})$.

When A is transitive, one has the commutative diagram of exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \xrightarrow{\iota} & A & \xrightarrow{\rho} & \Gamma(T\mathcal{M}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi_L} & & \downarrow{\scriptstyle \phi} & & \| & & \\
0 & \longrightarrow & \mathbf{A}(\mathcal{E}) & \xrightarrow{\iota} & \mathfrak{D}(\mathcal{E}) & \xrightarrow{\sigma} & \Gamma(T\mathcal{M}) & \longrightarrow & 0
\end{array}$$

$\phi_L : L \to \mathbf{A}(\mathcal{E})$ is a morphism of Lie algebras.

Reminder:

- Principal fiber bundles: representation theory is played by associated vector bundles.
- Noncommutative geometry: representation theory is played by modules.

Example 2: Atiyah Lie algebroids

Example 2: Atiyah Lie algebroids

$\mathcal{P} \xrightarrow{\pi} \mathcal{M}$ a G-principal fiber bundle, $\mathfrak{g}$ the Lie algebra of G.
$\mathrm{R}_g : \mathcal{P} \to \mathcal{P}$, $\mathrm{R}_g(p) = p{\cdot}g$, the right action of G on $\mathcal{P}$.

$$\Gamma_G(T\mathcal{P}) = \{\mathfrak{X} \in \Gamma(T\mathcal{P}) \,/\, \mathrm{R}_{g\,*}\mathfrak{X} = \mathfrak{X} \text{ for all } g \in G\}$$
$$\Gamma_G(\mathcal{P}, \mathfrak{g}) = \{v : P \to \mathfrak{g} \,/\, v(p{\cdot}g) = \mathrm{Ad}_{g^{-1}} v(p) \text{ for all } g \in G\}$$

Both are Lie algebras and $C^\infty(\mathcal{M})$-modules.

Example 2: Atiyah Lie algebroids

$\mathcal{P} \xrightarrow{\pi} \mathcal{M}$ a G-principal fiber bundle, $\mathfrak{g}$ the Lie algebra of G.
$\mathrm{R}_g : \mathcal{P} \to \mathcal{P}$, $\mathrm{R}_g(p) = p \cdot g$, the right action of G on $\mathcal{P}$.

$$\Gamma_G(T\mathcal{P}) = \{\mathfrak{X} \in \Gamma(T\mathcal{P}) \,/\, \mathrm{R}_{g\,*}\mathfrak{X} = \mathfrak{X} \text{ for all } g \in G\}$$
$$\Gamma_G(\mathcal{P}, \mathfrak{g}) = \{v : P \to \mathfrak{g} \,/\, v(p \cdot g) = \mathrm{Ad}_{g^{-1}} v(p) \text{ for all } g \in G\}$$

Both are Lie algebras and $C^\infty(\mathcal{M})$-modules.

$\Gamma_G(T\mathcal{P})$ is the space of π_*-projectable vector fields in $\Gamma(T\mathcal{P})$ ➞ $\pi_* : \Gamma_G(T\mathcal{P}) \to \Gamma(T\mathcal{M})$.

$$\iota : \Gamma_G(\mathcal{P}, \mathfrak{g}) \to \Gamma_G(T\mathcal{P}) \quad \text{defined by} \quad \iota(v)_{|p} = v(p)^{\mathcal{P}}_{|p} = \left(\frac{d}{dt} p \cdot \exp(tv(p)) \right)_{|t=0}$$

$\mathfrak{g} \ni v \mapsto v^{\mathcal{P}}$ the fundamental vector field on $\mathcal{P}$.

Example 2: Atiyah Lie algebroids

$\mathcal{P} \xrightarrow{\pi} \mathcal{M}$ a G-principal fiber bundle, $\mathfrak{g}$ the Lie algebra of G.
$\mathrm{R}_g : \mathcal{P} \to \mathcal{P}$, $\mathrm{R}_g(p) = p{\cdot}g$, the right action of G on $\mathcal{P}$.

$$\Gamma_G(T\mathcal{P}) = \{\mathfrak{X} \in \Gamma(T\mathcal{P}) \,/\, \mathrm{R}_{g\,*}\mathfrak{X} = \mathfrak{X} \text{ for all } g \in G\}$$
$$\Gamma_G(\mathcal{P}, \mathfrak{g}) = \{v : P \to \mathfrak{g} \,/\, v(p{\cdot}g) = \mathrm{Ad}_{g^{-1}} v(p) \text{ for all } g \in G\}$$

Both are Lie algebras and $C^\infty(\mathcal{M})$-modules.

