Gauge Theories on Transitive Lie Algebroids Inspired by Noncommutative Gauge Theories

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Abstract

Inspired by noncommutative methods, transitive Lie algebroids can be used as a framework for gauge fields theories. I will present the necessary tools to write down gauge theory action functionals: forms, integration, Hodge star operators... These gauge field theories are of Yang-Mills-Higgs type, and I will show how ordinary Yang-Mills theories are included in these theories. Comparisons with some noncommutative gauge field theories will be given.

Contents

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How to construct a gauge theory?

The basic ingredients are

1. a space of local symmetries (space-time dependence): a **gauge group**.
2. an implementation of the symmetry on matter fields: a **representation theory**.
3. a place for derivations: some **differential structures**.
4. a replacement of ordinary derivations: a notion of **covariant derivative**.
5. a way to write a gauge invariant Lagrangien density: **action functional**.

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Ordinary differential geometry

Given a G-principal fiber bundle $\mathcal{P}$ over $\mathcal{M}$, the ingredients are

gauge group: $\mathcal{G}(\mathcal{P})$ is the group of vertical automorphisms of $\mathcal{P}$.

representation theory: associated vector bundles, on which $\mathcal{G}(\mathcal{P})$ acts naturally.

differential structures: de Rham differential calculus in ordinary geometry.

covariant derivatives: a connection 1-form ω on $\mathcal{P}$ induces covariant derivatives on sections of associated vector bundles.

action functional: integration on the base manifold $\mathcal{M}$, Killing form on the Lie algebra $\mathfrak{g}$ of G, Hodge star operator, curvature of ω.

The gauge theories obtained are massless gauge theories.
They are the prototype of theories used in the standard model of particle physics.
The mathematical structures are now accustomed and popular.

Noncommutative geometry

Given an associative algebra $\mathbf{A}$, the ingredients are

representation theory: a right module $\boldsymbol{M}$ over $\mathbf{A}$.

gauge group: $\text{Aut}(\boldsymbol{M})$, the group of automorphisms of the right module.

differential structures: any differential calculus defined on top of $\mathbf{A}$.

➸ many choices: derivation-based differential calculus, spectral triples…

covariant derivatives: noncommutative connections are defined on $\boldsymbol{M}$ with the help of the chosen differential calculus.

action functional: depends on the differential calculus.

- derivation-based differential calculus: noncommutative integration, Hodge star operator, curvature of the connection…
- spectral triples: Dixmier trace, spectral action…

The gauge theories obtained in NCG can be realistic or exotic.
There are noncommutative gauge theories of Yang-Mills-Higgs type.
The mathematical structures can be very involved.
NCG contains ordinary differential geometry
➸ NCG gauge theories contain (ordinary) gauge theories.

What have we learnt from NCG?

- Many geometrical structures can be (re)written in terms of algebraic structures.
- Geometric spaces can be supplemented by purely "algebraic spaces".
  ➸ Kind of "algebraic fibrations".
- Combining geometric and algebraic structures produces interesting gauge theories.
  ➸ The root of, and the route to, Yang-Mills-Higgs theories…
- For applications in physics, there is no reason to rest at a purely geometric stage.

The framework of transitive Lie algebroids gives us

- a mixture of geometric and algebraic structures,
- the mathematical structures are close to ordinary geometry,
  ➸ Accustomed and popular structures.
- natural gauge theories of Yang-Mills-Higgs type.
  ➸ The place of pure Yang-Mills theories is well understood…

1 – Lie algebroids and their representations

Generalities on Lie algebroids

Let $\mathcal{M}$ be a smooth manifold. $\Gamma(T\mathcal{M})$ the Lie algebra and $C^\infty(\mathcal{M})$-module of vector fields.

The following definition is given in the “language” of NCG: algebras and modules.

Definition 1 (Lie algebroids – Algebraic version)**.** A Lie algebroid A is a finite projective module over $C^\infty(\mathcal{M})$ equipped with a Lie bracket $[-,-]$ and a $C^\infty(\mathcal{M})$-linear Lie morphism $\rho: A \to \Gamma(T\mathcal{M})$ such that

$$[\mathfrak{X}, f\mathfrak{Y}] = f[\mathfrak{X},\mathfrak{Y}] + (\rho(\mathfrak{X})\cdot f)\mathfrak{Y}$$

for any $\mathfrak{X},\mathfrak{Y} \in A$ and $f \in C^\infty(\mathcal{M})$.

ρ is the anchor of A.

The usual definition uses the vector bundle $\mathcal{A}$ such that $A = \Gamma(\mathcal{A})$.

Transitive Lie algebroids

Definition 2 (Transitive Lie algebroids)**.** A Lie algebroid $A \xrightarrow{\rho} \Gamma(T\mathcal{M})$ is transitive if ρ is surjective.

Proposition 3 (The kernel of a transitive Lie algebroid)**.** *Let A be a transitive Lie algebroid.*

- *$L = \operatorname{Ker}\rho$ is a Lie algebroid with null anchor on $\mathcal{M}$.*
- *The vector bundle $\mathcal{L}$ such that $L = \Gamma(\mathcal{L})$ is a locally trivial bundle in Lie algebras.*
    ➡ *gives the Lie structure on L.*

One has the short exact sequence of Lie algebras and $C^\infty(\mathcal{M})$-modules

$$0 \longrightarrow L \xrightarrow{\iota} A \xrightarrow{\rho} \Gamma(T\mathcal{M}) \longrightarrow 0$$

L is called the kernel of A.

This short exact sequence is the key structure for various considerations.

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Example 1: Derivations of a vector bundle

$\mathcal{E}$ a vector bundle over $\mathcal{M}$.
$\mathrm{End}(\mathcal{E})$ the fiber bundle of endomorphisms of $\mathcal{E}$.
$\mathrm{Diff}^1(\mathcal{E})$ the space of first order differential operators on $\mathcal{E}$.
Symbol map $\sigma : \mathrm{Diff}^1(\mathcal{E}) \to \mathsf{Hom}_{C^\infty(\mathcal{M})}(T^*\mathcal{M}, \mathrm{End}(\mathcal{E}))$.

By duality: $\mathsf{Hom}_{C^\infty(\mathcal{M})}(T^*\mathcal{M}, \mathrm{End}(\mathcal{E})) \simeq \Gamma(T\mathcal{M} \otimes \mathrm{End}(\mathcal{E}))$.
One has: $\Gamma(T\mathcal{M}) \simeq \Gamma(T\mathcal{M}) \otimes \mathbb{1} \subset \Gamma(T\mathcal{M} \otimes \mathrm{End}(\mathcal{E}))$.
➡ $\Gamma(T\mathcal{M}) \subset \mathsf{Hom}_{C^\infty(\mathcal{M})}(T^*\mathcal{M}, \mathrm{End}(\mathcal{E}))$.

$$\mathfrak{D}(\mathcal{E}) = \sigma^{-1}(\Gamma(T\mathcal{M}))$$

is the transitive Lie algebroid of derivations of $\mathcal{E}$:

$$0 \longrightarrow \mathbf{A}(\mathcal{E}) \xrightarrow{\iota} \mathfrak{D}(\mathcal{E}) \xrightarrow{\sigma} \Gamma(T\mathcal{M}) \longrightarrow 0$$

with $\mathbf{A}(\mathcal{E}) = \Gamma(\mathrm{End}(\mathcal{E}))$.
The kernel $\mathbf{A}(\mathcal{E})$ is an associative algebra (Lie structure is the commutator).

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Representation of a Lie algebroid

$A \xrightarrow{\rho} \Gamma(T\mathcal{M})$ a Lie algebroid and $\mathcal{E} \to \mathcal{M}$ a vector bundle.

Definition 4 (Representation of a Lie algebroid)**.** A representation of A on $\mathcal{E}$ is a morphism of Lie algebroids $\phi : A \to \mathfrak{D}(\mathcal{E})$.

