Connections on transitive Lie algebroids: relationships with derivation-based NCG

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S. Lazzarini, T. M. Connections on Lie algebroids and on derivation-based non-commutative geometry. J. Geom. Phys., *62:387–402, 2012*

Motivations and objectives

Noncommutative geometry: (NCG)

➡ manage differential geometry and differential algebras (and operator algebras).

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Objectives of this talk:

- Facts about Lie algebroids (representation theory, connections).
- Differential structures and the notion of generalized connections.
- Two examples related to the ordinary theory of fiber bundles:
  - The Atiyah Lie algebroid of a principal fiber bundle.
  - The Lie algebroid of derivations of the endomorphisms algebra of a $SL(n)$-vector bundle.
- NCG of the endomorphisms algebra of a $SL(n)$-vector bundle.
- Compare NC connections with generalized connections on Lie algebroids.

Contents

Lie algebroids and differential calculi

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Definition (Lie algebroids – Geometric version)

A Lie algebroid $\mathcal{A}$ on $\mathcal{M}$ is a vector bundle $\mathcal{A} \xrightarrow{\pi} \mathcal{M}$ with a vector bundle map $\rho : \mathcal{A} \to T\mathcal{M}$ (the anchor of $\mathcal{A}$) and a Lie bracket $[-,-]$ on $\Gamma(\mathcal{A})$ (smooth sections of $\mathcal{A}$) fulfilling

$$[\mathfrak{X}, f\mathfrak{Y}] = f[\mathfrak{X},\mathfrak{Y}] + (\rho(\mathfrak{X})\cdot f)\mathfrak{Y} \qquad \rho([\mathfrak{X},\mathfrak{Y}]) = [\rho(\mathfrak{X}),\rho(\mathfrak{Y})]$$

for any $\mathfrak{X},\mathfrak{Y} \in \Gamma(\mathcal{A})$ and $f \in C^\infty(\mathcal{M})$.

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With $\mathsf{A} = \Gamma(\mathcal{A})$ one has:

Definition (Lie algebroids – Algebraic version)

A Lie algebroid A is a finite projective module over $C^\infty(\mathcal{M})$ equipped with a Lie bracket $[-,-]$ and a $C^\infty(\mathcal{M})$-linear Lie morphism $\rho : \mathsf{A} \to \Gamma(T\mathcal{M})$ such that

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Definition closed to some algebraic structures encountered in NCG.

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Proposition (The kernel of a transitive Lie algebroid)

Let A *be a transitive Lie algebroid.*

- $\mathsf{L} = \operatorname{Ker}\rho$ *is a Lie algebroid with null anchor on* $\mathcal{M}$*.*
- *The vector bundle* $\mathcal{L}$ *such that* $\mathsf{L} = \Gamma(\mathcal{L})$ *is a locally trivial bundle in Lie algebras.*
  ➡ *gives the Lie structure on* L*.*

One has the short exact sequence of Lie algebras and $C^\infty(\mathcal{M})$*-modules*

$$0 \longrightarrow \mathsf{L} \xrightarrow{\iota} \mathsf{A} \xrightarrow{\rho} \Gamma(T\mathcal{M}) \longrightarrow 0$$

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This short exact sequence is the key structure for various considerations.

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Symbol map: $\sigma : \mathrm{Diff}^1(\mathcal{E}) \to \mathrm{Hom}_{C^\infty(\mathcal{M})}(T^*\mathcal{M}, \mathrm{End}(\mathcal{E}))$ defined by

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$$\mathfrak{D}(\mathcal{E}) = \sigma^{-1}(\Gamma(T\mathcal{M}))$$

is a transitive Lie algebroid, with anchor σ and kernel $\mathbf{A}(\mathcal{E}) = \Gamma(\text{End}(\mathcal{E}))$:

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This is the Lie algebroid of derivations of $\mathcal{E}$.

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The kernel $\mathbf{A}(\mathcal{E})$ is an associative algebra and the Lie structure is the commutator.

Representations

$\mathsf{A} \xrightarrow{\rho} \Gamma(T\mathcal{M})$ a Lie algebroid and $\mathcal{E} \to \mathcal{M}$ a vector bundle.

Definition (Representation of a Lie algebroid)

A representation of A on $\mathcal{E}$ is a morphism of Lie algebroids $\phi : \mathsf{A} \to \mathfrak{D}(\mathcal{E})$.

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$\phi_{\mathsf{L}} : \mathsf{L} \to \mathbf{A}(\mathcal{E})$ is a $C^\infty(\mathcal{M})$-linear morphism of Lie algebras
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A representation for a Lie algebroid is like a right module for an algebra in NCG.

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Similar situation in NCG: gauge group defined only for a given right module.

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$\Omega^0(A) = C^\infty(\mathcal{M})$.
$\Omega^\bullet(A) = \bigoplus_{p\geq 0} \Omega^p(A)$ gets a differential $\widehat{d}_A : \Omega^p(A) \to \Omega^{p+1}(A)$:

$$(\widehat{d}_A \omega)(\mathfrak{X}_1, \dots, \mathfrak{X}_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} \rho(\mathfrak{X}_i) \cdot \omega(\mathfrak{X}_1, \dots \overset{i}{\vee} \dots, \mathfrak{X}_{p+1}) + \sum_{1\leq i<j\leq p+1} (-1)^{i+j} \omega([\mathfrak{X}_i, \mathfrak{X}_j], \mathfrak{X}_1, \dots \overset{i}{\vee} \dots \overset{j}{\vee} \dots, \mathfrak{X}_{p+1})$$

and a graded commutative product:

$$(\omega\eta)(\mathfrak{X}_1, \dots, \mathfrak{X}_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in \mathfrak{S}_{p+q}} (-1)^{|\sigma|} \omega(\mathfrak{X}_{\sigma(1)}, \dots, \mathfrak{X}_{\sigma(p)}) \eta(\mathfrak{X}_{\sigma(p+1)}, \dots, \mathfrak{X}_{\sigma(p+q)})$$

$(\Omega^\bullet(A), \widehat{d}_A)$ is a graded commutative differential algebra.

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$(\Omega^\bullet(A), \widehat{d}_A)$ is a graded commutative differential algebra.

This is the space of forms with values in $C^\infty(\mathcal{M})$.
Objects in this algebra will be called forms on A.

Forms with values in the kernel

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Definition (Forms with values in the kernel)

$\forall p \in \mathbb{N}$, $\Omega^p(\mathsf{A},\mathsf{L})$ is the space of $C^\infty(\mathcal{M})$-multilinear antisymmetric maps $\mathsf{A}^p \to \mathsf{L}$.

