

Generalized connections on transitive Lie algebroids

Relationships with NCG

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Orsay, June 3rd, 2010

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arXiv:1003.6106



Motivations and objectives

Noncommutative geometry: (NCG)

→ manage differential geometry and differential algebras (and operator algebras).

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Objectives of this talk:

- Facts about Lie algebroids (representation theory, connections).
- Differential structures and the notion of generalized connections.
- Two examples related to the ordinary theory of fiber bundles:
 - The Atiyah Lie algebroid of a principal fiber bundle.
 - The Lie algebroid of derivations of the endomorphisms algebra of a $SL(n)$ -vector bundle.
- NCG of the endomorphisms algebra of a $SL(n)$ -vector bundle.
- Compare NC connections with generalized connections on Lie algebroids.

Contents

- 1 Lie algebroids and differential calculi
- 2 Connections and their generalizations
- 3 Two examples
- 4 Relationships with noncommutative geometry

Lie algebroids and differential calculi

1 Lie algebroids and differential calculi

- General definitions
- Representations
- Differential structures
- Cartan operations

2 Connections and their generalizations

3 Two examples

4 Relationships with noncommutative geometry

Lie algebroids

Let \mathcal{M} be a smooth manifold. $\Gamma(\mathcal{M})$ Lie algebra of vector fields.

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Definition (Lie algebroids – Geometric version)

A Lie algebroid \mathcal{A} on \mathcal{M} is a vector bundle $\mathcal{A} \xrightarrow{\pi} \mathcal{M}$ with a vector bundle map $\rho : \mathcal{A} \rightarrow T\mathcal{M}$ (the anchor of \mathcal{A}) and a Lie bracket $[-, -]$ on $\Gamma(\mathcal{A})$ (smooth sections of \mathcal{A}) fulfilling

$$[\mathfrak{X}, f\mathfrak{Y}] = f[\mathfrak{X}, \mathfrak{Y}] + (\rho(\mathfrak{X}) \cdot f)\mathfrak{Y} \qquad \rho([\mathfrak{X}, \mathfrak{Y}]) = [\rho(\mathfrak{X}), \rho(\mathfrak{Y})]$$

for any $\mathfrak{X}, \mathfrak{Y} \in \Gamma(\mathcal{A})$ and $f \in C^\infty(\mathcal{M})$.

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for any $\mathfrak{X}, \mathfrak{Y} \in \Gamma(\mathcal{A})$ and $f \in C^\infty(\mathcal{M})$.

With $\mathbf{A} = \Gamma(\mathcal{A})$ one has:

Definition (Lie algebroids – Algebraic version)

A Lie algebroid \mathbf{A} is a finite projective module over $C^\infty(\mathcal{M})$ equipped with a Lie bracket $[-, -]$ and a $C^\infty(\mathcal{M})$ -linear Lie morphism $\rho: \mathbf{A} \rightarrow \Gamma(\mathcal{M})$ such that

$$[\mathfrak{X}, f\mathfrak{Y}] = f[\mathfrak{X}, \mathfrak{Y}] + (\rho(\mathfrak{X}) \cdot f)\mathfrak{Y}$$

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Proposition (The kernel of a transitive Lie algebroid)

Let A be a transitive Lie algebroid.

- $L = \text{Ker } \rho$ is a Lie algebroid with null anchor on \mathcal{M} .
- The vector bundle \mathcal{L} such that $L = \Gamma(\mathcal{L})$ is a locally trivial bundle in Lie algebras.
 \rightarrow gives the Lie structure on L .

One has the short exact sequence of Lie algebras and $C^\infty(\mathcal{M})$ -modules

$$0 \longrightarrow L \xrightarrow{i} A \xrightarrow{\rho} \Gamma(\mathcal{M}) \longrightarrow 0$$

L is called the kernel of A .

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Symbol map: $\sigma : \text{Diff}^1(\mathcal{E}) \rightarrow \text{Hom}_{C^\infty(\mathcal{M})}(T^*\mathcal{M}, \text{End}(\mathcal{E}))$ defined by

$$\sigma(D)(fdg)(s) = f(D(gs) - gD(s)) = f[D, g]s$$

for any $f, g \in C^\infty(\mathcal{M})$ and any $s \in \Gamma(\mathcal{E})$.

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$$\mathfrak{D}(\mathcal{E}) = \sigma^{-1}(\Gamma(\mathcal{M}))$$

is a transitive Lie algebroid, with anchor σ and kernel $\mathbf{A}(\mathcal{E}) = \Gamma(\text{End}(\mathcal{E}))$:

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This is the Lie algebroid of derivations of \mathcal{E} .

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The kernel $\mathbf{A}(\mathcal{E})$ is an associative algebra and the Lie structure is the commutator.

Representations

$A \xrightarrow{\rho} \Gamma(\mathcal{M})$ a Lie algebroid and $\mathcal{E} \rightarrow \mathcal{M}$ a vector bundle.

Definition (Representation of a Lie algebroid)

A representation of A on \mathcal{E} is a morphism of Lie algebroids $\phi : A \rightarrow \mathcal{D}(\mathcal{E})$.

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If A is a transitive Lie algebroid, one has the commutative diagram of exact rows:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & L & \xrightarrow{\iota} & A & \xrightarrow{\rho} & \Gamma(\mathcal{M}) \longrightarrow 0 \\
 & & \downarrow \phi_L & & \downarrow \phi & & \parallel \\
 0 & \longrightarrow & \mathbf{A}(\mathcal{E}) & \xrightarrow{\iota} & \mathcal{D}(\mathcal{E}) & \xrightarrow{\sigma} & \Gamma(\mathcal{M}) \longrightarrow 0
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$\phi_L : L \rightarrow \mathbf{A}(\mathcal{E})$ is a $C^\infty(\mathcal{M})$ -linear morphism of Lie algebras

→ geometric relationship between \mathcal{L} ($L = \Gamma(\mathcal{L})$) and $\text{End}(\mathcal{E})$.

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➔ geometric relationship between \mathcal{L} ($\mathbf{L} = \Gamma(\mathcal{L})$) and $\text{End}(\mathcal{E})$.

A representation for a Lie algebroid is like a right module for an algebra in NCG.

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The gauge group can't be defined with existing objects on the transitive Lie algebroid \mathbf{A} .

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Definition (Infinitesimal gauge transformations)

An infinitesimal gauge transformation on the transitive Lie algebroid \mathbf{A} is an element $\xi \in \mathbf{L}$. Such an infinitesimal gauge transformation acts vertically on \mathcal{E} by $\phi_L : \mathbf{L} \rightarrow \mathbf{A}(\mathcal{E})$.