$\Gamma_G(T\mathcal{P})$ is the space of π_*-projectable vector fields in $\Gamma(T\mathcal{P})$ ➡ $\pi_* : \Gamma_G(T\mathcal{P}) \to \Gamma(T\mathcal{M})$.

$$\iota : \Gamma_G(\mathcal{P}, \mathfrak{g}) \to \Gamma_G(T\mathcal{P}) \quad \text{defined by} \quad \iota(v)_{|p} = v(p)^{\mathcal{P}}_{|p} = \left(\frac{d}{dt} p{\cdot} \exp(t v(p)) \right)_{|t=0}$$

$\mathfrak{g} \ni v \mapsto v^{\mathcal{P}}$ the fundamental vector field on $\mathcal{P}$.

This induces a s.e.s. of Lie algebras and $C^\infty(\mathcal{M})$-modules

$$0 \longrightarrow \Gamma_G(\mathcal{P}, \mathfrak{g}) \xrightarrow{\iota} \Gamma_G(T\mathcal{P}) \xrightarrow{\pi_*} \Gamma(T\mathcal{M}) \longrightarrow 0\,.$$

$\Gamma_G(T\mathcal{P})$ is the **Atiyah (transitive) Lie algebroid** associated to $\mathcal{P}$.

The representations of $\Gamma_G(T\mathcal{P})$ are given by the associated vector bundles to $\mathcal{P}$.

Example 2: Atiyah Lie algebroids

$\mathcal{P} \xrightarrow{\pi} \mathcal{M}$ a G-principal fiber bundle, $\mathfrak{g}$ the Lie algebra of G.
$\mathrm{R}_g : \mathcal{P} \to \mathcal{P}$, $\mathrm{R}_g(p) = p{\cdot}g$, the right action of G on $\mathcal{P}$.

$$\Gamma_G(T\mathcal{P}) = \{\mathfrak{X} \in \Gamma(T\mathcal{P}) \,/\, \mathrm{R}_{g\,*}\mathfrak{X} = \mathfrak{X} \text{ for all } g \in G\}$$
$$\Gamma_G(\mathcal{P}, \mathfrak{g}) = \{v : P \to \mathfrak{g} \,/\, v(p{\cdot}g) = \mathrm{Ad}_{g^{-1}} v(p) \text{ for all } g \in G\}$$

Both are Lie algebras and $C^\infty(\mathcal{M})$-modules.

$\Gamma_G(T\mathcal{P})$ is the space of π_*-projectable vector fields in $\Gamma(T\mathcal{P})$ ➡ $\pi_* : \Gamma_G(T\mathcal{P}) \to \Gamma(T\mathcal{M})$.

$$\iota : \Gamma_G(\mathcal{P}, \mathfrak{g}) \to \Gamma_G(T\mathcal{P}) \quad \text{defined by} \quad \iota(v)_{|p} = v(p)^{\mathcal{P}}_{|p} = \left(\frac{d}{dt} p{\cdot}\exp(tv(p))\right)_{|t=0}$$

$\mathfrak{g} \ni v \mapsto v^{\mathcal{P}}$ the fundamental vector field on $\mathcal{P}$.

This induces a s.e.s. of Lie algebras and $C^\infty(\mathcal{M})$-modules

$$0 \longrightarrow \Gamma_G(\mathcal{P}, \mathfrak{g}) \xrightarrow{\iota} \Gamma_G(T\mathcal{P}) \xrightarrow{\pi_*} \Gamma(T\mathcal{M}) \longrightarrow 0\,.$$

$\Gamma_G(T\mathcal{P})$ is the **Atiyah (transitive) Lie algebroid** associated to $\mathcal{P}$.

The representations of $\Gamma_G(T\mathcal{P})$ are given by the associated vector bundles to $\mathcal{P}$.

Atiyah Lie algebroids permit to embed ordinary gauge theories in this framework.