$\Omega^\bullet(\mathsf{A},\mathsf{L}) = \bigoplus_{p\geq 0} \Omega^p(\mathsf{A},\mathsf{L})$ gets a differential $\widehat{\mathrm{d}} : \Omega^p(\mathsf{A},\mathsf{L}) \to \Omega^{p+1}(\mathsf{A},\mathsf{L})$:

$$\begin{aligned}(\widehat{\mathrm{d}}\omega)(\mathfrak{X}_1, \dots, \mathfrak{X}_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} \mathrm{ad}_{\mathfrak{X}_i} \omega(\mathfrak{X}_1, \dots \overset{i}{\vee} \dots, \mathfrak{X}_{p+1}) \\ + \sum_{1\leq i<j\leq p+1} (-1)^{i+j} \omega([\mathfrak{X}_i, \mathfrak{X}_j], \mathfrak{X}_1, \dots \overset{i}{\vee} \dots \overset{j}{\vee} \dots, \mathfrak{X}_{p+1})\end{aligned}$$

and a graded Lie bracket $\Omega^p(\mathsf{A},\mathsf{L}) \otimes \Omega^q(\mathsf{A},\mathsf{L}) \to \Omega^{p+q}(\mathsf{A},\mathsf{L})$:

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Objects in this complex will be called forms on A with values in L.

Forms on A with values in $\mathsf{A}(\mathcal{E})$

$\phi : \mathsf{A} \to \mathfrak{D}(\mathcal{E})$ representation of the transitive Lie algebroid $0 \longrightarrow \mathsf{L} \xrightarrow{\iota} \mathsf{A} \xrightarrow{\rho} \Gamma(T\mathcal{M}) \longrightarrow 0$.

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This is the space of forms on A with values in $\mathbf{A}(\mathcal{E})$.
There are relations between $\Omega^\bullet(\mathsf{A})$, $\Omega^\bullet(\mathsf{A}, \mathsf{L})$ and $\Omega^\bullet(\mathsf{A}, \mathbf{A}(\mathcal{E}))$.

Example: Trivial Lie algebroids

Models for local trivializations of transitive Lie algebroids.

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Definition (Trivial Lie algebroid)

$\mathsf{A} = \Gamma(T\mathcal{M} \oplus (\mathcal{M} \times \mathfrak{g}))$ is a Lie algebroid for the following structures:

$$\rho(X \oplus \gamma) = X \qquad [X \oplus \gamma, Y \oplus \eta] = [X, Y] \oplus (X \cdot \eta - Y \cdot \gamma + [\gamma, \eta])$$

for any $X, Y \in \Gamma(T\mathcal{M})$ and $\gamma, \eta \in \Gamma(\mathcal{M} \times \mathfrak{g})$.

Notation $\mathsf{A} = \mathsf{TLA}(\mathcal{M}, \mathfrak{g})$ for this Lie algebroid (Trivial Lie Algebroid).

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The s.e.s.

$$0 \longrightarrow \mathsf{L} \xrightarrow{\iota} \mathsf{A} \xrightarrow{\rho} \Gamma(T\mathcal{M}) \longrightarrow 0$$

splits as Lie algebras and $C^\infty(\mathcal{M})$-modules.

Example: Trivial Lie algebroids (cont'd)

Let $\mathsf{A} = \mathsf{TLA}(\mathcal{M}, \mathfrak{g})$ and $\mathsf{L} = \Gamma(\mathcal{M} \times \mathfrak{g})$.
Then $\Omega^\bullet(\mathsf{A}, \mathsf{L}) = \Omega^\bullet_{\mathsf{TLA}}(\mathcal{M}, \mathfrak{g})$ is the total complex of the bicomplex

$$\Omega^{\bullet,\bullet}_{\mathsf{TLA}}(\mathcal{M}, \mathfrak{g}) = \Omega^\bullet(\mathcal{M}) \otimes \textstyle\bigwedge^\bullet \mathfrak{g}^* \otimes \mathfrak{g}$$

with the two differentials:

$$\mathsf{d} : \Omega^{\bullet,\bullet}_{\mathsf{TLA}}(\mathcal{M}, \mathfrak{g}) \to \Omega^{\bullet+1,\bullet}_{\mathsf{TLA}}(\mathcal{M}, \mathfrak{g})$$

de Rham diff. on $\Omega^\bullet(\mathcal{M})$

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$\widehat{\mathrm{d}} = \widehat{\mathrm{d}}_{\mathsf{TLA}} = \mathrm{d} + \mathrm{s}$ on $\Omega^\bullet_{\mathsf{TLA}}(\mathcal{M}, \mathfrak{g})$ is given by

$$\begin{aligned}
(\widehat{\mathrm{d}}_{\mathsf{TLA}}\omega)(X_1 \oplus \gamma_1, \dots, X_{p+1} \oplus \gamma_{p+1}) = \\
&\sum_{i=1}^{p+1} (-1)^{i+1} X_i \cdot \omega(X_1 \oplus \gamma_1, \dots \overset{i}{\vee} \dots, X_{p+1} \oplus \gamma_{p+1}) \\
&+ \sum_{i=1}^{p+1} (-1)^{i+1} [\gamma_i, \omega(X_1 \oplus \gamma_1, \dots \overset{i}{\vee} \dots, X_{p+1} \oplus \gamma_{p+1})] \\
&+ \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} \omega([X_i \oplus \gamma_i, X_j \oplus \gamma_j], X_1 \oplus \gamma_1, \dots \overset{i}{\vee} \dots \overset{j}{\vee} \dots, X_{p+1} \oplus \gamma_{p+1})
\end{aligned}$$

for any $\omega \in \Omega^p_{\mathsf{TLA}}(\mathcal{M}, \mathfrak{g})$ and any $X_i \oplus \gamma_i \in \mathsf{TLA}(\mathcal{M}, \mathfrak{g})$.

Example: Trivial Lie algebroids (cont'd)

Let $\mathsf{A} = \mathsf{TLA}(\mathcal{M}, \mathfrak{g})$ and $\mathsf{L} = \Gamma(\mathcal{M} \times \mathfrak{g})$.
E vector space and $\eta(\xi) : E \to E$ a linear representation of $\mathfrak{g}$.
$\mathcal{L}(E)$ algebra of endomorphisms of E, then $\eta : \mathfrak{g} \to \mathcal{L}(E)$.

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There is a natural representation of A on $\mathcal{E} = \mathcal{M} \times E$:
For $X \oplus \gamma \in \mathsf{A}$, $\phi(X \oplus \gamma)s = X{\cdot}s + \eta(\gamma)s$ for any $s \in \Gamma(\mathcal{E})$.