A representation is given by the commutative diagram of exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{\iota} & A & \xrightarrow{\rho} & \Gamma(T\mathcal{M}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi_L} & & \downarrow{\scriptstyle \phi} & & \| & & \\ 0 & \longrightarrow & \mathbf{A}(\mathcal{E}) & \xrightarrow{\iota} & \mathfrak{D}(\mathcal{E}) & \xrightarrow{\sigma} & \Gamma(T\mathcal{M}) & \longrightarrow & 0 \end{array}$$

$\phi_L : L \to \mathbf{A}(\mathcal{E})$ is a morphism of Lie algebras.

- Noncommutative geometry: representation theory is played by modules.
- Principal fiber bundles: representation theory is played by associated vector bundles.

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Example 2: Atiyah Lie algebroids

$\mathcal{P}\xrightarrow{\pi}\mathcal{M}$ a G-principal fiber bundle, $\mathfrak{g}$ the Lie algebra of G.
$\mathsf{R}_g:\mathcal{P}\to\mathcal{P}$, $\mathsf{R}_g(p)=p\cdot g$, the right action of G on $\mathcal{P}$.

$$\Gamma_G(T\mathcal{P})=\{\mathfrak{X}\in\Gamma(T\mathcal{P})\,/\,\mathsf{R}_{g*}\mathfrak{X}=\mathfrak{X}\text{ for all }g\in G\}$$
$$\Gamma_G(\mathcal{P},\mathfrak{g})=\{v:P\to\mathfrak{g}\,/\,v(p\cdot g)=\mathrm{Ad}_{g^{-1}}v(p)\text{ for all }g\in G\}$$

are Lie algebras and $C^\infty(\mathcal{M})$-modules.

$\Gamma_G(T\mathcal{P})$ is the space of π_*-projectable vector fields in $\Gamma(T\mathcal{P})$ ➡ $\pi_*:\Gamma_G(T\mathcal{P})\to\Gamma(T\mathcal{M})$.

Define $\iota:\Gamma_G(\mathcal{P},\mathfrak{g})\to\Gamma_G(T\mathcal{P})$ by:

$$\iota(v)_{|p}=v(p)^{\mathcal{P}}_{|p}=\left(\frac{d}{dt}p\cdot\exp(tv(p))\right)_{|t=0}$$

$\mathfrak{g}\ni v\mapsto v^{\mathcal{P}}$ the fundamental vector field on $\mathcal{P}$.
The s.e.s. of Lie algebras and $C^\infty(\mathcal{M})$-modules

$$0\longrightarrow\Gamma_G(\mathcal{P},\mathfrak{g})\xrightarrow{\iota}\Gamma_G(T\mathcal{P})\xrightarrow{\pi_*}\Gamma(T\mathcal{M})\longrightarrow 0$$

defines $\Gamma_G(T\mathcal{P})$ as a transitive Lie algebroid over $\mathcal{M}$.
This is the Lie algebroid of Atiyah associated to $\mathcal{P}$.

The representations of $\Gamma_G(T\mathcal{P})$ are given by the associated vector bundles to $\mathcal{P}$.

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Example 3: Trivial Lie algebroids

Trivial Lie algebroid = Atiyah Lie algebroid of a trivial principal bundle $\mathcal{P}=\mathcal{M}\times G$.

Concrete description in terms of the bundle $T\mathcal{M}\oplus(\mathcal{M}\times\mathfrak{g})$:

- Module: $\mathrm{TLA}(\mathcal{M},\mathfrak{g})\equiv A=\Gamma(T\mathcal{M}\oplus(\mathcal{M}\times\mathfrak{g}))$.
- Bracket: $[X\oplus\gamma,Y\oplus\eta]=[X,Y]\oplus(X\cdot\eta-Y\cdot\gamma+[\gamma,\eta])$
- Anchor: $\rho(X\oplus\gamma)=X$.
- Kernel: $L=\Gamma(\mathcal{M}\times\mathfrak{g})$ (section of a trivial bundle).

Proposition 5. *Every transitive Lie algebroid A is locally of the form $TLA(\mathcal{U},\mathfrak{g})$ for $\mathcal{U}\subset\mathcal{M}$ open subset.*

Trivialization of an Atiyah Lie algebroid $\longleftrightarrow$ Trivialization of the principal fiber bundle.

The global picture so far

- Transitive Lie algebroids as a general structure.

$$0 \longrightarrow L \xrightarrow{\iota} A \xrightarrow{\rho} \Gamma(T\mathcal{M}) \longrightarrow 0$$

- Local description of transitive Lie algebroids as trivial Lie algebroids.
- Representation theory on derivations of a vector bundle.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{\iota} & A & \xrightarrow{\rho} & \Gamma(T\mathcal{M}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi_L} & & \downarrow{\scriptstyle \phi} & & \| & & \\ 0 & \longrightarrow & \mathbf{A}(\mathcal{E}) & \xrightarrow{\iota} & \mathfrak{D}(\mathcal{E}) & \xrightarrow{\sigma} & \Gamma(T\mathcal{M}) & \longrightarrow & 0 \end{array}$$

- Principal fiber bundle ➙ canonical Atiyah Lie algebroid.

2 – Differential structures

Differential forms: general definition

$0 \longrightarrow L \xrightarrow{\iota} A \xrightarrow{\rho} \Gamma(T\mathcal{M}) \longrightarrow 0$ a transitive Lie algebroid.
$\phi : A \to \mathfrak{D}(\mathcal{E})$ a representation of A on $\mathcal{E}$.

Definition 6 (Differential forms)**.** For $p \in \mathbb{N}$, let $\Omega^p(A, \mathcal{E})$ be the linear space of $C^\infty(\mathcal{M})$-multilinear antisymmetric maps from A^p to $\Gamma(\mathcal{E})$ (smooth sections).
For $p = 0$, let $\Omega^0(A, \mathcal{E}) = \Gamma(\mathcal{E})$.
$\Omega^\bullet(A, \mathcal{E}) = \bigoplus_{p \geq 0} \Omega^p(A, \mathcal{E})$ is equipped with the natural differential

$$(\widehat{\mathrm{d}}_\phi \widehat{\omega})(\mathfrak{X}_1, \dots, \mathfrak{X}_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} \phi(\mathfrak{X}_i) \cdot \widehat{\omega}(\mathfrak{X}_1, \dots \overset{i}{\vee} \dots, \mathfrak{X}_{p+1}) \\ + \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} \widehat{\omega}([\mathfrak{X}_i, \mathfrak{X}_j], \mathfrak{X}_1, \dots \overset{i}{\vee} \dots \overset{j}{\vee} \dots, \mathfrak{X}_{p+1})$$

$\phi(\mathfrak{X}) \cdot \varphi$ is the action of the first order diff. op. $\phi(\mathfrak{X})$ on $\varphi \in \Gamma(\mathcal{E})$.
One has $\widehat{\mathrm{d}}_\phi^2 = 0$ (ϕ is a morphism of Lie algebras).

Differential forms: two examples

Let $\mathcal{E} = \mathcal{M} \times \mathbb{C}$. Then $\Gamma(\mathcal{E}) = C^\infty(\mathcal{M})$.
The anchor map is a representation of A on $C^\infty(\mathcal{M})$.

Definition 7 (Forms with values in $C^\infty(\mathcal{M})$)**.** $(\Omega^\bullet(A), \widehat{\mathrm{d}}_A)$ is the graded commutative differential algebra of forms on A with values in $C^\infty(\mathcal{M})$ associated to the anchor as a representation.

$\mathcal{E} = \mathcal{L}$ the vector bundle such that $L = \Gamma(\mathcal{L})$.
For $\mathfrak{X} \in A$ and $\ell \in L$, define $\mathrm{ad}_\mathfrak{X}(\ell) \in L$ such that $\iota(\mathrm{ad}_\mathfrak{X}(\ell)) = [\mathfrak{X}, \iota(\ell)]$.
This is the adjoint representation of A on $\mathcal{L}$.

Definition 8 (Forms with values in the kernel)**.** $(\Omega^\bullet(A, L), \widehat{\mathrm{d}})$ is the graded differential Lie algebra of forms on A with values in the kernel L associated to the adjoint representation.