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Similar situation in NCG: gauge group defined only for a given right module.

Forms with values in $C^\infty(\mathcal{M})$

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Definition (Forms with values in $C^\infty(\mathcal{M})$)

$\forall p \in \mathbb{N}$, $\Omega^p(A)$ is the space of $C^\infty(\mathcal{M})$ -multilinear antisymmetric maps $A^p \rightarrow C^\infty(\mathcal{M})$.

$\Omega^0(A) = C^\infty(\mathcal{M})$.

$\Omega^\bullet(A) = \bigoplus_{p \geq 0} \Omega^p(A)$ gets a differential $\widehat{d}_A : \Omega^p(A) \rightarrow \Omega^{p+1}(A)$:

$$\begin{aligned} (\widehat{d}_A \omega)(\mathfrak{X}_1, \dots, \mathfrak{X}_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i+1} \rho(\mathfrak{X}_i) \cdot \omega(\mathfrak{X}_1, \dots, \overset{i}{\underset{\cdot}{\vee}} \dots, \mathfrak{X}_{p+1}) \\ &\quad + \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} \omega([\mathfrak{X}_i, \mathfrak{X}_j], \mathfrak{X}_1, \dots, \overset{i}{\underset{\cdot}{\vee}} \dots, \overset{j}{\underset{\cdot}{\vee}} \dots, \mathfrak{X}_{p+1}) \end{aligned}$$

and a graded commutative product:

$$(\omega \eta)(\mathfrak{X}_1, \dots, \mathfrak{X}_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in \mathfrak{S}_{p+q}} (-1)^{|\sigma|} \omega(\mathfrak{X}_{\sigma(1)}, \dots, \mathfrak{X}_{\sigma(p)}) \eta(\mathfrak{X}_{\sigma(p+1)}, \dots, \mathfrak{X}_{\sigma(p+q)})$$

$(\Omega^\bullet(A), \widehat{d}_A)$ is a graded commutative differential algebra.

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$(\Omega^\bullet(A), \widehat{d}_A)$ is a graded commutative differential algebra.

This is the space of forms with values in $C^\infty(\mathcal{M})$.

Objects in this algebra will be called forms on A .

Forms with values in the kernel

$0 \longrightarrow \mathcal{L} \xrightarrow{\iota} \mathcal{A} \xrightarrow{\rho} \Gamma(\mathcal{M}) \longrightarrow 0$ a transitive Lie algebroid.

Forms with values in the kernel

$0 \longrightarrow L \xrightarrow{\iota} A \xrightarrow{\rho} \Gamma(\mathcal{M}) \longrightarrow 0$ a transitive Lie algebroid.

Natural action of A on L : $\forall \mathfrak{X} \in A$ and $\forall \ell \in L$,

$[\mathfrak{X}, \ell] \in L$ is the unique element in L such that $\iota([\mathfrak{X}, \ell]) = [\mathfrak{X}, \iota(\ell)]$.

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and a graded Lie bracket $\Omega^p(A, L) \otimes \Omega^q(A, L) \rightarrow \Omega^{p+q}(A, L)$:

$$[\omega, \eta](\mathfrak{X}_1, \dots, \mathfrak{X}_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in \mathfrak{S}_{p+q}} (-1)^{|\sigma|} [\omega(\mathfrak{X}_{\sigma(1)}, \dots, \mathfrak{X}_{\sigma(p)}), \eta(\mathfrak{X}_{\sigma(p+1)}, \dots, \mathfrak{X}_{\sigma(p+q)})]$$

$(\Omega^\bullet(A, L), \widehat{d})$ is a graded differential Lie algebra.

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and a graded Lie bracket $\Omega^p(A, L) \otimes \Omega^q(A, L) \rightarrow \Omega^{p+q}(A, L)$:

$$[\omega, \eta](\mathfrak{X}_1, \dots, \mathfrak{X}_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in \mathfrak{S}_{p+q}} (-1)^{|\sigma|} [\omega(\mathfrak{X}_{\sigma(1)}, \dots, \mathfrak{X}_{\sigma(p)}), \eta(\mathfrak{X}_{\sigma(p+1)}, \dots, \mathfrak{X}_{\sigma(p+q)})]$$

$(\Omega^\bullet(A, L), \widehat{d})$ is a graded differential Lie algebra.

This is the space of forms on A with values in the kernel.

Objects in this complex will be called forms on A with values in L .

Forms on A with values in $A(\mathcal{E})$

$\phi: A \rightarrow \mathcal{D}(\mathcal{E})$ representation of the transitive Lie algebroid $0 \rightarrow L \xrightarrow{l} A \xrightarrow{p} \Gamma(\mathcal{M}) \rightarrow 0$.

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$(\Omega^\bullet(A, \mathbf{A}(\mathcal{E})), \widehat{d}_\mathcal{E})$ is a graded differential algebra.

This is the space of forms on A with values in $\mathbf{A}(\mathcal{E})$.

There are relations between $\Omega^\bullet(A)$, $\Omega^\bullet(A, L)$ and $\Omega^\bullet(A, \mathbf{A}(\mathcal{E}))$.

Example: Trivial Lie algebroids

Models for local trivializations of transitive Lie algebroids.

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$A = \Gamma(T\mathcal{M} \oplus (\mathcal{M} \times \mathfrak{g}))$ is a Lie algebroid for the following structures:

$$\rho(X \oplus \gamma) = X \quad [X \oplus \gamma, Y \oplus \eta] = [X, Y] \oplus (X \cdot \eta - Y \cdot \gamma + [\gamma, \eta])$$

for any $X, Y \in \Gamma(\mathcal{M})$ and $\gamma, \eta \in \Gamma(\mathcal{M} \times \mathfrak{g})$.

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The s.e.s.

$$0 \longrightarrow L \xrightarrow{i} A \xrightarrow{\rho} \Gamma(\mathcal{M}) \longrightarrow 0$$

splits as Lie algebras and $C^\infty(\mathcal{M})$ -modules.

Example: Trivial Lie algebroids (cont'd)

Let $A = \text{TLA}(\mathcal{M}, \mathfrak{g})$ and $L = \Gamma(\mathcal{M} \times \mathfrak{g})$.