Example 3: Trivial Lie algebroids

Trivial Lie algebroid = Atiyah Lie algebroid of a trivial principal bundle $\mathcal{P} = \mathcal{M} \times G$.
Compact notation: $\mathsf{TLA}(\mathcal{M}, \mathfrak{g}) \equiv A = \Gamma(T\mathcal{M} \oplus (\mathcal{M} \times \mathfrak{g}))$.

Example 3: Trivial Lie algebroids

Trivial Lie algebroid = Atiyah Lie algebroid of a trivial principal bundle $\mathcal{P} = \mathcal{M} \times G$.
Compact notation: $\mathsf{TLA}(\mathcal{M}, \mathfrak{g}) \equiv A = \Gamma(T\mathcal{M} \oplus (\mathcal{M} \times \mathfrak{g}))$.

Proposition

Every transitive Lie algebroid A is locally of the form $TLA(\mathcal{U}, \mathfrak{g})$ for $\mathcal{U} \subset \mathcal{M}$ open subset.

Trivialization of an Atiyah Lie algebroid $\longleftrightarrow$ Trivialization of the principal fiber bundle.

Differential forms: general definition

$0 \longrightarrow L \xrightarrow{\iota} A \xrightarrow{\rho} \Gamma(T\mathcal{M}) \longrightarrow 0$ a transitive Lie algebroid.
$\phi : A \to \mathfrak{D}(\mathcal{E})$ a representation of A on $\mathcal{E}$.

- For $p = 0$: $\Omega^0(A, \mathcal{E}) = \Gamma(\mathcal{E})$.
- For $p \geq 1$: $\Omega^p(A, \mathcal{E})$ the linear space of $C^\infty(\mathcal{M})$-multilinear antisymmetric maps from A^p to $\Gamma(\mathcal{E})$.
- $\Omega^\bullet(A, \mathcal{E}) = \bigoplus_{p \geq 0} \Omega^p(A, \mathcal{E})$ is equipped with the differential

$$(\widehat{\mathrm{d}}_\phi \widehat{\omega})(\mathfrak{X}_1, \dots, \mathfrak{X}_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} \phi(\mathfrak{X}_i) \cdot \widehat{\omega}(\mathfrak{X}_1, \dots \overset{i}{\vee} \dots, \mathfrak{X}_{p+1})$$
$$+ \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} \widehat{\omega}([\mathfrak{X}_i, \mathfrak{X}_j], \mathfrak{X}_1, \dots \overset{i}{\vee} \dots \overset{j}{\vee} \dots, \mathfrak{X}_{p+1})$$

  $\phi(\mathfrak{X}) \cdot \varphi$ is the action of the first order diff. op. $\phi(\mathfrak{X})$ on $\varphi \in \Gamma(\mathcal{E})$.
- $\widehat{\mathrm{d}}_\phi^2 = 0$ (ϕ is a morphism of Lie algebras).

Differential forms: two examples

Let $\mathcal{E} = \mathcal{M} \times \mathbb{C}$, then $\Gamma(\mathcal{E}) = C^\infty(\mathcal{M})$.
The anchor map ρ is a representation of A on $C^\infty(\mathcal{M})$.

Forms with values in $C^\infty(\mathcal{M})$: $(\Omega^\bullet(A), \widehat{\mathrm{d}}_A)$ is the graded commutative differential algebra of forms on A with values in $C^\infty(\mathcal{M})$ associated to the representation ρ.

Differential forms: two examples

Let $\mathcal{E} = \mathcal{M} \times \mathbb{C}$, then $\Gamma(\mathcal{E}) = C^\infty(\mathcal{M})$.
The anchor map ρ is a representation of A on $C^\infty(\mathcal{M})$.

Forms with values in $C^\infty(\mathcal{M})$: $(\Omega^\bullet(A), \widehat{\mathrm{d}}_A)$ is the graded commutative differential algebra of forms on A with values in $C^\infty(\mathcal{M})$ associated to the representation ρ.

$\mathcal{E} = \mathcal{L}$ the vector bundle such that $L = \Gamma(\mathcal{L})$.
For $\mathfrak{X} \in A$ and $\ell \in L$, define $\mathrm{ad}_{\mathfrak{X}}(\ell) \in L$ such that $\iota(\mathrm{ad}_{\mathfrak{X}}(\ell)) = [\mathfrak{X}, \iota(\ell)]$.
This is the adjoint representation of A on $\mathcal{L}$.