This differential space is a graded Lie algebra and a graded differential module on the graded commutative differential algebra $\Omega^\bullet(A)$.

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Differential forms on trivial Lie algebroids

$A = \mathrm{TLA}(\mathcal{M}, \mathfrak{g})$ a trivial Lie algebroid.

$\Omega^\bullet(A)$ is the total complex of the bigraded commutative algebra $\Omega^\bullet(\mathcal{M}) \otimes \bigwedge^\bullet \mathfrak{g}^*$.
$\widehat{\mathrm{d}}_A = \mathrm{d} + \mathrm{s}$ with

$$\mathrm{d} : \Omega^\bullet(\mathcal{M}) \otimes \textstyle\bigwedge^\bullet \mathfrak{g}^* \to \Omega^{\bullet+1}(\mathcal{M}) \otimes \textstyle\bigwedge^\bullet \mathfrak{g}^* \qquad \text{de Rham differential}$$
$$\mathrm{s} : \Omega^\bullet(\mathcal{M}) \otimes \textstyle\bigwedge^\bullet \mathfrak{g}^* \to \Omega^\bullet(\mathcal{M}) \otimes \textstyle\bigwedge^{\bullet+1} \mathfrak{g}^* \qquad \text{Chevalley-Eilenberg differential}$$

$\Omega^\bullet(A, L)$ is the total complex of the bigraded Lie algebra $\Omega^\bullet(\mathcal{M}) \otimes \bigwedge^\bullet \mathfrak{g}^* \otimes \mathfrak{g}$.
$\widehat{\mathrm{d}} = \mathrm{d} + \mathrm{s}'$ with s' the Chevalley-Eilenberg differential on $\bigwedge^\bullet \mathfrak{g}^* \otimes \mathfrak{g}$ (for the ad rep.).
Compact notation $(\Omega^\bullet_{\mathrm{TLA}}(\mathcal{M}, \mathfrak{g}), \widehat{\mathrm{d}}_{\mathrm{TLA}})$ for this graded differential Lie algebra.

This is a model for trivializations of forms on any transitive Lie algebroid.

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Differential forms on Atiyah Lie algebroids

A the Atiyah Lie algebroid of the G-principal fiber bundle $\mathcal{P} \xrightarrow{\pi} \mathcal{M}$.
Denote by $(\Omega^\bullet_{\mathrm{Lie}}(\mathcal{P}, \mathfrak{g}), \widehat{\mathrm{d}})$ the complex of forms with values in the kernel.

$$\mathfrak{g}_{\mathrm{equ}} = \{\xi^{\mathcal{P}} \oplus \xi \ / \ \xi \in \mathfrak{g}\} \subset \mathrm{TLA}(\mathcal{P}, \mathfrak{g}) = \Gamma(T\mathcal{P} \oplus (\mathcal{P} \times \mathfrak{g}))$$

is a Lie sub-algebra, which defines a Cartan operation on $(\Omega^\bullet_{\mathrm{TLA}}(\mathcal{P}, \mathfrak{g}), \widehat{\mathrm{d}}_{\mathrm{TLA}})$.
$(\Omega^\bullet_{\mathrm{TLA}}(\mathcal{P}, \mathfrak{g})_{\mathfrak{g}_{\mathrm{equ}}}, \widehat{\mathrm{d}}_{\mathrm{TLA}})$ the differential graded subcomplex of basic elements.

Theorem 9 (S. Lazzarini, T.M.)**.** *If G is connected and simply connected then* $(\Omega^\bullet_{\mathrm{Lie}}(\mathcal{P}, \mathfrak{g}), \widehat{\mathrm{d}})$ *is isomorphic to* $(\Omega^\bullet_{TLA}(\mathcal{P}, \mathfrak{g})_{\mathfrak{g}_{\mathrm{equ}}}, \widehat{\mathrm{d}}_{TLA})$

➞ $\Omega^\bullet_{\mathrm{Lie}}(\mathcal{P}, \mathfrak{g}) \subset \Omega^\bullet_{\mathrm{TLA}}(\mathcal{P}, \mathfrak{g}) \simeq \Omega^\bullet(\mathcal{P}) \otimes \bigwedge^\bullet \mathfrak{g}^* \otimes \mathfrak{g}$.

When G is connected and simply connected, forms can be described as:

- a $\mathfrak{g}_{\mathrm{equ}}$-basic elements in $\Omega^\bullet_{\mathrm{TLA}}(\mathcal{P}, \mathfrak{g})$;
- a form in $\Omega^\bullet_{\mathrm{Lie}}(\mathcal{P}, \mathfrak{g})$;
- a family of local trivializations in $\Omega^\bullet_{\mathrm{TLA}}(\mathcal{U}, \mathfrak{g})$ with gluing relations.

Similar to different “levels” of description for structures on $\mathcal{P}$.

3 – Connections and covariant derivatives

Ordinary connections on transitive Lie algebroids

$0 \longrightarrow L \xrightarrow{\iota} A \xrightarrow{\rho} \Gamma(T\mathcal{M}) \longrightarrow 0$ a transitive Lie algebroid.

Definition 10 (Connection on a transitive Lie algebroid)**.** A connection on A is a splitting $\nabla : \Gamma(T\mathcal{M}) \to A$ as $C^\infty(\mathcal{M})$-modules of the s.e.s.

$$0 \longrightarrow L \underset{\iota}{\overset{\omega^\nabla}{\rightleftarrows}} A \underset{\rho}{\overset{\nabla}{\rightleftarrows}} \Gamma(T\mathcal{M}) \longrightarrow 0$$

The curvature of ∇ is defined to be the obstruction to be a morphism of Lie algebras:

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \in \iota(L)$$

For any $\mathfrak{X} \in A$, with $X = \rho(\mathfrak{X})$, $\mathfrak{X} - \nabla_X \in \operatorname{Ker} \rho$ ➡ $\exists!\ \omega^\nabla(\mathfrak{X}) \in L$ such that

$$\mathfrak{X} = \nabla_X - \iota \circ \omega^\nabla(\mathfrak{X})$$

Proposition 11. *One has $\omega^\nabla \in \Omega^1(A,L)$ and $\omega^\nabla \circ \iota(\ell) = -\ell$ for any $\ell \in L$ (normalization on L).*
The 2-form $R^\nabla \in \Omega^2(A,L)$ defined by $R^\nabla(\mathfrak{X},\mathfrak{Y}) = (\widehat{\mathrm{d}}\omega^\nabla)(\mathfrak{X},\mathfrak{Y}) + [\omega^\nabla(\mathfrak{X}), \omega^\nabla(\mathfrak{Y})]$ vanishes when $\mathfrak{X}$ or $\mathfrak{Y}$ is in $\iota(L)$, and one has $\iota \circ R^\nabla(\mathfrak{X},\mathfrak{Y}) = R(X,Y)$.

ω^∇ is the connection 1-form associated to ∇.

Ordinary connections on Atiyah Lie algebroid

$$0 \longrightarrow \Gamma_G(\mathcal{P},\mathfrak{g}) \xrightarrow{\iota} \Gamma_G(T\mathcal{P}) \xrightarrow{\pi_*} \Gamma(T\mathcal{M}) \longrightarrow 0$$

Proposition 12 (Connections)**.** *Ordinary connection on the Atiyah Lie algebroid = connection on $\mathcal{P}$.*
The notions of curvature coincide.