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The differential calculus $\Omega^\bullet_{\mathsf{TLA}}(\mathcal{M}, \mathfrak{g}, E) = \Omega^\bullet(\mathsf{A}, \mathbf{A}(\mathcal{E}))$ is the total complex of $\Omega^\bullet(\mathcal{M}) \otimes \bigwedge^\bullet \mathfrak{g}^* \otimes \mathcal{L}(E)$ with differential

$$
\begin{aligned}
(\widehat{\mathrm{d}}_{\mathsf{TLA},\mathcal{E}}\,\omega)(X_1 \oplus \gamma_1, \dots, X_{p+1} \oplus \gamma_{p+1}) = \\
\sum_{i=1}^{p+1} (-1)^{i+1} X_i{\cdot}\omega(X_1 \oplus \gamma_1, \dots \overset{i}{\vee} \dots, X_{p+1} \oplus \gamma_{p+1}) \\
+ \sum_{i=1}^{p+1} (-1)^{i+1} [\eta(\gamma_i), \omega(X_1 \oplus \gamma_1, \dots \overset{i}{\vee} \dots, X_{p+1} \oplus \gamma_{p+1})] \\
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Inner product and Lie derivative

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Definition (Inner product and Lie derivative)

For any $\mathfrak{X} \in \mathsf{A}$ and $p \geq 1$, we define the inner product

$$i_{\mathfrak{X}} : \Omega^p(\mathsf{A}, \mathsf{L}) \to \Omega^{p-1}(\mathsf{A}, \mathsf{L})$$

by $(i_{\mathfrak{X}}\omega)(\mathfrak{X}_1, \dots, \mathfrak{X}_p) = \omega(\mathfrak{X}, \mathfrak{X}_1, \dots, \mathfrak{X}_p)$.
$i_{\mathfrak{X}}$ is zero on $\Omega^0(\mathsf{A}, \mathsf{L}) = \mathsf{L}$.
For any $p \geq 0$,

$$L_{\mathfrak{X}} = \widehat{\mathrm{d}} i_{\mathfrak{X}} + i_{\mathfrak{X}} \widehat{\mathrm{d}} : \Omega^p(\mathsf{A}, \mathsf{L}) \to \Omega^p(\mathsf{A}, \mathsf{L})$$

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$$(L_{\mathfrak{X}}\omega)(\mathfrak{X}_1,\dots,\mathfrak{X}_p) = \mathrm{ad}_{\mathfrak{X}}\omega(\mathfrak{X}_1,\dots,\mathfrak{X}_p) - \sum_{i=1}^{p} \omega(\mathfrak{X}_1,\dots,[\mathfrak{X},\mathfrak{X}_i],\dots,\mathfrak{X}_p)$$

Cartan operation

For any $\mathfrak{X}, \mathfrak{Y} \in A$ one has

$$i_{\mathfrak{X}} i_{\mathfrak{Y}} + i_{\mathfrak{Y}} i_{\mathfrak{X}} = 0 \qquad [L_{\mathfrak{X}}, i_{\mathfrak{Y}}] = i_{[\mathfrak{X}, \mathfrak{Y}]} \qquad [L_{\mathfrak{X}}, L_{\mathfrak{Y}}] = L_{[\mathfrak{X}, \mathfrak{Y}]}$$

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$\mathfrak{h}$ a Lie algebra.

Definition (Cartan operation of $\mathfrak{h}$)

A Cartan operation of $\mathfrak{h}$ on $(\Omega^\bullet(\mathsf{A}, \mathsf{L}), \widehat{\mathrm{d}})$ is given by the following data:
$\forall \mathfrak{X} \in \mathfrak{h}$, $\forall p \geq 1$, there is a map $i_{\mathfrak{X}} : \Omega^p(\mathsf{A}, \mathsf{L}) \to \Omega^{p-1}(\mathsf{A}, \mathsf{L})$ such that the above relations hold for any $\mathfrak{X}, \mathfrak{Y} \in \mathfrak{h}$, with $L_{\mathfrak{X}} = \widehat{\mathrm{d}} i_{\mathfrak{X}} + i_{\mathfrak{X}} \widehat{\mathrm{d}}$.

Such a Cartan operation on $(\Omega^\bullet(\mathsf{A}, \mathsf{L}), \widehat{\mathrm{d}})$ is denoted by $(\mathfrak{h}, i, L)$.

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Example (Cartan operation of the kernel)

As a Lie algebroid, $\iota \circ \mathsf{L} \subset \mathsf{A}$
➡ one can restrict the operations i and L defined with $\mathfrak{X} \in \mathsf{A}$ to $\mathfrak{X} \in \iota \circ \mathsf{L}$.
This defines the Cartan operation of L (as a Lie algebra) on $(\Omega^\bullet(\mathsf{A},\mathsf{L}),\widehat{\mathrm{d}})$.

Horizontal, invariant and basic elements

Definition (Horizontal, invariant and basic elements)

Let $(\mathfrak{h}, i, L)$ be a Cartan operation on $(\Omega^\bullet(\mathsf{A}, \mathsf{L}), \widehat{\mathrm{d}})$. Then

- The horizontal elements in $\Omega^\bullet(\mathsf{A}, \mathsf{L})$ are the forms ω such that $i_{\mathfrak{X}}\omega = 0$ for any $\mathfrak{X} \in \mathfrak{h}$. $\Omega^\bullet(\mathsf{A}, \mathsf{L})_{\mathrm{Hor}}$ is the graded subspace of horizontal elements.
- The invariant elements in $\Omega^\bullet(\mathsf{A}, \mathsf{L})$ are the forms ω such that $L_{\mathfrak{X}}\omega = 0$ for any $\mathfrak{X} \in \mathfrak{h}$. $\Omega^\bullet(\mathsf{A}, \mathsf{L})_{\mathrm{Inv}}$ is the graded subspace of invariant elements.
- The basic elements in $\Omega^\bullet(\mathsf{A}, \mathsf{L})$ are the forms which are both horizontal and invariant. $\Omega^\bullet(\mathsf{A}, \mathsf{L})_{\mathrm{Basic}}$ is the graded subspace of basic elements.

$\Omega^\bullet(\mathsf{A}, \mathsf{L})_{\mathrm{Inv}}$ and $\Omega^\bullet(\mathsf{A}, \mathsf{L})_{\mathrm{Basic}}$ are differential subcomplexes of $(\Omega^\bullet(\mathsf{A}, \mathsf{L}), \widehat{\mathrm{d}})$.

Connections and their generalizations

Ordinary connections : definition

$0 \longrightarrow \mathsf{L} \xrightarrow{\iota} \mathsf{A} \xrightarrow{\rho} \Gamma(T\mathcal{M}) \longrightarrow 0$ a transitive Lie algebroid.

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Definition (Connection on a transitive Lie algebroid)

A connection on A is a splitting $\nabla : \Gamma(T\mathcal{M}) \to \mathsf{A}$ as $C^\infty(\mathcal{M})$-modules of the s.e.s.

$$0 \longrightarrow \mathsf{L} \xrightarrow[\iota]{} \mathsf{A} \underset{\nabla}{\overset{\rho}{\rightleftarrows}} \Gamma(T\mathcal{M}) \longrightarrow 0$$

Its curvature is the obstruction of being a morphism of Lie algebras:

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

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One can associate to ∇ an algebraic object which permits concrete computations.