Then $\Omega^\bullet(A, L) = \Omega_{\text{TLA}}^\bullet(\mathcal{M}, \mathfrak{g})$ is the total complex of the bicomplex

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$$d : \Omega_{\text{TLA}}^{\bullet, \bullet}(\mathcal{M}, \mathfrak{g}) \rightarrow \Omega_{\text{TLA}}^{\bullet+1, \bullet}(\mathcal{M}, \mathfrak{g})$$

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Then the differential is $\widehat{d} = \widehat{d}_{\text{TLA}} = d + s$ on $\Omega_{\text{TLA}}^\bullet(\mathcal{M}, \mathfrak{g})$. It is given by

$$\begin{aligned} (\widehat{d}_{\text{TLA}} \omega)(X_1 \oplus \gamma_1, \dots, X_{p+1} \oplus \gamma_{p+1}) = & \\ & \sum_{i=1}^{p+1} (-1)^{i+1} X_i \cdot \omega(X_1 \oplus \gamma_1, \dots, \overset{i}{\dot{\vee}} \dots, X_{p+1} \oplus \gamma_{p+1}) \\ & + \sum_{i=1}^{p+1} (-1)^{i+1} [\gamma_i, \omega(X_1 \oplus \gamma_1, \dots, \overset{i}{\dot{\vee}} \dots, X_{p+1} \oplus \gamma_{p+1})] \\ & + \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} \omega([X_i \oplus \gamma_i, X_j \oplus \gamma_j], X_1 \oplus \gamma_1, \dots, \overset{i}{\dot{\vee}} \dots \overset{j}{\dot{\vee}} \dots, X_{p+1} \oplus \gamma_{p+1}) \end{aligned}$$

for any $\omega \in \Omega_{\text{TLA}}^p(\mathcal{M}, \mathfrak{g})$ and any $X_i \oplus \gamma_i \in \text{TLA}(\mathcal{M}, \mathfrak{g})$.

Example: Trivial Lie algebroids (cont'd)

Let $A = \text{TLA}(\mathcal{M}, \mathfrak{g})$ and $L = \Gamma(\mathcal{M} \times \mathfrak{g})$.

E vector space and $\eta(\xi) : E \rightarrow E$ a linear representation of \mathfrak{g} .

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There is a natural representation of A on $\mathcal{E} = \mathcal{M} \times E$:

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The differential calculus $\Omega_{\text{TLA}}^\bullet(\mathcal{M}, \mathfrak{g}, E) = \Omega^\bullet(A, \mathbf{A}(\mathcal{E}))$ is the total complex of $\Omega^\bullet(\mathcal{M}) \otimes \wedge^\bullet \mathfrak{g}^* \otimes \mathcal{L}(E)$ with differential

$$\begin{aligned}
 (\widehat{d}_{\text{TLA}, \mathcal{E}} \omega)(X_1 \oplus \gamma_1, \dots, X_{p+1} \oplus \gamma_{p+1}) = & \\
 & \sum_{i=1}^{p+1} (-1)^{i+1} X_i \cdot \omega(X_1 \oplus \gamma_1, \dots, \overset{i}{\underset{\cdot}{\vee}} \dots, X_{p+1} \oplus \gamma_{p+1}) \\
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Inner product and Lie derivative

$0 \longrightarrow \mathcal{L} \xrightarrow{\iota} \mathcal{A} \xrightarrow{\rho} \Gamma(\mathcal{M}) \longrightarrow 0$ a transitive Lie algebroid.

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Definition (Inner product and Lie derivative)

For any $\mathfrak{X} \in \mathcal{A}$ and $p \geq 1$, we define the inner product

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by $(i_{\mathfrak{X}}\omega)(\mathfrak{X}_1, \dots, \mathfrak{X}_p) = \omega(\mathfrak{X}, \mathfrak{X}_1, \dots, \mathfrak{X}_p)$.

$i_{\mathfrak{X}}$ is zero on $\Omega^0(\mathcal{A}, \mathcal{L}) = \mathcal{L}$.

For any $p \geq 0$,

$$L_{\mathfrak{X}} = \widehat{d}i_{\mathfrak{X}} + i_{\mathfrak{X}}\widehat{d} : \Omega^p(\mathcal{A}, \mathcal{L}) \rightarrow \Omega^p(\mathcal{A}, \mathcal{L})$$

is the Lie derivative associated in the \mathfrak{X} direction.

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$$(L_{\mathfrak{X}}\omega)(\mathfrak{X}_1, \dots, \mathfrak{X}_p) = [\mathfrak{X}, \omega(\mathfrak{X}_1, \dots, \mathfrak{X}_p)] - \sum_{i=1}^p \omega(\mathfrak{X}_1, \dots, [\mathfrak{X}, \mathfrak{X}_i], \dots, \mathfrak{X}_p)$$

Cartan operation

For any $\mathfrak{X}, \mathfrak{Y} \in \mathfrak{A}$ one has

$$i_{\mathfrak{X}}i_{\mathfrak{Y}} + i_{\mathfrak{Y}}i_{\mathfrak{X}} = 0$$

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\mathfrak{h} a Lie algebra.

Definition (Cartan operation of \mathfrak{h})

A Cartan operation of \mathfrak{h} on $(\Omega^\bullet(A, L), \widehat{d})$ is given by the following data:

$\forall \mathfrak{X} \in \mathfrak{h}, \forall p \geq 1$, there is a map $i_{\mathfrak{X}} : \Omega^p(A, L) \rightarrow \Omega^{p-1}(A, L)$ such that the above relations hold for any $\mathfrak{X}, \mathfrak{Y} \in \mathfrak{h}$, with $L_{\mathfrak{X}} = \widehat{d}i_{\mathfrak{X}} + i_{\mathfrak{X}}\widehat{d}$.

Such a Cartan operation on $(\Omega^\bullet(A, L), \widehat{d})$ is denoted by (\mathfrak{h}, i, L) .

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Example (Cartan operation of the kernel)

As a Lie algebroid, $\iota \circ L \subset A$

→ one can restrict the operations i and L defined with $\mathfrak{X} \in A$ to $\mathfrak{X} \in \iota \circ L$.

This defines the Cartan operation of L (as a Lie algebra) on $(\Omega^\bullet(A, L), \widehat{d})$.