This example explains the terminology "ordinary connection".
The geometric equivalence: A connection on $\mathcal{P}$ defines a horizontal lift $\Gamma(T\mathcal{M}) \to \Gamma_G(T\mathcal{P})$, $X \mapsto X^h$.
The algebraic equivalence:
Suppose G is connected and simply connected.
$\omega^{\mathcal{P}} \in \Omega^1(\mathcal{P}) \otimes \mathfrak{g}$ a connection 1-form on $\mathcal{P}$.
$\theta \in \mathfrak{g}^* \otimes \mathfrak{g}$ the Maurer-Cartan 1-form on G.

$$\widehat{\omega}_{\mathfrak{g}_{\mathrm{equ}}} = \omega^{\mathcal{P}} - \theta \in \Omega^1_{\mathrm{TLA}}(\mathcal{P},\mathfrak{g}) \subset \Omega^\bullet(\mathcal{P}) \otimes {\textstyle\bigwedge}^\bullet \mathfrak{g}^* \otimes \mathfrak{g}$$

is $\mathfrak{g}_{\mathrm{equ}}$-basic.
It corresponds to the connection 1-form $\omega^\nabla \in \Omega^1_{\mathrm{Lie}}(\mathcal{P},\mathfrak{g})$ associated to $\omega^{\mathcal{P}}$.

Generalized connections on transitive Lie algebroids

Definition 13 (Generalized connection)**.** A generalized connection on a transitive Lie algebroid A is a 1-form $\widehat{\omega} \in \Omega^1(A, L)$.
The curvature of $\widehat{\omega}$ is the 2-form $\widehat{R} = \widehat{\mathrm{d}}\widehat{\omega} + \frac{1}{2}[\widehat{\omega}, \widehat{\omega}] \in \Omega^2(A, L)$.

An ordinary connection is a generalized connection for which $\widehat{\omega} \circ \iota = -\mathrm{Id}_L$.
Consider a representation of A on $\mathcal{E}$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \underset{\iota}{\overset{\widehat{\omega}}{\rightleftarrows}} & A & \xrightarrow{\rho} & \Gamma(T\mathcal{M}) & \longrightarrow & 0 \\ & & \phi_L \downarrow & & \widehat{\nabla} \downarrow \phi & & \| & & \\ 0 & \longrightarrow & \mathbf{A}(\mathcal{E}) & \xrightarrow{\iota} & \mathfrak{D}(\mathcal{E}) & \xrightarrow{\sigma} & \Gamma(T\mathcal{M}) & \longrightarrow & 0 \end{array}$$

$\widehat{\omega}$ defines a map $\widehat{\nabla}$ given by $\widehat{\nabla}_{\mathfrak{X}} \varphi = \phi(\mathfrak{X}) \cdot \varphi + \phi_L(\widehat{\omega}(\mathfrak{X}))\varphi$.
This is the covariant derivative on $\mathcal{E}$ associated to $\widehat{\omega}$.
$[\widehat{\nabla}_{\mathfrak{X}}, \widehat{\nabla}_{\mathfrak{Y}}] - \widehat{\nabla}_{[\mathfrak{X},\mathfrak{Y}]} = \iota \circ \phi_L \circ \widehat{R}(\mathfrak{X}, \mathfrak{Y})$ ➞ $\widehat{\nabla}$ is not a representation in general.

Generalized connections on Atiyah Lie algebroids

$$0 \longrightarrow \Gamma_G(\mathcal{P}, \mathfrak{g}) \xrightarrow{\iota} \Gamma_G(T\mathcal{P}) \xrightarrow{\pi_*} \Gamma(T\mathcal{M}) \longrightarrow 0$$

To simplify the presentation: suppose G is connected and simply connected.

A connection $\widehat{\omega}$ on $\Gamma_G(T\mathcal{P})$ is a $\mathfrak{g}_{\mathrm{equ}}$-basic 1-form $\widehat{\omega}_{\mathfrak{g}_{\mathrm{equ}}} \in \Omega^1_{\mathrm{TLA}}(\mathcal{P}, \mathfrak{g})$.
$\widehat{\omega}_{\mathfrak{g}_{\mathrm{equ}}} = \omega + \varphi \in (\Omega^1(\mathcal{P}) \otimes \mathfrak{g}) \oplus (C^\infty(\mathcal{P}) \otimes \mathfrak{g}^* \otimes \mathfrak{g})$.

If $\varphi = -\theta$, then $\widehat{\omega}$ is an ordinary connection on $\Gamma_G(T\mathcal{P})$.
➞ ω is an (ordinary) connection 1-form on $\mathcal{P}$.

Otherwise, $\varphi + \theta$ measures the deviation of $\widehat{\omega}$ from an ordinary connection.

Generalized connections and NCG

$\mathcal{E}$ a $SL(n)$-vector bundle over $\mathcal{M}$ with fiber $\mathbb{C}^n$, $\mathbf{A}=\Gamma(\mathrm{End}(\mathcal{E}))$ (denoted $\mathbf{A}(\mathcal{E})$ before).
$\mathrm{Der}(\mathbf{A})$ the space of derivations of $\mathbf{A}$: Lie algebra and $C^\infty(\mathcal{M})$-module.
$\mathrm{Int}(\mathbf{A})\subset \mathrm{Der}(\mathbf{A})$ inner derivations, $\mathbf{A}_0\simeq \mathrm{Int}(\mathbf{A})$ by $a\mapsto \mathrm{ad}_a$, with $\mathbf{A}_0=$ traceless elements.
$\rho:\mathrm{Der}(\mathbf{A})\to \mathrm{Der}(\mathbf{A})/\mathrm{Int}(\mathbf{A})=\mathrm{Out}(\mathbf{A})\simeq\Gamma(T\mathcal{M})$.
The short exact sequence

$$0 \longrightarrow \mathbf{A}_0 \xrightarrow{\mathrm{ad}} \mathrm{Der}(\mathbf{A}) \xrightarrow{\rho} \Gamma(T\mathcal{M}) \longrightarrow 0$$

defines $\mathrm{Der}(\mathbf{A})$ as a transitive Lie algebroid over $\mathcal{M}$, with $\iota=\mathrm{ad}$.
$\mathcal{P}$ the $SL(n)$-principal bundle to which $\mathcal{E}$ is associated.

Theorem 14 (S. Lazzarini, T.M.)**.** *The following three spaces are isomorphic:*

1. *The space of generalized connections on* $\mathrm{Der}(\mathbf{A})$.
2. *The space of generalized connections on* $\Gamma_G(T\mathcal{P})$.
3. *The space of traceless noncommutative connections on the right* $\mathbf{A}$*-module* $\boldsymbol{M}=\mathbf{A}$

These isomorphisms are compatible with curvatures and gauge transformations.

Connections: a summary

- Ordinary connections on transitive Lie algebroids are splittings:

$$0 \longrightarrow L \underset{\iota}{\overset{\omega^\nabla}{\rightleftarrows}} A \underset{\rho}{\overset{\nabla}{\rightleftarrows}} \Gamma(T\mathcal{M}) \longrightarrow 0$$

  ∇ ➡ $\omega^\nabla\in\Omega^1(A,L)$ connection 1-form, curvature as a 2-form.

- Generalized connections are any 1-forms $\widehat{\omega}\in\Omega^1(A,L)$.
  - ➡ Covariant derivatives on representations.
  - ➡ Notion of curvature.
- Ordinary connection = normalized generalized connection:
  $\widehat{\omega}\circ\iota(\ell)=-\ell$ for any $\ell\in L$
- For Atiyah Lie algebroids:
  - the space of generalized connections contains the space of ordinary connections on $\mathcal{P}$;
  - connection 1-forms and curvatures are directly related in $\Omega^\bullet_{\mathrm{TLA}}(\mathcal{P},\mathfrak{g})$.
- Generalized connections are noncommutative connections in some specific cases.

4 – The gauge group

Gauge group of a representation

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \xrightarrow{\iota} & A & \xrightarrow{\rho} & \Gamma(T\mathcal{M}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi_L} & & \downarrow{\scriptstyle \phi} & & \| & & \\
0 & \longrightarrow & \mathbf{A}(\mathcal{E}) & \xrightarrow{\iota} & \mathfrak{D}(\mathcal{E}) & \xrightarrow{\sigma} & \Gamma(T\mathcal{M}) & \longrightarrow & 0
\end{array}$$

a representation of a transitive Lie algebroid A on $\mathcal{E}$.