Forms with values in the kernel: $(\Omega^\bullet(A, L), \widehat{\mathrm{d}})$ is the graded differential Lie algebra of forms on A with values in the kernel L associated to the adjoint representation.
➞ graded Lie algebra and graded differential module on $\Omega^\bullet(A)$.

Differential forms on trivial Lie algebroids

$A = \mathsf{TLA}(\mathcal{M}, \mathfrak{g})$ a trivial Lie algebroid.

$\Omega^\bullet(A)$ is the total complex of the bigraded commutative algebra $\Omega^\bullet(\mathcal{M}) \otimes \bigwedge^\bullet \mathfrak{g}^*$.
$\widehat{\mathrm{d}}_A = \mathrm{d} + \mathrm{s}$ with

$\mathrm{d} : \Omega^\bullet(\mathcal{M}) \otimes \bigwedge^\bullet \mathfrak{g}^* \to \Omega^{\bullet+1}(\mathcal{M}) \otimes \bigwedge^\bullet \mathfrak{g}^*$ de Rham differential

$\mathrm{s} : \Omega^\bullet(\mathcal{M}) \otimes \bigwedge^\bullet \mathfrak{g}^* \to \Omega^\bullet(\mathcal{M}) \otimes \bigwedge^{\bullet+1} \mathfrak{g}^*$ Chevalley-Eilenberg differential

Differential forms on trivial Lie algebroids

$A = \mathsf{TLA}(\mathcal{M}, \mathfrak{g})$ a trivial Lie algebroid.

$\Omega^\bullet(A)$ is the total complex of the bigraded commutative algebra $\Omega^\bullet(\mathcal{M}) \otimes \bigwedge^\bullet \mathfrak{g}^*$.
$\widehat{\mathrm{d}}_A = \mathrm{d} + \mathrm{s}$ with

$$\mathrm{d} : \Omega^\bullet(\mathcal{M}) \otimes \textstyle\bigwedge^\bullet \mathfrak{g}^* \to \Omega^{\bullet+1}(\mathcal{M}) \otimes \bigwedge^\bullet \mathfrak{g}^* \qquad \text{de Rham differential}$$

$$\mathrm{s} : \Omega^\bullet(\mathcal{M}) \otimes \textstyle\bigwedge^\bullet \mathfrak{g}^* \to \Omega^{\bullet}(\mathcal{M}) \otimes \bigwedge^{\bullet+1} \mathfrak{g}^* \qquad \text{Chevalley-Eilenberg differential}$$

$\Omega^\bullet(A, L)$ is the total complex of the bigraded Lie algebra $\Omega^\bullet(\mathcal{M}) \otimes \bigwedge^\bullet \mathfrak{g}^* \otimes \mathfrak{g}$.
$\widehat{\mathrm{d}} = \mathrm{d} + \mathrm{s}'$ with s' the Chevalley-Eilenberg differential on $\bigwedge^\bullet \mathfrak{g}^* \otimes \mathfrak{g}$ (for the ad rep.).
Compact notation: $(\Omega^\bullet_{\mathsf{TLA}}(\mathcal{M}, \mathfrak{g}), \widehat{\mathrm{d}}_{\mathsf{TLA}}) = (\Omega^\bullet(A, L), \widehat{\mathrm{d}})$.

They are models for trivializations of forms on any transitive Lie algebroid.

Differential forms on Atiyah Lie algebroids

A the Atiyah Lie algebroid of a G-principal fiber bundle $\mathcal{P} \xrightarrow{\pi} \mathcal{M}$.

Compact notation: $\Omega^\bullet_{\text{Lie}}(\mathcal{P}, \mathfrak{g}) = \Omega^\bullet(A, L)$.

$$\mathfrak{g}_{\text{equ}} = \{\xi^{\mathcal{P}} \oplus \xi \,/\, \xi \in \mathfrak{g}\} \subset \mathsf{TLA}(\mathcal{P}, \mathfrak{g}) = \Gamma(T\mathcal{P} \oplus (\mathcal{P} \times \mathfrak{g}))$$

is a Lie algebra, which defines a Cartan operation on $(\Omega^\bullet_{\text{TLA}}(\mathcal{P}, \mathfrak{g}), \widehat{\mathrm{d}}_{\text{TLA}})$.