$\mathfrak{X} \in \mathsf{A}$, let $X = \rho(\mathfrak{X})$

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- ➡ $\mathfrak{X} - \nabla_X \in \operatorname{Ker} \rho$
- ➡ $\exists!\ \alpha(\mathfrak{X}) \in \mathsf{L}$ such that

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The map $\alpha : \mathsf{A} \to \mathsf{L}$ is a morphism of $C^\infty(\mathcal{M})$-modules.

There is a notion of (infinitesimal) gauge transformations on connections.

Ordinary connections : the connection 1-form

∇ connection on the transitive Lie algebroid A, $\mathfrak{X} = \nabla_X - \iota \circ \alpha(\mathfrak{X})$.

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Proposition (Connection 1-form and curvature in algebraic terms)

The associated morphism $\alpha : \mathsf{A} \to \mathsf{L}$ *is an element of* $\Omega^1(\mathsf{A}, \mathsf{L})$ *normalized on* $\iota \circ \mathsf{L}$ *by*

$$\alpha \circ \iota(\ell) = -\ell$$

for any $\ell \in \mathsf{L}$.

Any $\alpha \in \Omega^1(\mathsf{A}, \mathsf{L})$ *normalized as before defines a unique connection on* A.

The 2*-form*

$$\widehat{R} = \widehat{\mathrm{d}}\alpha + \tfrac{1}{2}[\alpha, \alpha]$$

is horizontal for the Cartan operation of L *on* $(\Omega^\bullet(\mathsf{A}, \mathsf{L}), \widehat{\mathrm{d}})$. *With usual notations:*

$$\iota \circ \widehat{R}(\mathfrak{X}, \mathfrak{Y}) = \iota\left((\widehat{\mathrm{d}}\alpha)(\mathfrak{X}, \mathfrak{Y}) + [\alpha(\mathfrak{X}), \alpha(\mathfrak{Y})] \right) = R(X, Y)$$

$\widehat{R}$ *satisfies the Bianchi identity*

$$\widehat{\mathrm{d}}\widehat{R} + [\alpha, \widehat{R}] = 0$$

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$\alpha \in \Omega^1(\mathsf{A}, \mathsf{L})$ is the connection 1-form of ∇
("connection reform" in the book by Mackenzie).
$\widehat{R} \in \Omega^2(\mathsf{A}, \mathsf{L})$ is the curvature 2-form of ∇.

A-connections on $\mathcal{E}$

First generalization of connections

$\phi : \mathsf{A} \to \mathfrak{D}(\mathcal{E})$ a representation of the transitive Lie algebroid A.

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An A-connection on $\mathcal{E}$ is a linear map $\widehat{\nabla}^{\mathcal{E}} : \mathsf{A} \to \mathfrak{D}(\mathcal{E})$ compatible with the anchor maps such that

$$\widehat{\nabla}^{\mathcal{E}}_{f\mathfrak{X}} s = f \widehat{\nabla}^{\mathcal{E}}_{\mathfrak{X}} s \qquad \widehat{\nabla}^{\mathcal{E}}_{\mathfrak{X}}(s_1 + s_2) = \widehat{\nabla}^{\mathcal{E}}_{\mathfrak{X}} s_1 + \widehat{\nabla}^{\mathcal{E}}_{\mathfrak{X}} s_2 \qquad \widehat{\nabla}^{\mathcal{E}}_{\mathfrak{X}}(fs) = (\rho(\mathfrak{X})\cdot f)s + f \widehat{\nabla}^{\mathcal{E}}_{\mathfrak{X}} s$$

$\forall \mathfrak{X} \in \mathsf{A}$, $\forall f \in C^\infty(\mathcal{M})$ and $\forall s, s_1, s_2 \in \Gamma(\mathcal{E})$.

The curvature of $\widehat{\nabla}^{\mathcal{E}}$ is defined as:

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An A-connection on $\mathcal{E}$ can also be seen as a generalized "representation".
The curvature measures the obstruction to be a morphism of Lie algebras.
Notice: $\widehat{R}^{\mathcal{E}}(\mathfrak{X}, \mathfrak{Y}) \in \mathsf{L}$.

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Notice: $\widehat{R}^{\mathcal{E}}(\mathfrak{X}, \mathfrak{Y}) \in \mathsf{L}$.
There is a notion of gauge transformation on A-connections on $\mathcal{E}$: gauge group $\mathrm{Aut}(\mathcal{E})$.

Connection 1-form of a A-connection on $\mathcal{E}$

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\begin{array}{ccccccccc}
0 & \longrightarrow & \mathsf{L} & \xrightarrow{\iota} & \mathsf{A} & \xrightarrow{\rho} & \Gamma(T\mathcal{M}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi_{\mathsf{L}}} & & \downarrow{\scriptstyle \phi} & & \| & & \\
0 & \longrightarrow & \mathbf{A}(\mathcal{E}) & \xrightarrow{\iota} & \mathfrak{D}(\mathcal{E}) & \xrightarrow{\sigma} & \Gamma(T\mathcal{M}) & \longrightarrow & 0
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➞ $\exists!\ \omega^{\mathcal{E}} \in \Omega^1(\mathsf{A}, \mathbf{A}(\mathcal{E}))$ such that $\widehat{\nabla}^{\mathcal{E}}_{\mathfrak{X}} = \phi(\mathfrak{X}) + \iota \circ \omega^{\mathcal{E}}(\mathfrak{X})$.

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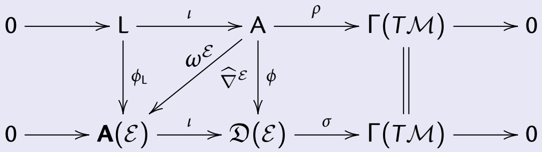

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Proposition (Connection 1-form and curvature 2-form)

An A-connection on $\mathcal{E}$, $\widehat{\nabla}^{\mathcal{E}} : \mathsf{A} \to \mathfrak{D}(\mathcal{E})$, is completely determined by the 1-form $\omega^{\mathcal{E}} \in \Omega^1(\mathsf{A}, \mathbf{A}(\mathcal{E}))$ with the above relation.
One has $\widehat{R}^{\mathcal{E}} \in \Omega^2(\mathsf{A}, \mathbf{A}(\mathcal{E}))$ with

$$\widehat{R}^{\mathcal{E}}(\mathfrak{X}, \mathfrak{Y}) = (\widehat{\mathrm{d}}_{\mathcal{E}} \omega^{\mathcal{E}})(\mathfrak{X}, \mathfrak{Y}) + [\omega^{\mathcal{E}}(\mathfrak{X}), \omega^{\mathcal{E}}(\mathfrak{Y})]$$

$\widehat{R}^{\mathcal{E}}$ satisfies the Bianchi identity

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$\omega^{\mathcal{E}}$ is the connection 1-form of $\widehat{\nabla}^{\mathcal{E}}$ and $\widehat{R}^{\mathcal{E}}$ its curvature 2-form.