Definition 15 (Gauge group of a representation)**.** The gauge group of $\mathcal{E}$ is the group $\mathrm{Aut}(\mathcal{E})$ (vertical automorphisms of $\mathcal{E}$).

Notice that $\mathrm{Aut}(\mathcal{E}) \subset \mathbf{A}(\mathcal{E})$ and $\mathbf{A}(\mathcal{E})$ are the infinitesimal gauge transformations on $\mathcal{E}$.
Any $\xi \in L$ defines an infinitesimal gauge transformation on $\Gamma(\mathcal{E})$ by $\varphi \mapsto \phi_L(\xi)\varphi$.

Definition 16 (Infinitesimal gauge transformations)**.** An infinitesimal gauge transformation on A is an element $\xi \in L$.

No notion of gauge transformation at the level of A (similar situation in NCG).
One can "exponentiate" $\phi_L(\xi) \in \mathbf{A}(\mathcal{E})$ into $\exp \circ \phi_L(\xi) \in \mathrm{Aut}(\mathcal{E})$.

Gauge transformations

$\widehat{\omega}$ a connection on A, and $\widehat{\nabla}$ its associated covariant derivative on $\mathcal{E}$:

$$\widehat{\nabla}_{\mathfrak{X}}\varphi = \phi(\mathfrak{X})\cdot\varphi + \phi_L(\widehat{\omega}(\mathfrak{X}))\varphi$$

$\xi \in L$ an infinitesimal gauge transformation.
$g = \exp \circ \phi_L(\xi)$ its associated gauge transformation on $\mathcal{E}$.
The first order differential operator $\widehat{\nabla}^g_{\mathfrak{X}} = g^{-1} \circ \widehat{\nabla}_{\mathfrak{X}} \circ g$ on $\mathcal{E}$ can be written as

$$\widehat{\nabla}^g_{\mathfrak{X}} s = \phi(\mathfrak{X})\cdot s + \phi_L(\widehat{\omega}(\mathfrak{X}))s + \phi_L(\widehat{\mathrm{d}}\xi(\mathfrak{X}) + [\widehat{\omega}(\mathfrak{X}), \xi])s + O(\xi^2)s$$

Definition 17 (Infinitesimal gauge variation)**.** The infinitesimal gauge variation of $\widehat{\omega}$ induced by ξ is defined to be $\widehat{\mathrm{d}}\xi + [\widehat{\omega}, \xi]$.

Proposition 18 (Infinitesimal gauge action on curvature)**.** *The infinitesimal gauge variation of the curvature $\widehat{R}$ of $\widehat{\omega}$ is $[\widehat{R}, \xi]$.*

The (local) gauge principle is implemented, at least at the infinitesimal level.

Gauge transformations on Atiyah Lie algebroids

$$0 \longrightarrow \Gamma_G(\mathcal{P}, \mathfrak{g}) \xrightarrow{\iota} \Gamma_G(T\mathcal{P}) \xrightarrow{\pi_*} \Gamma(T\mathcal{M}) \longrightarrow 0$$

$\mathcal{G}(\mathcal{P})$ the gauge group of $\mathcal{P}$ (vertical automorphisms of $\mathcal{P}$).
$u \in \mathcal{G}(\mathcal{P})$ is a G-equivariant map $u: \mathcal{P} \to G$, $u(p \cdot g) = g^{-1} u(p) g$.

- ➥ $L = \Gamma_G(\mathcal{P}, \mathfrak{g})$ is the Lie algebra of $\mathcal{G}(\mathcal{P})$.
- ➥ Infinitesimal (usual) gauge transformations are elements in L.

$\widehat{\omega} \in \Omega^1_{\mathrm{Lie}}(\mathcal{P}, \mathfrak{g})$ and $u \in \mathcal{G}(\mathcal{P})$.
Define $\widehat{\omega}^u(\mathfrak{X}) = u^{-1} \widehat{\omega}(\mathfrak{X}) u + u^{-1}(\mathfrak{X} \cdot u)$ for any $\mathfrak{X} \in \Gamma_G(T\mathcal{P})$.

- ➥ $\widehat{\omega}^u \in \Omega^1_{\mathrm{Lie}}(\mathcal{P}, \mathfrak{g})$.
- ➥ This is the gauge action of u on $\widehat{\omega}$.

Proposition 19 (Gauge transformations on Atiyah Lie algebroids)**.** *The infinitesimal gauge transformations on generalized connections on $\Gamma_G(T\mathcal{P})$ are the (ordinary) infinitesimal gauge transformations on $\mathcal{P}$ induced by $\widehat{\omega} \mapsto \widehat{\omega}^u$.*

5 – Structures to construct an action functional

Metrics on transitive Lie algebroids

Definition 20 (Metric on a Lie algebroid)**.** A metric on A is a symmetric, $C^\infty(\mathcal{M})$-linear map $\widehat{g}: A \otimes_{C^\infty(\mathcal{M})} A \to C^\infty(\mathcal{M})$.

$\widehat{g}$ defines a metric $h = \iota^* \widehat{g}$ on L given by $h(\gamma, \eta) = \widehat{g}(\iota(\gamma), \iota(\eta))$ for any $\gamma, \eta \in L$.

Definition 21 (Inner non degenerate metric)**.** $\widehat{g}$ is inner non degenerate if the induced metric h on L is non degenerate.

Proposition 22 (C. Fournel, S. Lazzarini, T.M.)**.** *An inner non degenerate metric $\widehat{g}$ on A is equivalent to a triple (g, h, ∇) where*

- *g is a (possibly degenerate) metric on $\mathcal{M}$;*
- *h is a non degenerate metric on L;*
- *∇ is an ordinary connection on A, with $\mathfrak{a} \in \Omega^1(A, L)$ its connection 1-form;*
- *$\widehat{g}(\mathfrak{X}, \mathfrak{Y}) = g(\rho(\mathfrak{X}), \rho(\mathfrak{Y})) + h(\mathfrak{a}(\mathfrak{X}), \mathfrak{a}(\mathfrak{Y}))$.*
- *$\widehat{g}(\nabla_X, \iota(\gamma)) = 0$ for any $X \in \Gamma(T\mathcal{M})$ and $\gamma \in L$.*

Local description of forms and the mixed basis

$\mathfrak{g}$ the Lie algebra fiber of $\mathcal{L}$, where $L = \Gamma(\mathcal{L})$. $\{E_a\}_{1 \leq a \leq n}$ basis of $\mathfrak{g}$, $\{\theta^a\}_{1 \leq a \leq n}$ its dual basis.
$\mathcal{U} \subset \mathcal{M}$ open subset which trivializes A.
$\widehat{\omega} \in \Omega^p(A, L)$ and $\widehat{\omega}_{\text{loc}} \in \Omega^p_{\text{TLA}}(\mathcal{U}, \mathfrak{g})$ its local description:

$$\widehat{\omega}_{\text{loc}} = \sum_{r+s=p} \widehat{\omega}^{\theta}_{\mu_1 \dots \mu_r a_1 \dots a_s} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_r} \wedge \theta^{a_1} \wedge \cdots \wedge \theta^{a_s} \quad \text{with} \quad \widehat{\omega}^{\theta}_{\mu_1 \dots \mu_r a_1 \dots a_s} : \mathcal{U} \to \mathfrak{g}$$

∇ ordinary connection on A, $\mathfrak{a}$ its connection 1-form, $\mathfrak{a}_{\text{loc}} = (A^a - \theta^a) E_a$, $A^a \in \Omega^1(\mathcal{U}) \otimes \mathfrak{g}$.

Definition 23 (Mixed basis)**.** The local 1-forms $\mathfrak{a}^a = A^a - \theta^a \in \Omega^1_{\text{TLA}}(\mathcal{U})$ define the mixed basis in $\Omega^1_{\text{TLA}}(\mathcal{U})$ relative to the ordinary connection ∇ and to the basis $\{E_a\}_{1 \leq a \leq n}$ of $\mathfrak{g}$.