$(\Omega^\bullet_{\text{TLA}}(\mathcal{P}, \mathfrak{g})_{\mathfrak{g}_{\text{equ}}}, \widehat{\mathrm{d}}_{\text{TLA}})$ the differential graded subcomplex of basic elements.

Theorem (S. Lazzarini, T.M.)

If G is connected and simply connected then
$(\Omega^\bullet_{\text{Lie}}(\mathcal{P}, \mathfrak{g}), \widehat{\mathrm{d}})$ *is isomorphic to* $(\Omega^\bullet_{TLA}(\mathcal{P}, \mathfrak{g})_{\mathfrak{g}_{\text{equ}}}, \widehat{\mathrm{d}}_{TLA})$

$$\rightarrow \Omega^\bullet_{\text{Lie}}(\mathcal{P}, \mathfrak{g}) \subset \Omega^\bullet_{\text{TLA}}(\mathcal{P}, \mathfrak{g}) \simeq \Omega^\bullet(\mathcal{P}) \otimes {\textstyle\bigwedge}^\bullet \mathfrak{g}^* \otimes \mathfrak{g}.$$

When G is connected and simply connected, a form can be described as:

- a $\mathfrak{g}_{\text{equ}}$-basic elements in $\Omega^\bullet_{\text{TLA}}(\mathcal{P}, \mathfrak{g})$;
- a form in $\Omega^\bullet_{\text{Lie}}(\mathcal{P}, \mathfrak{g})$;
- a family of local trivializations in $\Omega^\bullet_{\text{TLA}}(\mathcal{U}, \mathfrak{g})$ with gluing relations.

Ordinary connections on transitive Lie algebroids

$0 \longrightarrow L \xrightarrow{\iota} A \xrightarrow{\rho} \Gamma(T\mathcal{M}) \longrightarrow 0$ a transitive Lie algebroid.

Connection on a transitive Lie algebroid: splitting $\nabla : \Gamma(T\mathcal{M}) \to A$ as $C^\infty(\mathcal{M})$-modules of the s.e.s.

$$0 \longrightarrow L \xrightarrow[\iota]{} A \underset{\rho}{\overset{\nabla}{\rightleftarrows}} \Gamma(T\mathcal{M}) \longrightarrow 0$$

Curvature: obstruction to be a morphism of Lie algebras:

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \in \iota(L)$$

Ordinary connections on transitive Lie algebroids

$0 \longrightarrow L \xrightarrow{\iota} A \xrightarrow{\rho} \Gamma(T\mathcal{M}) \longrightarrow 0$ a transitive Lie algebroid.

Connection on a transitive Lie algebroid: splitting $\nabla : \Gamma(T\mathcal{M}) \to A$ as $C^\infty(\mathcal{M})$-modules of the s.e.s.

$$0 \longrightarrow L \underset{\iota}{\overset{\omega^\nabla}{\rightleftarrows}} A \underset{\rho}{\overset{\nabla}{\rightleftarrows}} \Gamma(T\mathcal{M}) \longrightarrow 0$$

Curvature: obstruction to be a morphism of Lie algebras:

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \in \iota(L)$$

$$\mathfrak{X} = \nabla_X - \iota \circ \omega^\nabla(\mathfrak{X})$$

- $\omega^\nabla \in \Omega^1(A, L)$ and $\omega^\nabla \circ \iota(\ell) = -\ell$ for any $\ell \in L$ (normalization on L).
- $R^\nabla(\mathfrak{X}, \mathfrak{Y}) = (\widehat{\mathrm{d}}\omega^\nabla)(\mathfrak{X}, \mathfrak{Y}) + [\omega^\nabla(\mathfrak{X}), \omega^\nabla(\mathfrak{Y})]$, $R^\nabla \in \Omega^2(A, L)$, $\iota \circ R^\nabla(\mathfrak{X}, \mathfrak{Y}) = R(X, Y)$ (vanishes when $\mathfrak{X}$ or $\mathfrak{Y}$ is in $\iota(L)$).

ω^∇ is the **connection 1-form** associated to ∇.

Ordinary connections on Atiyah Lie algebroid

$$0 \longrightarrow \Gamma_G(\mathcal{P}, \mathfrak{g}) \xrightarrow{\iota} \Gamma_G(T\mathcal{P}) \xrightarrow{\pi_*} \Gamma(T\mathcal{M}) \longrightarrow 0$$

Proposition (Connections)

Ordinary connection on the Atiyah Lie algebroid = connection on $\mathcal{P}$.
The notions of curvature coincide.