Horizontal, invariant and basic elements

Definition (Horizontal, invariant and basic elements)

Let (\mathfrak{h}, i, L) be a Cartan operation on $(\Omega^\bullet(A, L), \widehat{d})$. Then

- The horizontal elements in $\Omega^\bullet(A, L)$ are the forms ω such that $i_{\mathfrak{X}}\omega = 0$ for any $\mathfrak{X} \in \mathfrak{h}$. $\Omega^\bullet(A, L)_{\text{Hor}}$ is the graded subspace of horizontal elements.
- The invariant elements in $\Omega^\bullet(A, L)$ are the forms ω such that $L_{\mathfrak{X}}\omega = 0$ for any $\mathfrak{X} \in \mathfrak{h}$. $\Omega^\bullet(A, L)_{\text{Inv}}$ is the graded subspace of invariant elements.
- The basic elements in $\Omega^\bullet(A, L)$ are the forms which are both horizontal and invariant. $\Omega^\bullet(A, L)_{\text{Basic}}$ is the graded subspace of basic elements.

$\Omega^\bullet(A, L)_{\text{Inv}}$ and $\Omega^\bullet(A, L)_{\text{Basic}}$ are differential subcomplexes of $(\Omega^\bullet(A, L), \widehat{d})$.

Connections and their generalizations

1 Lie algebroids and differential calculi

2 **Connections and their generalizations**

- Ordinary connections
- Generalized connections

3 Two examples

4 Relationships with noncommutative geometry

Ordinary connections : definition

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Definition (Connection on a transitive Lie algebroid)

A connection on A is a splitting $\nabla : \Gamma(\mathcal{M}) \rightarrow A$ as $C^\infty(\mathcal{M})$ -modules of the s.e.s.

$$0 \longrightarrow L \xrightarrow{\iota} A \xleftarrow[\rho]{\nabla} \Gamma(\mathcal{M}) \longrightarrow 0$$

Its curvature is the obstruction of being a morphism of Lie algebras:

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$$

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∇ a connection. There is an algebraic object which permits concrete computations:

$\mathfrak{X} \in A$, let $X = \rho(\mathfrak{X})$

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A connection on A is a splitting $\nabla : \Gamma(\mathcal{M}) \rightarrow A$ as $C^\infty(\mathcal{M})$ -modules of the s.e.s.

$$0 \longrightarrow L \xrightarrow{\iota} A \xleftarrow[\rho]{\nabla} \Gamma(\mathcal{M}) \longrightarrow 0$$

Its curvature is the obstruction of being a morphism of Lie algebras:

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$$

∇ a connection. There is an algebraic object which permits concrete computations:

$\mathfrak{X} \in A$, let $X = \rho(\mathfrak{X})$

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There is a notion of gauge transformation on connections.

Ordinary connections : the connection 1-form

∇ connection on the transitive Lie algebroid A , $\mathfrak{X} = \nabla_X - \iota \circ \alpha(\mathfrak{X})$.

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The associated morphism $\alpha : A \rightarrow L$ is an element of $\Omega^1(A, L)$ normalized on $\iota \circ L$ by

$$\alpha \circ \iota(\ell) = -\ell$$

for any $\ell \in L$.

Any $\alpha \in \Omega^1(A, L)$ normalized as before defines a unique connection on A .

The 2-form

$$\widehat{R} = \widehat{d}\alpha + \frac{1}{2}[\alpha, \alpha]$$

is horizontal for the Cartan operation of L on $(\Omega^\bullet(A, L), \widehat{d})$. With usual notations:

$$\iota \circ \widehat{R}(\mathfrak{X}, \mathfrak{Y}) = \iota \left((\widehat{d}\alpha)(\mathfrak{X}, \mathfrak{Y}) + [\alpha(\mathfrak{X}), \alpha(\mathfrak{Y})] \right) = R(X, Y)$$

\widehat{R} satisfies the Bianchi identity

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$\alpha \in \Omega^1(A, L)$ is the connection 1-form of ∇ ("connection reform" in Mackenzie).

$\widehat{R} \in \Omega^2(A, L)$ is the curvature 2-form of ∇ .

The covariant derivative

α the connection 1-form of a connection ∇ on A .

Proposition (Covariant derivative on $\Omega^\bullet(A, L)$)

$$D: \Omega^\bullet(A, L) \rightarrow \Omega^{\bullet+1}(A, L)$$

$$D\eta = \widehat{d}\eta + [\alpha, \eta]$$

satisfies

$$D^2\eta = [\widehat{R}, \eta]$$

for any $\eta \in \Omega^\bullet(A, L)$.

The Bianchi identity on \widehat{R} is equivalent to $D\widehat{R} = 0$.

D is the covariant derivative of α on $\Omega^\bullet(A, L)$.

A-connections on \mathcal{E}

First generalization of connections

$\phi : A \rightarrow \mathcal{D}(\mathcal{E})$ a representation of the transitive Lie algebroid A .

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$$\widehat{\nabla}_{f\mathfrak{X}}^{\mathcal{E}}s = f\widehat{\nabla}_{\mathfrak{X}}^{\mathcal{E}}s \quad \widehat{\nabla}_{\mathfrak{X}}^{\mathcal{E}}(s_1 + s_2) = \widehat{\nabla}_{\mathfrak{X}}^{\mathcal{E}}s_1 + \widehat{\nabla}_{\mathfrak{X}}^{\mathcal{E}}s_2 \quad \widehat{\nabla}_{\mathfrak{X}}^{\mathcal{E}}(fs) = (\rho(\mathfrak{X}) \cdot f)s + f\widehat{\nabla}_{\mathfrak{X}}^{\mathcal{E}}s$$

$\forall \mathfrak{X} \in A, \forall f \in C^\infty(\mathcal{M})$ and $\forall s, s_1, s_2 \in \Gamma(\mathcal{E})$.

The curvature of $\widehat{\nabla}^{\mathcal{E}}$ is defined as:

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An A-connection on \mathcal{E} can also be seen as a generalized "representation".

The curvature measures the obstruction to be a morphism of Lie algebras.

Notice: $\widehat{R}^{\mathcal{E}}(\mathfrak{X}, \mathfrak{Y}) \in L$.

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Notice: $\widehat{R}^{\mathcal{E}}(\mathfrak{X}, \mathfrak{Y}) \in L$.

There is a notion of gauge transformation on A-connections on \mathcal{E} .