Covariant derivative: ordinary connections

$\nabla^{\mathcal{E}}$ (ordinary) connection on $\mathfrak{D}(\mathcal{E})$:

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Proposition (Connections on $\mathcal{E}$ as A-connections on $\mathcal{E}$)

$\nabla^{\mathcal{E}} \mapsto \widehat{\nabla}^{\mathcal{E}}$ *injects the space of connections on* $\mathfrak{D}(\mathcal{E})$ *into the space of* A*-connections on* $\mathcal{E}$. *This injection is compatible with the notions of curvature and gauge transformations.*

The notion of A-connections on $\mathcal{E}$ generalizes the notion of connections on $\mathfrak{D}(\mathcal{E})$.

Generalized connections

Second generalization of connections

$\phi : \mathsf{A} \to \mathfrak{D}(\mathcal{E})$ a representation of the transitive Lie algebroid A.

Definition (Induced connections)

An A-connection on $\mathcal{E}$ is an *induced connection* if its associated 1-form $\omega^{\mathcal{E}} \in \Omega^1(\mathsf{A}, \mathbf{A}(\mathcal{E}))$ can be factorized through ϕ_{L} as $\omega^{\mathcal{E}} = \phi_{\mathsf{L}} \circ \omega$ for a 1-form $\omega \in \Omega^1(\mathsf{A}, \mathsf{L})$.

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Definition (Generalized connections)

A 1-form ω is called a generalized connection 1-form on the transitive Lie algebroid A. Its curvature is defined as the 2-form $\widehat{R} = \widehat{\mathrm{d}}\omega + [\omega, \omega] \in \Omega^2(\mathsf{A}, \mathsf{L})$.

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The curvatures are related by $\widehat{R}^{\mathcal{E}} = \phi_{\mathsf{L}}(\widehat{R})$.

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The curvatures are related by $\widehat{R}^{\mathcal{E}} = \phi_{\mathsf{L}}(\widehat{R})$.

A generalized connection can induce A-connections on any representations. Compare this with connection on a principal fiber bundle which gives rise to covariant derivatives on associated vector bundles.

Covariant derivative, ordinary connections

$\omega \in \Omega^1(\mathsf{A},\mathsf{L})$ a generalized connection on A.

Proposition (Covariant derivative of a generalized connection)

$$\widehat{D} : \Omega^\bullet(\mathsf{A},\mathsf{L}) \to \Omega^{\bullet+1}(\mathsf{A},\mathsf{L}) \qquad \widehat{D}\eta = \widehat{\mathrm{d}}\omega + [\omega,\eta]$$

satisfies $\widehat{D}^2\eta = [\widehat{R},\eta]$*. The Bianchi identity is equivalent to* $\widehat{D}\widehat{R} = 0$*.*

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Proposition (Ordinary connections as generalized connections)

As a 1-form in $\Omega^1(\mathsf{A}, \mathsf{L})$*,* α^∇ *defines a generalized connection on* A*.*
The curvature of ∇ *identifies as a 2-form in* $\Omega^2(\mathsf{A}, \mathsf{L})$ *with the curvature 2-form of the generalized connection.*
Gauge transformations on the connection ∇ *are compatible with this identification.*

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The notion of generalized connections extends the notion of connections on A.
Contrary to connections on $\mathfrak{D}(\mathcal{E})$, there is no "$\widehat{\nabla}$" operator.

Two examples

The Atiyah Lie algebroid

$\mathcal{P} \xrightarrow{\pi} \mathcal{M}$ a G-principal fiber bundle. $\mathfrak{g}$ the Lie algebra of G.
$\mathrm{R}_g : \mathcal{P} \to \mathcal{P}$, $\mathrm{R}_g(p) = p{\cdot}g$, the right action of G on $\mathcal{P}$.

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$$\Gamma_G(T\mathcal{P}) = \{\mathsf{X} \in \Gamma(T\mathcal{P}) \,/\, \mathsf{R}_{g*}\mathsf{X} = \mathsf{X} \text{ for all } g \in G\}$$
$$\Gamma_G(\mathcal{P}, \mathfrak{g}) = \{v : P \to \mathfrak{g} \,/\, v(p{\cdot}g) = \mathrm{Ad}_{g^{-1}} v(p) \text{ for all } g \in G\}$$

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$$\iota : \Gamma_G(\mathcal{P}, \mathfrak{g}) \to \Gamma_G(T\mathcal{P})$$

induced by $\mathfrak{g} \ni \xi \mapsto \xi^{\mathcal{P}}$ (fundamental vertical vector field on $\mathcal{P}$ associated to $\xi \in \mathfrak{g}$ for R).
$\Gamma_G(\mathcal{P}, \mathfrak{g})$ is a subspace of *vertical vector fields* in $\Gamma_G(T\mathcal{P})$.

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Proposition (Differential on the space of forms)

For any $\mathsf{X}\in\Gamma_G(T\mathcal{P})$ and any $v\in\Gamma_G(\mathcal{P},\mathfrak{g})$, one has $[\mathsf{X},\iota(v)]=\iota(\mathsf{X}\cdot v)$.
➔ *the differential $\widehat{\mathrm{d}}$ on $\Omega^\bullet_{\mathrm{Lie}}(\mathcal{P},\mathfrak{g})$ takes the explicit form*

$$(\widehat{\mathrm{d}}\omega)(\mathsf{X}_1,\dots,\mathsf{X}_{p+1}) = \sum_{i=1}^{p+1}(-1)^{i+1}\mathsf{X}_i\cdot\omega(\mathsf{X}_1,\dots\overset{i}{\vee}\dots,\mathsf{X}_{p+1}) + \sum_{1\leq i<j\leq p+1}(-1)^{i+j}\omega([\mathsf{X}_i,\mathsf{X}_j],\mathsf{X}_1,\dots\overset{i}{\vee}\dots\overset{j}{\vee}\dots,\mathsf{X}_{p+1})$$

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Proposition (Identification of the differential calculus)

As differential graded complexes, $(\Omega^\bullet_{\mathsf{Lie}}(\mathcal{P}, \mathfrak{g}), \widehat{\mathrm{d}})$ *and* $(\Omega^\bullet_{\mathsf{TLA}}(\mathcal{P}, \mathfrak{g})_{\mathfrak{g}_{\text{equ}}}, \widehat{\mathrm{d}}_{\mathsf{TLA}})$ *are isomorphic.*

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Our result is more interesting:

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- it has practical applications on connections.

Connections and generalized connections

Proposition (Connections on the Atiyah Lie algebroid)

*Connections on the Atiyah Lie algebroid $\Gamma_G(T\mathcal{P})$ are exactly connections on $\mathcal{P}$.
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Generalized connections are written more generally as $\mathfrak{g}_{\mathrm{equ}}$-basic 1-forms

$$\widehat{\omega}=\omega+\phi \in \Omega^1_{\mathrm{TLA}}(\mathcal{P},\mathfrak{g}) = (\Omega^1(\mathcal{P})\otimes\mathfrak{g})\oplus(C^\infty(\mathcal{P})\otimes\bigwedge^1\mathfrak{g}^*\otimes\mathfrak{g})$$

The normalization $\omega(\xi^{\mathcal{P}})=\xi$ is no more assumed.