This example explains the terminology "ordinary connection".

Ordinary connections on Atiyah Lie algebroid

$$0 \longrightarrow \Gamma_G(\mathcal{P}, \mathfrak{g}) \xrightarrow{\iota} \Gamma_G(T\mathcal{P}) \xrightarrow{\pi_*} \Gamma(T\mathcal{M}) \longrightarrow 0$$

Proposition (Connections)

Ordinary connection on the Atiyah Lie algebroid = connection on $\mathcal{P}$.
The notions of curvature coincide.

This example explains the terminology “ordinary connection”.

The geometric equivalence: A connection on $\mathcal{P}$ defines a horizontal lift $\Gamma(T\mathcal{M}) \to \Gamma_G(T\mathcal{P}), X \mapsto X^h$.

Ordinary connections on Atiyah Lie algebroid

$$0 \longrightarrow \Gamma_G(\mathcal{P}, \mathfrak{g}) \xrightarrow{\iota} \Gamma_G(T\mathcal{P}) \xrightarrow{\pi_*} \Gamma(T\mathcal{M}) \longrightarrow 0$$

Proposition (Connections)

Ordinary connection on the Atiyah Lie algebroid = connection on $\mathcal{P}$.
The notions of curvature coincide.

This example explains the terminology "ordinary connection".

The geometric equivalence: A connection on $\mathcal{P}$ defines a horizontal lift $\Gamma(T\mathcal{M}) \to \Gamma_G(T\mathcal{P}), X \mapsto X^h$.

The algebraic equivalence: Suppose G is connected and simply connected.
$\omega^{\mathcal{P}} \in \Omega^1(\mathcal{P}) \otimes \mathfrak{g}$ a connection 1-form on $\mathcal{P}$.
$\theta \in \mathfrak{g}^* \otimes \mathfrak{g}$ the Maurer-Cartan 1-form on G.

$$\widehat{\omega}_{\mathfrak{g}_{\text{equ}}} = \omega^{\mathcal{P}} - \theta \in \Omega^1_{\text{TLA}}(\mathcal{P}, \mathfrak{g}) \subset \Omega^\bullet(\mathcal{P}) \otimes \bigwedge^\bullet \mathfrak{g}^* \otimes \mathfrak{g}$$

is $\mathfrak{g}_{\text{equ}}$-basic.
➞ It corresponds to the connection 1-form $\omega^\nabla \in \Omega^1_{\text{Lie}}(\mathcal{P}, \mathfrak{g})$ associated to $\omega^{\mathcal{P}}$.

Generalized connections on transitive Lie algebroids

A a transitive Lie algebroid.

Generalized connection: a 1-form $\widehat{\omega} \in \Omega^1(A, L)$.

Curvature: the 2-form $\widehat{R} = \widehat{d}\widehat{\omega} + \frac{1}{2}[\widehat{\omega}, \widehat{\omega}] \in \Omega^2(A, L)$.

An ordinary connection is a generalized connection for which $\widehat{\omega} \circ \iota = -\text{Id}_L$.

Generalized connections on transitive Lie algebroids

A a transitive Lie algebroid.

Generalized connection: a 1-form $\widehat{\omega} \in \Omega^1(A, L)$.

Curvature: the 2-form $\widehat{R} = \widehat{\mathrm{d}}\widehat{\omega} + \frac{1}{2}[\widehat{\omega}, \widehat{\omega}] \in \Omega^2(A, L)$.

An ordinary connection is a generalized connection for which $\widehat{\omega} \circ \iota = -\mathrm{Id}_L$.

Consider a representation of A on $\mathcal{E}$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \underset{\iota}{\overset{\widehat{\omega}}{\rightleftarrows}} & A & \xrightarrow{\rho} & \Gamma(T\mathcal{M}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi_L} & & \downarrow{\scriptstyle \phi} & & \| & & \\ 0 & \longrightarrow & \mathbf{A}(\mathcal{E}) & \xrightarrow{\iota} & \mathfrak{D}(\mathcal{E}) & \xrightarrow{\sigma} & \Gamma(T\mathcal{M}) & \longrightarrow & 0 \end{array}$$

Generalized connections on transitive Lie algebroids

A a transitive Lie algebroid.