Connection 1-form of a A-connection on \mathcal{E}

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$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbf{L} & \xrightarrow{\iota} & \mathbf{A} & \xrightarrow{\rho} & \Gamma(\mathcal{M}) \longrightarrow 0 \\
 & & \downarrow \phi_L & & \downarrow \phi & & \parallel \\
 0 & \longrightarrow & \mathbf{A}(\mathcal{E}) & \xrightarrow{\iota} & \mathcal{D}(\mathcal{E}) & \xrightarrow{\sigma} & \Gamma(\mathcal{M}) \longrightarrow 0
 \end{array}$$

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Connection 1-form of a A-connection on \mathcal{E}

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & L & \xrightarrow{\iota} & A & \xrightarrow{\rho} & \Gamma(\mathcal{M}) & \longrightarrow & 0 \\
 & & \downarrow \phi_L & \nearrow \omega^\mathcal{E} & \downarrow \widehat{\nabla}^\mathcal{E} & \downarrow \phi & \parallel & & \\
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$\widehat{\nabla}^\mathcal{E} : A \rightarrow \mathcal{D}(\mathcal{E})$ a A-connection on $\mathcal{E} \Rightarrow \widehat{\nabla}_{\mathfrak{X}}^\mathcal{E} - \phi(\mathfrak{X}) \in \text{Ker } \sigma$

$\Rightarrow \exists! \omega^\mathcal{E} \in \Omega^1(A, \mathbf{A}(\mathcal{E}))$ such that $\widehat{\nabla}_{\mathfrak{X}}^\mathcal{E} = \phi(\mathfrak{X}) + \iota \circ \omega^\mathcal{E}(\mathfrak{X})$.

Connection 1-form of a \mathbf{A} -connection on \mathcal{E}

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbf{L} & \xrightarrow{\iota} & \mathbf{A} & \xrightarrow{\rho} & \Gamma(\mathcal{M}) \longrightarrow 0 \\
 & & \downarrow \phi_L & \nearrow \omega^\mathcal{E} & \downarrow \widehat{\nabla}^\mathcal{E} & \downarrow \phi & \parallel \\
 0 & \longrightarrow & \mathbf{A}(\mathcal{E}) & \xrightarrow{\iota} & \mathcal{D}(\mathcal{E}) & \xrightarrow{\sigma} & \Gamma(\mathcal{M}) \longrightarrow 0
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Proposition (Connection 1-form and curvature 2-form)

An \mathbf{A} -connection on \mathcal{E} , $\widehat{\nabla}^\mathcal{E} : \mathbf{A} \rightarrow \mathcal{D}(\mathcal{E})$, is completely determined by the 1-form $\omega^\mathcal{E} \in \Omega^1(\mathbf{A}, \mathbf{A}(\mathcal{E}))$ with the above relation.

One has $\widehat{R}^\mathcal{E} \in \Omega^2(\mathbf{A}, \mathbf{A}(\mathcal{E}))$ with

$$\widehat{R}^\mathcal{E}(\mathfrak{X}, \mathfrak{Y}) = (\widehat{d}_\mathcal{E} \omega^\mathcal{E})(\mathfrak{X}, \mathfrak{Y}) + [\omega^\mathcal{E}(\mathfrak{X}), \omega^\mathcal{E}(\mathfrak{Y})]$$

$\widehat{R}^\mathcal{E}$ satisfies the Bianchi identity

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$\omega^\mathcal{E}$ is the connection 1-form of $\widehat{\nabla}^\mathcal{E}$ and $\widehat{R}^\mathcal{E}$ its curvature 2-form.

Covariant derivative, ordinary connections

$\omega^\mathcal{E}$ connection 1-form of a A-connection on \mathcal{E} .

Proposition (Covariant derivative on $\Omega^\bullet(A, L)$)

$$\widehat{D}^\mathcal{E} : \Omega^\bullet(A, \mathbf{A}(\mathcal{E})) \rightarrow \Omega^{\bullet+1}(A, \mathbf{A}(\mathcal{E}))$$

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Proposition (Connections on \mathcal{E} as A -connections on \mathcal{E})

$\nabla^\mathcal{E} \mapsto \widehat{\nabla}^\mathcal{E}$ injects the space of connections on $\mathfrak{D}(\mathcal{E})$ into the space of A -connections on \mathcal{E} .
 This injection is compatible with the notions of curvature and gauge transformations.

The notion of A -connections on \mathcal{E} generalizes the notion of connections on $\mathfrak{D}(\mathcal{E})$.

Generalized connections

Second generalization of connections

$\phi : A \rightarrow \mathcal{D}(\mathcal{E})$ a representation of the transitive Lie algebroid A .

Definition (Induced connections)

An A -connection on \mathcal{E} is an *induced connection* if its associated 1-form $\omega^{\mathcal{E}} \in \Omega^1(A, \mathbf{A}(\mathcal{E}))$ can be factorized through ϕ_L as $\omega^{\mathcal{E}} = \phi_L \circ \omega$ for a 1-form $\omega \in \Omega^1(A, L)$.

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Definition (Generalized connections)

A 1-form ω is called a *generalized connection 1-form* on the transitive Lie algebroid A . Its curvature is defined as the 2-form $\widehat{R} = \widehat{d}\omega + [\omega, \omega] \in \Omega^2(A, L)$.

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A generalized connection can induce A -connections on any representations.

→ Compare with connection on a principal fiber bundle which gives rise to covariant derivatives on associated vector bundles.

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$\nabla: \Gamma(\mathcal{M}) \rightarrow A$ an ordinary connection on A .

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As a 1-form in $\Omega^1(A, L)$, α^∇ defines a generalized connection on A .

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Contrary to connections on $\mathfrak{D}(\mathcal{E})$, there is no “ $\widehat{\nabla}$ ” operator.

Two examples

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1 Lie algebroids and differential calculi

2 Connections and their generalizations

3 Two examples

- The Atiyah Lie algebroid
- The Lie algebroids of derivations of an endomorphism fiber bundle

4 Relationships with noncommutative geometry

The Atiyah Lie algebroid

$\mathcal{P} \xrightarrow{\pi} \mathcal{M}$ a G -principal fiber bundle. \mathfrak{g} the Lie algebra of G .

$R_g : \mathcal{P} \rightarrow \mathcal{P}$, $R_g(p) = p \cdot g$, the right action of G on \mathcal{P} .

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These two spaces are Lie algebras and $C^\infty(\mathcal{M})$ -modules (map $f \in C^\infty(\mathcal{M})$ to $\pi^*f \in C^\infty(\mathcal{P})$).

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$$\iota: \Gamma_G(\mathcal{P}, \mathfrak{g}) \rightarrow \Gamma_G(\mathcal{P})$$

defined with $\mathfrak{g} \ni \xi \mapsto \xi^{\mathcal{P}}$ (fundamental vertical vector field on \mathcal{P} associated to $\xi \in \mathfrak{g}$ for R).