Representation theory

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$$\phi : \Gamma_G(T\mathcal{P}) \to \mathfrak{D}(\mathcal{E}) \qquad\qquad \phi(\mathsf{X})(s) = \mathsf{X} \cdot s$$

is a representation of the Lie algebroid $\Gamma_G(T\mathcal{P})$ on $\mathcal{E}$.

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$\mathcal{E} = \mathcal{P} \times_\ell E$ the associated vector bundle.

$$\Gamma(\mathcal{E}) = \{s : \mathcal{P} \to E \,/\, s(p\cdot g) = \ell_{g^{-1}} s(p) \text{ for all } g \in G\}$$

$$\phi : \Gamma_G(T\mathcal{P}) \to \mathfrak{D}(\mathcal{E}) \qquad\qquad \phi(X)(s) = X\cdot s$$

is a representation of the Lie algebroid $\Gamma_G(T\mathcal{P})$ on $\mathcal{E}$.

The kernel $\mathbf{A}(\mathcal{E})$ of $\mathfrak{D}(\mathcal{E})$ is

$$\mathbf{A}(\mathcal{E}) = \{a : \mathcal{P} \to \mathcal{L}(E) \,/\, a(p\cdot g) = \ell_{g^{-1}} \circ a(p) \circ \ell_g \text{ for all } g \in G\}$$

ϕ applied to $v \in \Gamma_G(\mathcal{P}, \mathfrak{g})$ is just $\eta(v) : \mathcal{P} \to \mathcal{L}(E)$.

Representation theory

E vector space and $\ell_g : E \to E$ a linear representation of G.
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The theory of representation of the transitive Lie algebroid $\Gamma_G(T\mathcal{P})$ is the theory of associated vector bundles.

Forms with values in $A(\mathcal{E})$

Denote as before $\Omega^\bullet_{\mathsf{TLA}}(\mathcal{P}, \mathfrak{g}, E)$ the total complex of $\Omega^\bullet(\mathcal{P}) \otimes \bigwedge^\bullet \mathfrak{g}^* \otimes \mathcal{L}(E)$, equipped with its differential $\widehat{\mathrm{d}}_{\mathsf{TLA},\mathcal{E}}$.

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$((\Omega^\bullet_{\mathrm{TLA}}(\mathcal{P}, \mathfrak{g}, E))_{\mathfrak{g}_{\mathrm{equ}}}, \widehat{\mathrm{d}}_{\mathrm{TLA},\mathcal{E}})$ the graded differential sub-algebra of basic elements.

Proposition (Identification of $(\Omega^\bullet(\Gamma_G(T\mathcal{P}), \mathbf{A}(\mathcal{E})), \widehat{\mathrm{d}}_{\mathcal{E}})$)

As differential graded algebras, $(\Omega^\bullet(\Gamma_G(T\mathcal{P}), \mathbf{A}(\mathcal{E})), \widehat{\mathrm{d}}_{\mathcal{E}}) \simeq ((\Omega^\bullet_{\mathrm{TLA}}(\mathcal{P}, \mathfrak{g}, E))_{\mathfrak{g}_{\mathrm{equ}}}, \widehat{\mathrm{d}}_{\mathrm{TLA},\mathcal{E}})$.

Derivations of an associative algebra

$\mathbf{A}$ associative algebra with unit denoted by $\mathbb{1}$. $\mathcal{Z}(\mathbf{A})$ its center.

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Definition (Vector space of derivations of A)

The vector space of derivations of $\mathbf{A}$ is the space
$\operatorname{Der}(\mathbf{A}) = \{\mathfrak{X} : \mathbf{A} \to \mathbf{A} \;/\; \mathfrak{X} \text{ linear}, \mathfrak{X}\cdot(ab) = (\mathfrak{X}\cdot a)b + a(\mathfrak{X}\cdot b), \forall a, b \in \mathbf{A}\}$

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Proposition (Structure of Der(A))

- $\text{Der}(\mathbf{A})$ *is a Lie algebra for the bracket* $[\mathfrak{X}, \mathfrak{Y}]\cdot a = \mathfrak{X}\cdot\mathfrak{Y}\cdot a - \mathfrak{Y}\cdot\mathfrak{X}\cdot a$ *for any* $\mathfrak{X}, \mathfrak{Y} \in \text{Der}(\mathbf{A})$.
- *It is a* $\mathcal{Z}(\mathbf{A})$*-module for the product* $(f\mathfrak{X})\cdot a = f(\mathfrak{X}\cdot a)$ *for any* $f \in \mathcal{Z}(\mathbf{A})$ *and* $\mathfrak{X} \in \text{Der}(\mathbf{A})$.
- *The subspace* $\text{Int}(\mathbf{A}) = \{\text{ad}_a : b \mapsto [a, b] \ / \ a \in \mathbf{A}\} \subset \text{Der}(\mathbf{A})$ *(inner derivations) is a Lie ideal and a* $\mathcal{Z}(\mathbf{A})$*-submodule.*
- *With* $\text{Out}(\mathbf{A}) = \text{Der}(\mathbf{A})/\text{Int}(\mathbf{A})$*, there is a s.e.s. of Lie algebras and* $\mathcal{Z}(\mathbf{A})$*-modules*

$$0 \longrightarrow \text{Int}(\mathbf{A}) \longrightarrow \text{Der}(\mathbf{A}) \longrightarrow \text{Out}(\mathbf{A}) \longrightarrow 0$$

The endomorphisms algebra of a $SL(n)$-vector bundle

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The short exact sequence

$$0 \longrightarrow \mathbf{A}_0 \xrightarrow{\ \mathrm{ad}\ } \mathrm{Der}(\mathbf{A}) \xrightarrow{\ \rho\ } \Gamma(T\mathcal{M}) \longrightarrow 0$$

defines $\mathrm{Der}(\mathbf{A})$ as a transitive Lie algebroid over $\mathcal{M}$, with $\iota = \mathrm{ad}$.

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There is a natural representation of the Lie algebroid $\text{Der}(\mathbf{A})$ on $\mathcal{E}$ for which the restriction to the kernel is the inclusion $\phi_{\mathbf{A}_0} : \mathbf{A}_0 \hookrightarrow \mathbf{A}(\mathcal{E}) = \mathbf{A}$.

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Proposition (Der(A) and $\Gamma_G(T\mathcal{P})$)

As Lie algebroids $\text{Der}(\mathbf{A})$ *and* $\Gamma_G(T\mathcal{P})$ *are isomorphic.*
The representations on $\mathfrak{D}(\mathcal{E})$ *are the same.*

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What is different: the way to manage “vertical” vector fields.