Then one can write

$$\widehat{\omega}_{\text{loc}} = \sum_{r+s=p} \widehat{\omega}_{\mu_1 \dots \mu_r a_1 \dots a_s} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_r} \wedge \mathfrak{a}^{a_1} \wedge \cdots \wedge \mathfrak{a}^{a_s} \quad \text{with} \quad \widehat{\omega}_{\mu_1 \dots \mu_r a_1 \dots a_s} : \mathcal{U} \to \mathfrak{g}$$

Proposition 24 (Homogeneous transformations)**.** *In a change of local trivialization, the $\widehat{\omega}_{\mu_1 \dots \mu_r a_1 \dots a_s}$'s have homogeneous transformations.*

Integration along the kernel

We suppose that the vector bundle $\mathcal{L}$, where $L=\Gamma(\mathcal{L})$, is orientable.
h a metric on L, ∇ connection on A, $\mathfrak{a}\in\Omega^1(A,L)$ its connection 1-form.
$\mathcal{U}\subset\mathcal{M}$ an open subset which trivializes A.
h_{loc} trivialization of h over $\mathcal{U}$, $h_{ab}=h_{\mathrm{loc}}(E_a,E_b)\in C^\infty(\mathcal{U})$, $|h_{\mathrm{loc}}|=|\det(h_{ab})|$.

Proposition 25 (Volume form along L)**.**

$$\widehat{\omega}_{h,\mathfrak{a}\,\mathrm{loc}}=(-1)^n\sqrt{|h_{\mathrm{loc}}|}\,\mathfrak{a}^1_{\mathrm{loc}}\wedge\cdots\wedge\mathfrak{a}^n_{\mathrm{loc}}$$

defines a global form $\widehat{\omega}_{h,\mathfrak{a}}\in\Omega^\bullet(A)$ of maximal degree n in the L direction.

Any form $\widehat{\omega}\in\Omega^\bullet(A,L)$ of maximal degree n in the L direction can be written as

$$\widehat{\omega}=\omega^{\mathcal{M}}\,\widehat{\omega}_{h,\mathfrak{a}}+\widehat{\omega}^R$$

where $\omega^{\mathcal{M}}\in\Omega^{\bullet-n}(\mathcal{M},\mathcal{L})$ and $\widehat{\omega}^R$ of degrees $<n$ in the $\mathfrak{a}^a_{\mathrm{loc}}$'s.

Definition 26 (Integration along L)**.**

$$\int_{\mathrm{inner}}:\Omega^\bullet(A,L)\to\Omega^{\bullet-n}(\mathcal{M},\mathcal{L})\qquad\qquad\widehat{\omega}\mapsto\omega^{\mathcal{M}}$$

is the integration of $\widehat{\omega}$ along L. It does not depend on ∇.

Integration along L defines also a map $\int_{\mathrm{inner}}:\Omega^\bullet(A)\to\Omega^{\bullet-n}(\mathcal{M})$.

Integration on A

Definition 27 (Orientable transitive Lie algebroid)**.** A transitive Lie algebroid is orientable if it is inner orientable and if $\mathcal{M}$ is orientable.

A an orientable transitive Lie algebroid, h a metric on $\mathcal{L}$, ∇ a connection on A.

Definition 28 (Integration on A)**.** The integration on A of a form $\widehat{\omega}\in\Omega^\bullet(A)$ is defined by

$$\int_A\widehat{\omega}=\int_{\mathcal{M}}\int_{\mathrm{inner}}\widehat{\omega}\in\mathbb{C}.$$

Definition 29 (Scalar product of forms)**.** The scalar product of any 2 forms $\widehat{\omega}_1,\widehat{\omega}_2\in\Omega^\bullet(A,L)$ is defined by

$$\langle\widehat{\omega}_1,\widehat{\omega}_2\rangle=\int_A h(\widehat{\omega}_1,\widehat{\omega}_2)\in\mathbb{C}$$

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Hodge star operator

A an orientable transitive Lie algebroid.
$\widehat{g}=(g,h,\nabla)$ a metric on $\mathbf{A}$, and $\mathfrak{a}$ the connection 1-form of ∇.
Write $\widehat{\omega}\in\Omega^p(A,L)$ locally as

$$\widehat{\omega}_{\text{loc}}=\sum_{r+s=p}\widehat{\omega}_{\mu_1\dots\mu_r a_1\dots a_s}\mathrm{d}x^{\mu_1}\wedge\dots\wedge\mathrm{d}x^{\mu_r}\wedge\mathfrak{a}^{a_1}\wedge\dots\wedge\mathfrak{a}^{a_s}$$

We associate to $\widehat{\omega}_{\text{loc}}$ a local form in $\Omega^{m+n-p}_{\text{TLA}}(U,\mathfrak{g})$ given by

$$\begin{aligned}\star\widehat{\omega}_{\text{loc}}=\sum_{r+s=p}(-1)^{s(m-r)}\frac{1}{r!s!}\sqrt{|h_{\text{loc}}|}\sqrt{|g|}\,\widehat{\omega}_{\mu_1\dots\mu_r a_1\dots a_s}\,\epsilon_{\nu_1\dots\nu_m}\,\epsilon_{b_1\dots b_n}\\ \times g^{\mu_1\nu_1}\cdots g^{\mu_r\nu_r}\,h^{a_1b_1}\cdots h^{a_sb_s}\,\mathrm{d}x^{\nu_{r+1}}\wedge\dots\wedge\mathrm{d}x^{\nu_m}\wedge\mathfrak{a}^{b_{s+1}}\wedge\dots\wedge\mathfrak{a}^{b_n}\end{aligned}$$

where $\epsilon_{\nu_1\dots\nu_m}$ and $\epsilon_{b_1\dots b_n}$ are the totally antisymmetric Levi-Civita symbols,
$(g^{\mu\nu})$ and (h^{ab}) are the inverse matrices of $(g_{\mu\nu})$ and (h_{ab}).

Proposition 30 (Hodge star operator)**.** *The map* $\star:\Omega^p(A,L)\to\Omega^{m+n-p}(A,L)$ *is well defined globally.*

This is the Hodge star operator associated to $\widehat{g}$ on A.

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Metrics: a summary

A metric $\widehat{g}=(g,h,\nabla)$, with $\nabla\leftrightarrow\mathfrak{a}\in\Omega^1(A,L)$, gives us:

- h ➞ scalar product on L;
- $h,\mathfrak{a}$ ➞ integration along L;
- g ➞ integration on $\mathcal{M}$;
- $g,h,\mathfrak{a}$ ➞ Hodge star operator $\star$.

6 – Gauge theories

Gauge invariant action

A an orientable transitive Lie algebroid equipped with a metric $\widehat{g}=(g,h,\nabla)$,
such that g and h are both non degenerate.

Suppose that h is a Killing metric: $h([\xi,\gamma],\eta)+h(\gamma,[\xi,\eta])=0$ for any $\gamma,\eta,\xi\in L$.

$\widehat{\omega}\in\Omega^1(A,L)$ a connection on A and $\widehat{R}$ its curvature 2-form.

Proposition 31 (C. Fournel, S. Lazzarini, T.M.). *The action functional*

$$\mathcal{S}_{Gauge}[\widehat{\omega}]=\langle\widehat{R},\star\widehat{R}\rangle=\int_A h(\widehat{R},\star\widehat{R}).$$

is invariant under infinitesimal gauge transformations in L.

Example 32 (Atiyah Lie algebroid). A the Atiyah Lie algebroid of a G-principal fiber bundle $\mathcal{P}$.
h the metric on $\mathcal{L}$ induced by the Killing metric on $\mathfrak{g}$.
$\widehat{\omega}$ any generalized connection ➞ $\mathcal{S}_{\text{Gauge}}[\widehat{\omega}]$ is $\mathcal{G}(\mathcal{P})$-gauge invariant.
$\widehat{\omega}$ an ordinary connection on A ➞ $\mathcal{S}_{\text{Gauge}}[\widehat{\omega}]$ is the ordinary Yang-Mills action.