$\Gamma_G(\mathcal{P}, \mathfrak{g})$ is a subspace of *vertical vector fields* in $\Gamma_G(\mathcal{P})$.

The Atiyah Lie algebroid

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Proposition (Differential on the space of forms)

For any $X \in \Gamma_G(\mathcal{P})$ and any $v \in \Gamma_G(\mathcal{P}, \mathfrak{g})$, one has $[X, \iota(v)] = \iota(X \cdot v)$.

→ the differential \widehat{d} on $\Omega_{\text{Lie}}^\bullet(\mathcal{P}, \mathfrak{g})$ takes the explicit form

$$\begin{aligned} (\widehat{d}\omega)(X_1, \dots, X_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i+1} X_i \cdot \omega(X_1, \dots, \overset{i}{\dot{\vee}} \dots, X_{p+1}) \\ &\quad + \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \overset{i}{\dot{\vee}} \dots, \overset{j}{\dot{\vee}} \dots, X_{p+1}) \end{aligned}$$

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As differential graded complexes, $(\Omega_{\text{Lie}}^{\bullet}(\mathcal{P}, \mathfrak{g}), \widehat{\mathfrak{d}})$ and $(\Omega_{\text{TLA}}^{\bullet}(\mathcal{P}, \mathfrak{g})_{\mathfrak{g}_{\text{equ}}}, \widehat{\mathfrak{d}}_{\text{TLA}})$ are isomorphic.

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In “K. Mackenzie, *General Theory of Lie Groupoids and Lie Algebroids*, CUP 2005”,

$(\Omega_{\text{Lie}}^{\bullet}(\mathcal{P}, \mathfrak{g}), \widehat{\mathfrak{d}}) \simeq \{(R, \text{Ad})\text{-equivariant forms in } (\Omega^{\bullet}(\mathcal{P}) \otimes \mathfrak{g}, \mathfrak{d})\}$ ($\mathfrak{g}_{\text{equ}}$ -invariant forms).

Connections and generalized connections

Proposition (Connections on the Atiyah Lie algebroid)

*Connections on the Atiyah Lie algebroid $\Gamma_G(\mathcal{P})$ are exactly connections on \mathcal{P} .
The correspondence is compatible with curvature and gauge transformations.*

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Generalized connections are written more generally as $\mathfrak{g}_{\text{equ}}$ -basic 1-forms

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The normalization $\omega(\xi^{\mathcal{P}}) = \xi$ is no more required.

Representation theory

E vector space and $\ell_{\mathfrak{g}} : E \rightarrow E$ a linear representation of G .

$\mathcal{L}(E)$ algebra of endomorphisms of E .

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The theory of representation of the transitive Lie algebroid $\Gamma_G(\mathcal{P})$ is the theory of associated vector bundles.

Forms with values in $A(\mathcal{E})$

Denote as before $\Omega_{\text{TLA}}^\bullet(\mathcal{P}, \mathfrak{g}, E)$ the total complex of $\Omega^\bullet(\mathcal{P}) \otimes \wedge^\bullet \mathfrak{g}^* \otimes \mathcal{L}(E)$, equipped with its differential $\widehat{d}_{\text{TLA}, \mathcal{E}}$.

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Proposition (Identification of $(\Omega^\bullet(\Gamma_G(\mathcal{P}), \mathbf{A}(\mathcal{E})), \widehat{d}_{\mathcal{E}})$)

As differential graded algebras, $(\Omega^\bullet(\Gamma_G(\mathcal{P}), \mathbf{A}(\mathcal{E})), \widehat{d}_{\mathcal{E}}) \simeq ((\Omega_{\text{TLA}}^\bullet(\mathcal{P}, \mathfrak{g}, E))_{\mathfrak{g}_{\text{equ}}}, \widehat{d}_{\text{TLA}, \mathcal{E}})$.

Derivations of an associative algebra

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Definition (Vector space of derivations of \mathbf{A})

The vector space of derivations of \mathbf{A} is the space

$$\text{Der}(\mathbf{A}) = \{ \mathfrak{X} : \mathbf{A} \rightarrow \mathbf{A} / \mathfrak{X} \text{ linear, } \mathfrak{X} \cdot (ab) = (\mathfrak{X} \cdot a)b + a(\mathfrak{X} \cdot b), \forall a, b \in \mathbf{A} \}$$

Proposition (Structure of $\text{Der}(\mathbf{A})$)

- $\text{Der}(\mathbf{A})$ is a Lie algebra for the bracket $[\mathfrak{X}, \mathfrak{Y}] \cdot a = \mathfrak{X} \cdot \mathfrak{Y} \cdot a - \mathfrak{Y} \cdot \mathfrak{X} \cdot a$ for any $\mathfrak{X}, \mathfrak{Y} \in \text{Der}(\mathbf{A})$.
- It is a $\mathcal{Z}(\mathbf{A})$ -module for the product $(f\mathfrak{X}) \cdot a = f(\mathfrak{X} \cdot a)$ for any $f \in \mathcal{Z}(\mathbf{A})$ and $\mathfrak{X} \in \text{Der}(\mathbf{A})$.
- The subspace $\text{Int}(\mathbf{A}) = \{ \text{ad}_a : b \mapsto [a, b] / a \in \mathbf{A} \} \subset \text{Der}(\mathbf{A})$ (inner derivations) is a Lie ideal and a $\mathcal{Z}(\mathbf{A})$ -submodule.
- With $\text{Out}(\mathbf{A}) = \text{Der}(\mathbf{A}) / \text{Int}(\mathbf{A})$, there is a s.e.s. of Lie algebras and $\mathcal{Z}(\mathbf{A})$ -modules

$$0 \longrightarrow \text{Int}(\mathbf{A}) \longrightarrow \text{Der}(\mathbf{A}) \longrightarrow \text{Out}(\mathbf{A}) \longrightarrow 0$$

Two examples

The Lie algebroids of derivations of an endomorphism fiber bundle

The endomorphisms algebra of a $SL(n)$ -vector bundle

\mathcal{E} a $SL(n)$ -vector bundle over \mathcal{M} with fiber \mathbb{C}^n .

Two examples

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Proposition ($\text{Der}(\mathbf{A})$ and $\Gamma_G(\mathcal{P})$)

As Lie algebroids $\text{Der}(\mathbf{A})$ and $\Gamma_G(\mathcal{P})$ are isomorphic.