Generalized connection: a 1-form $\widehat{\omega} \in \Omega^1(A, L)$.

Curvature: the 2-form $\widehat{R} = \widehat{d}\widehat{\omega} + \frac{1}{2}[\widehat{\omega}, \widehat{\omega}] \in \Omega^2(A, L)$.

An ordinary connection is a generalized connection for which $\widehat{\omega} \circ \iota = -\mathrm{Id}_L$.

Consider a representation of A on $\mathcal{E}$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \underset{\iota}{\overset{\widehat{\omega}}{\rightleftarrows}} & A & \overset{\rho}{\longrightarrow} & \Gamma(T\mathcal{M}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi_L} & & {\scriptstyle \widehat{\nabla}}\downarrow{\scriptstyle \phi} & & \| & & \\
0 & \longrightarrow & \mathbf{A}(\mathcal{E}) & \overset{\iota}{\longrightarrow} & \mathfrak{D}(\mathcal{E}) & \overset{\sigma}{\longrightarrow} & \Gamma(T\mathcal{M}) & \longrightarrow & 0
\end{array}$$

$\widehat{\omega}$ defines a covariant derivative on $\mathcal{E}$:

$$\widehat{\nabla}_{\mathfrak{X}}\varphi = \phi(\mathfrak{X})\cdot\varphi + \phi_L(\widehat{\omega}(\mathfrak{X}))\varphi.$$

$[\widehat{\nabla}_{\mathfrak{X}}, \widehat{\nabla}_{\mathfrak{Y}}] - \widehat{\nabla}_{[\mathfrak{X},\mathfrak{Y}]} = \iota \circ \phi_L \circ \widehat{R}(\mathfrak{X}, \mathfrak{Y})$ ➜ $\widehat{\nabla}$ is not a representation in general.

Generalized connections on Atiyah Lie algebroids

Suppose G is connected and simply connected.

$$0 \longrightarrow \Gamma_G(\mathcal{P}, \mathfrak{g}) \xrightarrow{\iota} \Gamma_G(T\mathcal{P}) \xrightarrow{\pi_*} \Gamma(T\mathcal{M}) \longrightarrow 0$$

A connection $\widehat{\omega}$ on $\Gamma_G(T\mathcal{P})$ is a $\mathfrak{g}_{\text{equ}}$-basic 1-form $\widehat{\omega}_{\mathfrak{g}_{\text{equ}}} \in \Omega^1_{\text{TLA}}(\mathcal{P}, \mathfrak{g})$.

$\widehat{\omega}_{\mathfrak{g}_{\text{equ}}} = \omega + \varphi \in (\Omega^1(\mathcal{P}) \otimes \mathfrak{g}) \oplus (C^\infty(\mathcal{P}) \otimes \mathfrak{g}^* \otimes \mathfrak{g})$.

If $\varphi = -\theta$, then $\widehat{\omega}$ is an ordinary connection on $\Gamma_G(T\mathcal{P})$.
➡ ω is an (ordinary) connection 1-form on $\mathcal{P}$.

Otherwise, $\varphi + \theta$ measures the deviation of $\widehat{\omega}$ from an ordinary connection.

➡ φ contains new degrees of freedom for gauge field theories.

Gauge group of a representation

$\mathcal{E}$ a representation space of the transitive Lie algebroid A:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{\iota} & A & \xrightarrow{\rho} & \Gamma(T\mathcal{M}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi_L} & & \downarrow{\scriptstyle \phi} & & \| & & \\ 0 & \longrightarrow & \mathbf{A}(\mathcal{E}) & \xrightarrow{\iota} & \mathfrak{D}(\mathcal{E}) & \xrightarrow{\sigma} & \Gamma(T\mathcal{M}) & \longrightarrow & 0 \end{array}$$

The **gauge group of $\mathcal{E}$** is the group $\text{Aut}(\mathcal{E})$ (vertical automorphisms of $\mathcal{E}$).

$\text{Aut}(\mathcal{E}) \subset \mathbf{A}(\mathcal{E})$,
$\mathbf{A}(\mathcal{E})$ are the infinitesimal gauge transformations on $\mathcal{E}$.

➡ $\phi_L(\xi)$ in an infinitesimal gauge transformation for any $\xi \in L$.

Infinitesimal gauge transformations

Infinitesimal gauge transformation: any element $\xi \in L$.