Coupling to matter fields

Matter fields are sections $\varphi\in\Gamma(\mathcal{E})$ of a representation $\phi: A\to\mathfrak{D}(\mathcal{E})$ of A.

Definition 33 (ϕ_L-compatible metric). A metric $h^{\mathcal{E}}$ on $\mathcal{E}$ is ϕ_L-compatible if

$$h^{\mathcal{E}}(\phi_L(\xi)\varphi_1,\varphi_2)+h^{\mathcal{E}}(\varphi_1,\phi_L(\xi)\varphi_2)=0$$

for any $\varphi_1,\varphi_2\in\Gamma(\mathcal{E})$ and any $\xi\in L$.

Generalization of "Killing metric".
One can define a Hodge star operator on $\Omega^\bullet(A,\mathcal{E})$.

$\widehat{\omega}$ connection on A and $\widehat{\nabla}^{\mathcal{E}}$ the induced covariant derivative on $\Gamma(\mathcal{E})$.

Proposition 34. *The action functional*

$$\mathcal{S}_{Matter}[\varphi,\widehat{\omega}]=\int_A h^{\mathcal{E}}(\widehat{\nabla}^{\mathcal{E}}\varphi,\star\widehat{\nabla}^{\mathcal{E}}\varphi)$$

is invariant under infinitesimal gauge transformations in L.

The total action functional

$\widehat{g} = (g, h, \nabla)$ decomposes any connection $\widehat{\omega}$ on A as:

$$\widehat{\omega} \longleftrightarrow (\omega, \tau)$$

ω is an ordinary connection on A, τ is an algebraic object on L.

$$\begin{aligned} \mathcal{S}_{\text{Gauge}}[\widehat{\omega}] + \mathcal{S}_{\text{Matter}}[\varphi, \widehat{\omega}] = &\ (1) \text{ a Yang-Mills like term for } \omega \\ &\ (2) \text{ a covariant derivative for } \tau \text{ along } \omega \\ &\ (3) \text{ a potential for } \tau \\ &\ (4) \text{ a covariant derivative for } \varphi \text{ along } \omega \\ &\ (5) \text{ a coupling } \varphi \leftrightarrow \tau \end{aligned}$$

The potential (3) can vanish for $\tau \neq 0$.
A development around a solution $\tau_0 \neq 0$ induces:

- A mass terms for the ordinary connection ω in (2).
- A mass terms for φ in (5).

➜ Massive bosons (ω) coupled to massive particles (φ).
➜ Yang-Mills-Higgs type gauge theory.

Conclusion

- Gauge field theories can be generalized in at least two directions:
  - noncommutative geometry
  - transitive Lie algebroids
- Same idea in both cases: add some purely algebraic directions to space-time.
- We naturally get Yang-Mills-Higgs type gauge theories in both situations.
- Gauge theories on Atiyah Lie algebroids are close to Yang-Mills gauge theories.
  - ➙ They contain ordinary gauge theories used in physics.
  - ➙ They share some common mathematical structures.
- A lot more to investigate.
- There is life for gauge theories beyond connections on principal fiber bundles !

Additional material

Trivialization of transitive Lie algebroids

$0 \longrightarrow L \xrightarrow{\iota} A \xrightarrow{\rho} \Gamma(T\mathcal{M}) \longrightarrow 0$ a transitive Lie algebroid.
A local trivialization of A is a triple $(\mathcal{U}, \Psi, \nabla^0)$ where

- $\mathcal{U}$ is an open subset of $\mathcal{M}$;
- $\Psi : \Gamma(\mathcal{U} \times \mathfrak{g}) \xrightarrow{\simeq} L_{\mathcal{U}}$ is an isomorphism of Lie algebras and $C^\infty(\mathcal{U})$-modules;
- $\nabla^0 : \Gamma(T\mathcal{U}) \to A_{\mathcal{U}}$ is an injective morphism of Lie algebras and $C^\infty(\mathcal{U})$-modules compatible with the anchors;
- $[\nabla^0_X, \iota \circ \Psi(\gamma)] = \iota \circ \Psi(X \cdot \gamma)$ for any $X \in \Gamma(T\mathcal{U})$ and any $\gamma \in \Gamma(\mathcal{U} \times \mathfrak{g})$.

$S(X \oplus \gamma) = \nabla^0_X + \iota \circ \Psi(\gamma)$ is a isomorphism of Lie algebroids $S : \mathrm{TLA}(\mathcal{U}, \mathfrak{g}) \xrightarrow{\simeq} A_{\mathcal{U}}$.
Lie algebroid atlas for A: a family of local trivializations $(\mathcal{U}_i, \Psi_i, \nabla^{0,i})$ with $\bigcup_{i \in I} \mathcal{U}_i = \mathcal{M}$.
$\mathfrak{X} \in A$ is decomposed as $X^i \oplus \gamma^i \in \mathrm{TLA}(\mathcal{U}_i, \mathfrak{g})$ such that $S_i(X^i \oplus \gamma^i) = \mathfrak{X}_{|\mathcal{U}_i}$.
The X^i's are the restrictions to $\mathcal{U}_i$ of the global vector field $X = \rho(\mathfrak{X})$.
On $\mathcal{U}_{ij} = \mathcal{U}_i \cap \mathcal{U}_j \neq \varnothing$ one can define $\alpha^i_j = \Psi_i^{-1} \circ \Psi_j : \mathcal{U}_{ij} \to \mathrm{Aut}(\mathfrak{g})$.
There exists $\chi_{ij} \in \Omega^1(\mathcal{U}_{ij}) \otimes \mathfrak{g}$ such that $\gamma^i = \alpha^i_j(\gamma^j) + \chi_{ij}(X)$.
One has the cocycle relations

$$\alpha^i_k = \alpha^i_j \circ \alpha^j_k \qquad \alpha^i_j \circ \alpha^j_i = \mathrm{Id} \qquad \chi_{ik} = \alpha^i_j \circ \chi_{jk} + \chi_{ij} \qquad \alpha^i_j \circ \chi_{ji} + \chi_{ij} = 0$$

Trivialization of differential forms

$0 \longrightarrow L \xrightarrow{\iota} A \xrightarrow{\rho} \Gamma(T\mathcal{M}) \longrightarrow 0$ a transitive Lie algebroid.
$\{(\mathcal{U}_i, \Psi_i, \nabla^{0,i})\}_{i \in I}$ a Lie algebroid atlas for A.
To any q-form $\omega \in \Omega^q(A, L)$ we associate a family of local q-forms $\omega^i_{\mathrm{loc}} \in \Omega^q_{\mathrm{TLA}}(\mathcal{U}_i, \mathfrak{g})$

$$\omega^i_{\mathrm{loc}} = \Psi_i^{-1} \circ \omega \circ S_i$$

$s^j_i = S_j^{-1} \circ S_i : \mathrm{TLA}(\mathcal{U}_{ij}, \mathfrak{g}) \xrightarrow{\simeq} \mathrm{TLA}(\mathcal{U}_{ij}, \mathfrak{g})$ is an isomorphism of (trivial) Lie algebroids.
Define $\widehat{\alpha}^i_j : \Omega^q_{\mathrm{TLA}}(\mathcal{U}_{ij}, \mathfrak{g}) \to \Omega^q_{\mathrm{TLA}}(\mathcal{U}_{ij}, \mathfrak{g})$ by $\widehat{\alpha}^i_j(\omega^j_{\mathrm{loc}}) = \alpha^i_j \circ \omega^j_{\mathrm{loc}} \circ s^j_i$.

Proposition 35.