The representations on $\mathfrak{D}(\mathcal{E})$ are the same.

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What is different: the way to manage “vertical” vector fields.

Relationships with noncommutative geometry

1 Lie algebroids and differential calculi

2 Connections and their generalizations

3 Two examples

4 Relationships with noncommutative geometry

- Brief review on some noncommutative structures
- The algebra of endomorphisms of a vector fiber bundle

Derivation-based differential calculus

\mathbf{A} associative algebra with unit denoted by $\mathbb{1}$. $\mathcal{Z}(\mathbf{A})$ its center.

Definition (The derivation-based differential calculus)

$\forall p \in \mathbb{N}$, $\Omega_{\text{Der}}^p(\mathbf{A})$ is the space of $\mathcal{Z}(\mathbf{A})$ -multilinear antisymmetric maps $\text{Der}(\mathbf{A})^p \rightarrow \mathbf{A}$

$$\Omega_{\text{Der}}^0(\mathbf{A}) = \mathbf{A}.$$

$\Omega_{\text{Der}}^\bullet(\mathbf{A}) = \bigoplus_{p \geq 0} \Omega_{\text{Der}}^p(\mathbf{A})$ get a differential

$$\widehat{d}_{\text{NC}}\omega(\mathfrak{X}_1, \dots, \mathfrak{X}_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} \mathfrak{X}_i \cdot \omega(\mathfrak{X}_1, \dots, \overset{i}{\underset{\cdot}{\vee}} \dots, \mathfrak{X}_{p+1}) + \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} \omega([\mathfrak{X}_i, \mathfrak{X}_j], \dots, \overset{i}{\underset{\cdot}{\vee}} \dots \overset{j}{\underset{\cdot}{\vee}} \dots, \mathfrak{X}_{p+1})$$

and a graded product:

$$(\omega\eta)(\mathfrak{X}_1, \dots, \mathfrak{X}_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in \mathfrak{S}_{p+q}} (-1)^{|\sigma|} \omega(\mathfrak{X}_{\sigma(1)}, \dots, \mathfrak{X}_{\sigma(p)}) \eta(\mathfrak{X}_{\sigma(p+1)}, \dots, \mathfrak{X}_{\sigma(p+q)})$$

$(\Omega_{\text{Der}}^\bullet(\mathbf{A}), \widehat{d}_{\text{NC}})$ is a graded differential algebra.

Example I

Example (The algebra of smooth functions $\mathbf{A} = C^\infty(\mathcal{M})$)

\mathcal{M} a smooth manifold. $\mathbf{A} = C^\infty(\mathcal{M})$.

- $\mathcal{Z}(\mathbf{A}) = C^\infty(\mathcal{M})$.
- $\text{Der}(\mathbf{A}) = \Gamma(\mathcal{M})$. There are no inner derivations, $\text{Int}(\mathbf{A}) = 0$, so that $\text{Out}(\mathbf{A}) = \Gamma(\mathcal{M})$.
- This graded differential algebra coincides with the graded differential algebra of de Rham forms on \mathcal{M} : $\Omega_{\text{Der}}^\bullet(\mathbf{A}) = \Omega^\bullet(\mathcal{M})$

This example has motivated the general definitions.

Example II

Example (The matrix algebra $A = M_n(\mathbb{C}) = M_n$)

- $\mathcal{Z}(M_n) = \mathbb{C}$.
- $\text{Der}(M_n) = \text{Int}(M_n) \simeq \mathfrak{sl}_n = \mathfrak{sl}(n, \mathbb{C})$ (traceless matrices).
 $\gamma \in \mathfrak{sl}_n(\mathbb{C})$ defines the inner derivation $\text{ad}_\gamma : a \mapsto [\gamma, a]$.

- $\Omega_{\text{Der}}^\bullet(M_n) \simeq M_n \otimes \bigwedge^\bullet \mathfrak{sl}_n^*$.

The diff. \widehat{d}_{NC} is the Chevalley-Eilenberg diff. for the adjoint representation of \mathfrak{sl}_n on M_n .

- \exists canonical noncommutative 1-form $i\theta \in \Omega_{\text{Der}}^1(M_n)$: for any $\gamma \in M_n(\mathbb{C})$

$$i\theta(\text{ad}_\gamma) = \gamma - \frac{1}{n} \text{Tr}(\gamma) \mathbb{1}$$

It realizes the explicit isomorphism $\text{Int}(M_n(\mathbb{C})) \xrightarrow{\cong} \mathfrak{sl}_n$.

- $\widehat{d}_{\text{NC}}(i\theta) - (i\theta)^2 = 0 \implies i\theta$ looks very much like the Maurer-Cartan form in the geometry of Lie groups (here $SL_n(\mathbb{C})$).
- For any $a \in M_n$, one has $\widehat{d}_{\text{NC}} a = [i\theta, a] \in \Omega_{\text{Der}}^1(M_n)$. No longer true in higher degrees.

This example is purely noncommutative. It is well studied.

Noncommutative connections

Noncommutative connections are defined on a right \mathbf{A} -module \mathbf{M} .

Definition (Noncommutative connections and curvature)

A noncommutative connection on the right \mathbf{A} -module \mathbf{M} is a linear map $\widehat{\nabla}_{\mathfrak{X}} : \mathbf{M} \rightarrow \mathbf{M}$, defined for any $\mathfrak{X} \in \text{Der}(\mathbf{A})$, such that $\forall \mathfrak{X}, \mathfrak{Y} \in \text{Der}(\mathbf{A}), \forall a \in \mathbf{A}, \forall m \in \mathbf{M}, \forall f \in \mathcal{Z}(\mathbf{A})$:

$$\widehat{\nabla}_{\mathfrak{X}}(ma) = m(\mathfrak{X} \cdot a) + (\widehat{\nabla}_{\mathfrak{X}}m)a, \quad \widehat{\nabla}_{f\mathfrak{X}}m = f\widehat{\nabla}_{\mathfrak{X}}m, \quad \widehat{\nabla}_{\mathfrak{X}+\mathfrak{Y}}m = \widehat{\nabla}_{\mathfrak{X}}m + \widehat{\nabla}_{\mathfrak{Y}}m$$

The curvature of $\widehat{\nabla}$ is the linear map $\widehat{R}(\mathfrak{X}, \mathfrak{Y}) : \mathbf{M} \rightarrow \mathbf{M}$ defined by

$$\widehat{R}(\mathfrak{X}, \mathfrak{Y})m = [\widehat{\nabla}_{\mathfrak{X}}, \widehat{\nabla}_{\mathfrak{Y}}]m - \widehat{\nabla}_{[\mathfrak{X}, \mathfrak{Y}]}m$$

$\rightarrow \widehat{R}(\mathfrak{X}, \mathfrak{Y}) : \mathbf{M} \rightarrow \mathbf{M}$ is a right \mathbf{A} -module morphism.