➙ No notion of finite gauge transformation at the level of A ($\simeq$ NCG).

Gauge transformation of connection: $\widehat{\mathrm{d}}\xi + [\widehat{\omega}, \xi]$.

Gauge transformation of curvature: $[\widehat{R}, \xi]$.

➙ The (local) gauge principle is implemented at the infinitesimal level.

➙ BRST-like.

Infinitesimal gauge transformations

Infinitesimal gauge transformation: any element $\xi \in L$.

➡ No notion of finite gauge transformation at the level of A ($\simeq$ NCG).

Gauge transformation of connection: $\widehat{\mathrm{d}}\xi + [\widehat{\omega}, \xi]$.

Gauge transformation of curvature: $[\widehat{R}, \xi]$.

➡ The (local) gauge principle is implemented at the infinitesimal level.

➡ BRST-like.

Gauge transformations on Atiyah Lie algebroids: The infinitesimal gauge transformations on generalized connections on $\Gamma_G(T\mathcal{P})$ are the (ordinary) infinitesimal gauge transformations on $\mathcal{P}$.

➡ Here there is a notion of gauge group…

Assets and problems of gauge field theories on Lie algebroids

- The mathematical structures are (very) close to ordinary geometry.
  ➡ Natural extension of the ordinary differential geometry of gauge fields.
- The gauge theories are of Yang-Mills-Higgs type.
  ➡ We know where the Yang-Mills theories are:
  $\omega - \theta$ as a special case of $\omega + \varphi$…
- All Lie groups are accessible (Atiyah Lie algebroids).
- BRST-like differential structures.
  ➡ Work in progress to understand if this is only a coincidence…
- New framework: require some work to construct realistic models.

Before the conclusion: a unifying theorem

$SL(n)$-principal bundle $\mathcal{P}$ over $\mathcal{M}$.
$\mathcal{E}$ associated vector bundle with fiber $\mathbb{C}^n$.
$\mathbf{A} = \Gamma(\text{End}(\mathcal{E}))$ the algebra of endomorphisms of $\mathcal{E}$.

The short exact sequence

$$0 \longrightarrow \text{Int}(\mathbf{A}) \xrightarrow{\text{ad}} \text{Der}(\mathbf{A}) \xrightarrow{\rho} \Gamma(T\mathcal{M}) \longrightarrow 0$$

defines $\text{Der}(\mathbf{A})$ as a transitive Lie algebroid over $\mathcal{M}$, with $\iota = \text{ad}$.

Theorem (S. Lazzarini, T.M.)

The following three spaces are isomorphic:

1. *The space of generalized connections on* $\Gamma_G(T\mathcal{P})$.
2. *The space of generalized connections on* $\text{Der}(\mathbf{A})$.
3. *The space of traceless n.c. connections on the right* $\mathbf{A}$*-module* $\mathbf{M} = \mathbf{A}$

These isomorphisms are compatible with curvatures and gauge transformations.
All these spaces contain ordinary (Yang-Mills) connections on $\mathcal{P}$.

Conclusion

Conclusion

- Gauge field theories can be generalized in at least two directions:
  - noncommutative geometry,
  - transitive Lie algebroids.

  ➞ These generalizations can coincide in specific examples.

Conclusion

- Gauge field theories can be generalized in at least two directions:
  - noncommutative geometry,
  - transitive Lie algebroids.

  ➡ These generalizations can coincide in specific examples.
- Common feature: add some purely algebraic directions to space-time.

Conclusion

- Gauge field theories can be generalized in at least two directions:
  - noncommutative geometry,
  - transitive Lie algebroids.

  ➡ These generalizations can coincide in specific examples.
- Common feature: add some purely algebraic directions to space-time.
- We naturally get Yang-Mills-Higgs type gauge theories in both situations.

  ➡ They contain ordinary Yang-Mills gauge theories used in physics.

Conclusion

- Gauge field theories can be generalized in at least two directions:
  - noncommutative geometry,
  - transitive Lie algebroids.

  ➨ These generalizations can coincide in specific examples.
- Common feature: add some purely algebraic directions to space-time.
- We naturally get Yang-Mills-Higgs type gauge theories in both situations.

  ➨ They contain ordinary Yang-Mills gauge theories used in physics.
- A pattern for (realistic) gauge field theories:

Thank you for your attention