Relationships with noncommutative geometry

Derivation-based differential calculus

$\mathbf{A}$ associative algebra with unit denoted by $\mathbb{1}$. $\mathcal{Z}(\mathbf{A})$ its center.

Definition (The derivation-based differential calculus)

$\forall p \in \mathbb{N}$, $\Omega^p_{\text{Der}}(\mathbf{A})$ is the space of $\mathcal{Z}(\mathbf{A})$-multilinear antisymmetric maps $\text{Der}(\mathbf{A})^p \to \mathbf{A}$
$\Omega^0_{\text{Der}}(\mathbf{A}) = \mathbf{A}$.
$\Omega^\bullet_{\text{Der}}(\mathbf{A}) = \bigoplus_{p\geq 0} \Omega^p_{\text{Der}}(\mathbf{A})$ get a differential

$$\widehat{d}_{NC}\omega(\mathfrak{X}_1, \dots, \mathfrak{X}_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} \mathfrak{X}_i \cdot \omega(\mathfrak{X}_1, \dots \overset{i}{\vee} \dots, \mathfrak{X}_{p+1}) + \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} \omega([\mathfrak{X}_i, \mathfrak{X}_j], \dots \overset{i}{\vee} \dots \overset{j}{\vee} \dots, \mathfrak{X}_{p+1})$$

and a graded product:

$$(\omega\eta)(\mathfrak{X}_1, \dots, \mathfrak{X}_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in \mathfrak{S}_{p+q}} (-1)^{|\sigma|} \omega(\mathfrak{X}_{\sigma(1)}, \dots, \mathfrak{X}_{\sigma(p)}) \eta(\mathfrak{X}_{\sigma(p+1)}, \dots, \mathfrak{X}_{\sigma(p+q)})$$

$(\Omega^\bullet_{\text{Der}}(\mathbf{A}), \widehat{d}_{NC})$ is a graded differential algebra.

Example I

Example (The algebra of smooth functions $\mathbf{A} = C^\infty(\mathcal{M})$)

$\mathcal{M}$ a smooth manifold. $\mathbf{A} = C^\infty(\mathcal{M})$.

- $\mathcal{Z}(\mathbf{A}) = C^\infty(\mathcal{M})$.
- $\mathrm{Der}(\mathbf{A}) = \Gamma(T\mathcal{M})$. There are no inner derivations, $\mathrm{Int}(\mathbf{A}) = 0$, so that $\mathrm{Out}(\mathbf{A}) = \Gamma(T\mathcal{M})$.
- This graded differential algebra coincides with the graded differential algebra of de Rham forms on $\mathcal{M}$
$$\Omega^\bullet_{\mathrm{Der}}(\mathbf{A}) = \Omega^\bullet(\mathcal{M})$$

This example has motivated the general definitions.

Example II

Example (The matrix algebra $A = M_n(\mathbb{C}) = M_n$)

- $\mathcal{Z}(M_n) = \mathbb{C}$.
- $\mathrm{Der}(M_n) = \mathrm{Int}(M_n) \simeq \mathfrak{sl}_n = \mathfrak{sl}(n,\mathbb{C})$ (traceless matrices).
  $\gamma \in \mathfrak{sl}_n(\mathbb{C})$ defines the inner derivation $\mathrm{ad}_\gamma : a \mapsto [\gamma, a]$.
- $\Omega^\bullet_{\mathrm{Der}}(M_n) \simeq M_n \otimes \bigwedge^\bullet \mathfrak{sl}_n^*$.
  The differential $\widehat{\mathrm{d}}_{\mathrm{NC}}$ is the Chevalley-Eilenberg differential for the adjoint representation of $\mathfrak{sl}_n$ on M_n.
- $\exists$ canonical noncommutative 1-form $i\theta \in \Omega^1_{\mathrm{Der}}(M_n)$: for any $\gamma \in M_n(\mathbb{C})$
  $$i\theta(\mathrm{ad}_\gamma) = \gamma - \tfrac{1}{n}\mathrm{Tr}(\gamma)\mathbb{1}$$
  It realizes the explicit isomorphism $\mathrm{Int}(M_n(\mathbb{C})) \xrightarrow{\simeq} \mathfrak{sl}_n$.
- $\widehat{\mathrm{d}}_{\mathrm{NC}}(i\theta) - (i\theta)^2 = 0$ ➞ $i\theta$ looks very much like the Maurer-Cartan form in the geometry of Lie groups (here $SL_n(\mathbb{C})$).
- For any $a \in M_n$, one has $\widehat{\mathrm{d}}_{\mathrm{NC}} a = [i\theta, a] \in \Omega^1_{\mathrm{Der}}(M_n)$. No longer true in higher degrees.

This example is purely noncommutative. It is well studied.

Noncommutative connections

Noncommutative connections are defined on a right **A**-module ***M***.

Definition (Noncommutative connections and curvature)

A noncommutative connection on the right **A**-module ***M*** is a linear map $\widehat{\nabla}_{\mathfrak{X}} : \boldsymbol{M} \to \boldsymbol{M}$, defined for any $\mathfrak{X} \in \text{Der}(\mathbf{A})$, such that $\forall \mathfrak{X}, \mathfrak{Y} \in \text{Der}(\mathbf{A})$, $\forall a \in \mathbf{A}$, $\forall m \in \boldsymbol{M}$, $\forall f \in \mathcal{Z}(\mathbf{A})$:

$$\widehat{\nabla}_{\mathfrak{X}}(ma) = m(\mathfrak{X}\cdot a) + (\widehat{\nabla}_{\mathfrak{X}} m)a, \qquad \widehat{\nabla}_{f\mathfrak{X}} m = f \widehat{\nabla}_{\mathfrak{X}} m, \qquad \widehat{\nabla}_{\mathfrak{X}+\mathfrak{Y}} m = \widehat{\nabla}_{\mathfrak{X}} m + \widehat{\nabla}_{\mathfrak{Y}} m$$

The curvature of $\widehat{\nabla}$ is the linear map $\widehat{R}(\mathfrak{X}, \mathfrak{Y}) : \boldsymbol{M} \to \boldsymbol{M}$ defined by

$$\widehat{R}(\mathfrak{X}, \mathfrak{Y}) m = [\widehat{\nabla}_{\mathfrak{X}}, \widehat{\nabla}_{\mathfrak{Y}}] m - \widehat{\nabla}_{[\mathfrak{X}, \mathfrak{Y}]} m$$

➡ $\widehat{R}(\mathfrak{X}, \mathfrak{Y}) : \boldsymbol{M} \to \boldsymbol{M}$ is a right **A**-module morphism.

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Definition (The gauge group)

The gauge group of ***M*** is the group $\mathrm{Aut}(\boldsymbol{M})$ of right **A**-module automorphisms of ***M***.

There is a notion of gauge transformations on NC connections.