- *A family of local forms $\omega^i_{\mathrm{loc}} \in \Omega^\bullet_{TLA}(\mathcal{U}_i, \mathfrak{g})$ is a system of trivializations of a global form $\omega \in \Omega^\bullet(A, L)$ if and only if $\widehat{\alpha}^i_j(\omega^j_{\mathrm{loc}}) = \omega^i_{\mathrm{loc}}$ on any $\mathcal{U}_{ij} \neq \varnothing$.*
- *For any $\omega \in \Omega^\bullet(A, L)$ trivialized on $\mathcal{U}$ as ω_{loc}, one has $\widehat{\mathrm{d}}_{TLA}\omega_{\mathrm{loc}} = \Psi^{-1} \circ (\widehat{\mathrm{d}}\omega) \circ S$.*
- *$\widehat{\alpha}^i_j : \Omega^\bullet_{TLA}(\mathcal{U}_{ij}, \mathfrak{g}) \to \Omega^\bullet_{TLA}(\mathcal{U}_{ij}, \mathfrak{g})$ is an isomorphism of graded differential Lie algebras.*

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Gauge transformations on Atiyah Lie algebroids

Suppose G is connected and simply connected.
$\widehat{\omega} \in \Omega^1_{\mathrm{Lie}}(\mathcal{P}, \mathfrak{g})$ ➡ $\widehat{\omega}_{\mathfrak{g}_{\mathrm{equ}}} \in \Omega^1_{\mathrm{TLA}}(\mathcal{P}, \mathfrak{g})$, $\mathfrak{g}_{\mathrm{equ}}$-basic.

The gauge action $\widehat{\omega} \mapsto \widehat{\omega}^u$ induces

$$\widehat{\omega}_{\mathfrak{g}_{\mathrm{equ}}} \mapsto u \widehat{\omega}_{\mathfrak{g}_{\mathrm{equ}}} u^{-1} + u \widehat{\mathrm{d}}_{\mathrm{TLA}} u^{-1} \in \Omega^1_{\mathrm{TLA}}(\mathcal{P}, \mathfrak{g})$$

where

$$u \widehat{\mathrm{d}}_{\mathrm{TLA}} u^{-1} = u \mathrm{d} u^{-1} + u \theta u^{-1} - \theta$$

Notice that $u\theta u^{-1} - \theta = u[\theta, u^{-1}]$ is more or less "s" applied to u.

Proposition 36 (Ordinary gauge transformations)**.** *If $\widehat{\omega}_{\mathfrak{g}_{\mathrm{equ}}} = \omega^{\mathcal{P}} - \theta$ is an ordinary connection on $\Gamma_G(T\mathcal{P})$, this action reduces to the usual gauge transformation $\omega^{\mathcal{P}} \mapsto u \omega^{\mathcal{P}} u^{-1} + u \mathrm{d} u^{-1}$ on the (ordinay) connection 1-form $\omega^{\mathcal{P}}$.*

• •
Decomposition of a connection

$\widehat{\omega} \in \Omega^1(A, L)$ a generalized connection on A.

Definition 37 (Reduced kernel endomorphism)**.** The reduced kernel endomorphism $\tau \in \mathrm{End}(\mathcal{L})$ associated to $\widehat{\omega}$ is defined by

$$\tau = \widehat{\omega} \circ \iota + \mathrm{Id}_L.$$

τ vanishes iff $\widehat{\omega}$ is an ordinary connection on A ➡ measures the "non Yang-Mills" part.

τ is not a Lie morphism. Define $R_\tau(\gamma, \eta) = [\tau(\gamma), \tau(\eta)] - \tau([\gamma, \eta])$ for any $\gamma, \eta \in L$.

Let $\mathring{\omega} \in \Omega^1(A, L)$ be a fixed ordinary connection on A ("background connection").

Theorem 38. *$\widehat{\omega} \in \Omega^1(A, L)$ a connection and τ its reduced kernel endomorphism.*

$$\omega = \widehat{\omega} + \tau(\mathring{\omega})$$

is an ordinary connection on A.
The induced infinitesimal gauge action of L is the one on ordinary connections.

$\widehat{\omega}$ ordinary connection ➡ $\tau = 0$ ➡ $\omega = \widehat{\omega}$.
➡ $\mathring{\omega}$ only relevant for connections which are not ordinary connections.

Decompositions of curvature and covariant derivative

$\widehat{\omega} = \omega - \tau(\mathring{\omega})$ connection on A.
$\mathring{\nabla}, \nabla : \Gamma(T\mathcal{M}) \to A$ the splittings associated to the ordinary connections $\mathring{\omega}, \omega$.
$\mathring{R}, R \in \Omega^2(\mathcal{M}, \mathcal{L})$ the curvature 2-forms of $\mathring{\omega}, \omega$.
$\widehat{F} = R - \tau \circ \mathring{R} \in \Omega^2(\mathcal{M}, \mathcal{L})$ ➥ $\rho^* \widehat{F} \in \Omega^2(A, L)$.
For $X \in \Gamma(T\mathcal{M})$, define $\mathcal{D}_X \tau \in \mathrm{End}(\mathcal{L})$ by

$$(\mathcal{D}_X \tau)(\gamma) = [\nabla_X, \tau(\gamma)] - \tau([\mathring{\nabla}_X, \gamma])$$

for any $\gamma \in L$ ➥ $(\rho^* \mathcal{D}\tau) \circ \mathring{\omega} \in \Omega^2(A, L)$.
$\nabla^{\mathcal{E}}$ the (ordinary) covariant derivative induced on $\mathcal{E}$ by the (ordinary) connection ω.
For any $\varphi \in \Gamma(\mathcal{E})$, one has $\rho^* \phi(\nabla) \cdot \varphi = \rho^* \nabla^{\mathcal{E}} \varphi$.

Proposition 39 (Decomposition of the curvature and the covariant derivative)**.** *The curvature $\widehat{R} \in \Omega^2(A, L)$ of $\widehat{\omega}$ can be decomposed as*

$$\widehat{R} = \rho^* \widehat{F} - (\rho^* \mathcal{D}\tau) \circ \mathring{\omega} + R_\tau \circ \mathring{\omega}$$

The covariant derivative $\widehat{\nabla}^{\mathcal{E}} \varphi \in \Omega^1(A, \mathcal{E})$ can be decomposed as

$$\widehat{\nabla}^{\mathcal{E}} \varphi = \rho^* \phi(\nabla) \cdot \varphi - (\phi_L(\tau)\varphi) \circ \mathring{\omega}$$

Under infinitesimal gauge transformations, each term has homogeneous transformations.

"$\circ\mathring{\omega}$" ➥ along the mixed basis, "ρ^*" ➥ along $\Gamma(T\mathcal{M})$.

Decomposition of the action functional

$\widehat{\omega}$ connection on A, $\varphi \in \Gamma(\mathcal{E})$ matter field.

$$\mathcal{S}[\varphi, \widehat{\omega}] = \mathcal{S}_{\mathrm{Gauge}}[\widehat{\omega}] + \mathcal{S}_{\mathrm{Matter}}[\varphi, \widehat{\omega}] \quad \text{total action functional}$$

$\widehat{g} = (g, h, \nabla)$, with $\nabla \leftrightarrow \mathring{\omega} \in \Omega^1(A, L)$, metric on A.
The decomposition $\widehat{\omega} = \omega - \tau(\mathring{\omega})$ induces the decomposition:

$$\begin{aligned}
\mathcal{S}[\varphi, \widehat{\omega}] = \ & \langle \rho^* \widehat{F}, \star \rho^* \widehat{F} \rangle && \text{(1) spatial term: Yang-Mills like} \\
& + \langle (\rho^* \mathcal{D}\tau) \circ \mathring{\omega}, \star (\rho^* \mathcal{D}\tau) \circ \mathring{\omega} \rangle && \text{(2) mixed term: covariant derivative of } \tau \\
& + \langle R_\tau \circ \mathring{\omega}, \star R_\tau \circ \mathring{\omega} \rangle && \text{(3) algebraic term: potential for } \tau \\
& + \langle \rho^* \phi(\nabla) \cdot \varphi, \star \rho^* \phi(\nabla) \cdot \varphi \rangle && \text{(4) spatial term: covariant derivative of } \varphi \\
& + \langle (\phi_L(\tau)\varphi) \circ \mathring{\omega}, \star (\phi_L(\tau)\varphi) \circ \mathring{\omega} \rangle && \text{(5) algebraic term: coupling } \varphi \leftrightarrow \tau
\end{aligned}$$