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Definition (The gauge group)

The gauge group of \mathbf{M} is the group $\text{Aut}(\mathbf{M})$ of right \mathbf{A} -module automorphisms of \mathbf{M} .

There is a notion of gauge transformations on NC connections.

Noncommutative connections on $M = \mathbf{A}$

Consider the right \mathbf{A} -module $M = \mathbf{A}$.

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Let $\widehat{\nabla}_{\mathfrak{X}} : \mathbf{A} \rightarrow \mathbf{A}$ be a noncommutative connection.

Proposition (Noncommutative connections on $M = \mathbf{A}$)

$\widehat{\nabla}$ is completely determined by $\omega \in \Omega_{\text{Der}}^1(\mathbf{A})$ defined by $\omega(\mathfrak{X}) = \widehat{\nabla}_{\mathfrak{X}} \mathbb{1}$.

→ on any $a \in M = \mathbf{A}$:

$$\widehat{\nabla}_{\mathfrak{X}} a = \mathfrak{X} \cdot a + \omega(\mathfrak{X})a$$

Its curvature is the multiplication on the left on $M = \mathbf{A}$ by the noncommutative 2-form

$$\Omega(\mathfrak{X}, \mathfrak{Y}) = \widehat{d}_{\text{NC}}\omega(\mathfrak{X}, \mathfrak{Y}) + [\omega(\mathfrak{X}), \omega(\mathfrak{Y})]$$

The gauge group is identified with the invertible elements $g \in \mathbf{A}$ by $\Phi_g(a) = ga$.

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In that situation a NC connection is a noncommutative 1-form in $\Omega_{\text{Der}}^1(\mathbf{A})$.

The endomorphisms algebra of a $SL(n)$ -vector bundle

Noncommutative aspects

Consider the algebra $\mathbf{A} = \mathbf{A}(\mathcal{E})$ and its differential calculus $(\Omega_{\text{Der}}^\bullet(\mathbf{A}), \widehat{d}_{\text{NC}})$.

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Consider the algebra $\mathbf{A} = \mathbf{A}(\mathcal{E})$ and its differential calculus $(\Omega_{\text{Der}}^{\bullet}(\mathbf{A}), \widehat{d}_{\text{NC}})$.

Proposition (Connections on \mathcal{E})

Connections $\nabla^{\mathcal{E}}$ on \mathcal{E} are:

- splittings as $C^{\infty}(\mathcal{M})$ -modules of the s.e.s. $0 \rightarrow \mathbf{A}_0 \xrightarrow{\text{ad}} \text{Der}(\mathbf{A}) \xrightarrow{\rho} \Gamma(\mathcal{M}) \rightarrow 0$.
- noncommutative traceless 1-form $\alpha \in \Omega_{\text{Der}}^1(\mathbf{A})$ such that

$$\mathfrak{X} = \nabla_X - \text{ad}_{\alpha(\mathfrak{X})} \qquad \alpha(\text{ad}_Y) = -Y$$

(∇ is the connection on the vector bundle $\text{End}(\mathcal{E})$ induced by $\nabla^{\mathcal{E}}$)

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Proposition (Connections on \mathcal{E} as noncommutative connections)

The space of noncommutative connections on the right \mathbf{A} -module $\mathbf{M} = \mathbf{A}$ contains the space of connections on \mathcal{E} .

This inclusion is compatible with the notions of curvature and gauge transformations.

Similar to the situation for Lie algebroids.

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$(\Omega_{\text{Lie}}^\bullet(\mathbf{A}, \mathbf{A}_0), \widehat{d})$ the differential calculus defined on the transitive Lie algebroid $\text{Der}(\mathbf{A})$.

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$$\begin{aligned}
 (\widehat{d}\omega)(\mathfrak{X}_1, \dots, \mathfrak{X}_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i+1} \mathfrak{X}_i \cdot \omega(\mathfrak{X}_1, \dots, \overset{i}{\dot{\vee}} \dots, \mathfrak{X}_{p+1}) \\
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$(\Omega_{\text{Lie}}^\bullet(\mathbf{A}, \mathbf{A}_0), \widehat{d})$ (Lie algebroid structure) is included in the derivation-based differential calculus $(\Omega_{\text{Der}}^\bullet(\mathbf{A}), \widehat{d}_{\text{NC}})$ as traceless forms.

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Consider the natural representation of the Lie algebroid $\text{Der}(\mathbf{A})$ on \mathcal{E} .

Denote by $(\Omega_{\text{Lie}}^\bullet(\mathbf{A}), \widehat{d}_{\mathcal{E}})$ the graded differential algebra $(\Omega^\bullet(\text{Der}(\mathbf{A}), \mathbf{A}(\mathcal{E})), \widehat{d}_{\mathcal{E}})$.

Proposition (Identification of the derivation-based differential calculus)

$(\Omega_{\text{Lie}}^\bullet(\mathbf{A}), \widehat{d}_{\mathcal{E}}) \simeq (\Omega_{\text{Der}}^\bullet(\mathbf{A}), \widehat{d}_{\text{NC}})$ as graded differential algebras.

The spaces of connections

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Theorem (Lie algebroid connections and noncommutative connections)

The following three spaces are isomorphic:

- 1 The space of $\text{Der}(\mathbf{A})$ -connections on \mathcal{E} (Lie algebroid of derivations).
- 2 The space of $\Gamma_G(\mathcal{P})$ -connections on \mathcal{E} (Atiyah Lie algebroid of the $SL(n)$ -principal bundle).
- 3 The space of noncommutative connections on the right \mathbf{A} -module $\mathbf{M} = \mathbf{A}$ (noncommutative differential structure).

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Theorem (Generalized connections and traceless NC connections)

The following three spaces are isomorphic:

- 1 The space of generalized connections on $\text{Der}(\mathbf{A})$.
- 2 The space of generalized connections on $\Gamma_G(\mathcal{P})$.
- 3 The space of traceless noncommutative connections on the right \mathbf{A} -module $\mathbf{M} = \mathbf{A}$

These isomorphisms are compatible with curvatures and gauge transformations.

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Thank You