Noncommutative connections on $M = \mathbf{A}$

Consider the right $\mathbf{A}$-module $M = \mathbf{A}$.

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Let $\widehat{\nabla}_{\mathfrak{X}} : \mathbf{A} \to \mathbf{A}$ be a noncommutative connection.

Proposition (Noncommutative connections on $M = \mathbf{A}$)

$\widehat{\nabla}$ is completely determined by $\omega \in \Omega^1_{\mathrm{Der}}(\mathbf{A})$ defined by $\omega(\mathfrak{X}) = \widehat{\nabla}_{\mathfrak{X}} \mathbb{1}$.
➡ *on any $a \in \boldsymbol{M} = \mathbf{A}$:*

$$\widehat{\nabla}_{\mathfrak{X}} a = \mathfrak{X} \cdot a + \omega(\mathfrak{X}) a$$

Its curvature is the multiplication on the left on $\boldsymbol{M} = \mathbf{A}$ by the noncommutative 2-form

$$\Omega(\mathfrak{X}, \mathfrak{Y}) = \widehat{\mathrm{d}}_{NC}\omega(\mathfrak{X}, \mathfrak{Y}) + [\omega(\mathfrak{X}), \omega(\mathfrak{Y})]$$

The gauge group is identified with the invertible elements $g \in \mathbf{A}$ by $\Phi_g(a) = ga$.

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In that situation a NC connection is a noncommutative 1-form in $\Omega^1_{\mathrm{Der}}(\mathbf{A})$.

The endomorphisms algebra of a $SL(n)$-vector bundle

Noncommutative aspects

Consider the algebra $\mathbf{A} = \mathbf{A}(\mathcal{E})$ and it differential calculus $(\Omega^\bullet_{\mathrm{Der}}(\mathbf{A}), \widehat{\mathrm{d}}_{\mathrm{NC}})$.

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Proposition (Connections on $\mathcal{E}$)

Connections $\nabla^{\mathcal{E}}$ on $\mathcal{E}$ are:

- *splittings as $C^\infty(\mathcal{M})$-modules of the s.e.s.* $0 \longrightarrow \mathbf{A}_0 \xrightarrow{\mathrm{ad}} \mathrm{Der}(\mathbf{A}) \xrightarrow{\rho} \Gamma(T\mathcal{M}) \longrightarrow 0$.
- *noncommutative traceless 1-form $\alpha \in \Omega^1_{\mathrm{Der}}(\mathbf{A})$ such that*

$$\mathfrak{X} = \nabla_X - \mathrm{ad}_{\alpha(\mathfrak{X})} \qquad\qquad \alpha(\mathrm{ad}_\gamma) = -\gamma$$

(∇ is the connection on the vector bundle $\mathrm{End}(\mathcal{E})$ induced by $\nabla^{\mathcal{E}}$)

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α defines a noncommutative connection on the right $\mathbf{A}$-module $\boldsymbol{M} = \mathbf{A}$.

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Proposition (Connections on $\mathcal{E}$)

Connections $\nabla^{\mathcal{E}}$ on $\mathcal{E}$ are:

- *splittings as $C^\infty(\mathcal{M})$-modules of the s.e.s.* $0 \longrightarrow \mathbf{A}_0 \xrightarrow{\mathrm{ad}} \mathrm{Der}(\mathbf{A}) \xrightarrow{\rho} \Gamma(T\mathcal{M}) \longrightarrow 0$.
- *noncommutative traceless 1-form $\alpha \in \Omega^1_{\mathrm{Der}}(\mathbf{A})$ such that*

$$\mathfrak{X} = \nabla_X - \mathrm{ad}_{\alpha(\mathfrak{X})} \qquad\qquad \alpha(\mathrm{ad}_\gamma) = -\gamma$$

(∇ is the connection on the vector bundle $\mathrm{End}(\mathcal{E})$ induced by $\nabla^{\mathcal{E}}$)

α defines a noncommutative connection on the right $\mathbf{A}$-module $\boldsymbol{M} = \mathbf{A}$.

Proposition (Connections on $\mathcal{E}$ as noncommutative connections)

The space of noncommutative connections on the right $\mathbf{A}$-module $\mathbf{M} = \mathbf{A}$ contains the space of connections on $\mathcal{E}$.
This inclusion is compatible with the notions of curvature and gauge transformations.

Similar to the situation for Lie algebroids.

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$$(\widehat{\mathrm{d}}\omega)(\mathfrak{X}_1, \dots, \mathfrak{X}_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} \mathfrak{X}_i \cdot \omega(\mathfrak{X}_1, \dots \overset{i}{\vee} \dots, \mathfrak{X}_{p+1}) + \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} \omega([\mathfrak{X}_i, \mathfrak{X}_j], \mathfrak{X}_1, \dots \overset{i}{\vee} \dots \overset{j}{\vee} \dots, \mathfrak{X}_{p+1})$$

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Proposition (Relations between differential calculi)

$(\Omega^\bullet_{\mathrm{Lie}}(\mathbf{A}, \mathbf{A}_0), \widehat{\mathrm{d}})$ *(Lie algebroid structure) is included in the derivation-based differential calculus* $(\Omega^\bullet_{\mathrm{Der}}(\mathbf{A}), \widehat{\mathrm{d}}_{NC})$ *as traceless forms.*

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Consider the natural representation of the Lie algebroid $\mathrm{Der}(\mathbf{A})$ on $\mathcal{E}$.
Denote by $(\Omega^\bullet_{\mathrm{Lie}}(\mathbf{A}), \widehat{\mathrm{d}}_{\mathcal{E}})$ the graded differential algebra $(\Omega^\bullet(\mathrm{Der}(\mathbf{A}), \mathbf{A}(\mathcal{E})), \widehat{\mathrm{d}}_{\mathcal{E}})$.

Proposition (Identification of the derivation-based differential calculus)

$(\Omega^\bullet_{\mathrm{Lie}}(\mathbf{A}), \widehat{\mathrm{d}}_{\mathcal{E}}) \simeq (\Omega^\bullet_{\mathrm{Der}}(\mathbf{A}), \widehat{\mathrm{d}}_{NC})$ *as graded differential algebras.*

The spaces of connections

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Theorem (Lie algebroid connections and noncommutative connections)

The following three spaces are isomorphic:

1. *The space of* $\mathrm{Der}(\mathbf{A})$*-connections on* $\mathcal{E}$ *(Lie algebroid of derivations).*
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Theorem (Generalized connections and traceless NC connections)

The following three spaces are isomorphic:

1. *The space of generalized connections on* $\mathrm{Der}(\mathbf{A})$*.*
2. *The space of generalized connections on* $\Gamma_G(T\mathcal{P})$*.*
3. *The space of traceless noncommutative connections on the right* $\mathbf{A}$*-module* $\mathbf{M} = \mathbf{A}$

These isomorphisms are compatible with curvatures and gauge transformations.

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Thank You