Noncommutative analog and generalization of $SU(n)$ principal fiber bundles

Thierry Masson

Laboratoire de Physique Théorique
Université Paris XI

Oran, 14-18th April 2007

CENTRE NATIONAL
DE LA RECHERCHE
SCIENTIFIQUE

UNIVERSITÉ
PARIS-SUD 11

Outline of the lectures

- Introduce a particular associative algebra related naturally to some constructions in ordinary fiber bundle theory.
- Equip this algebra with its derivation-based differential calculus. Study this differential calculus, and show how it is related to some usual differential structures in fiber theory.
- Compare the two notions of (ordinary) connections and noncommutative connections. Show how one is the generalisation of the other. Use it to propose Yang-Mills-Higgs type models.
- Make some algebraic constructions based on this algebra to recover ordinary characteristic classes.
- Generalize and study the notion of invariant connections by the action of a Lie group to this noncommutative context.

Content of the lectures

❶ A brief review of ordinary fiber bundle theory

❷ Derivation-based noncommutative geometry

❸ The endomorphism algebra of a vector bundle

❹ Noncommutative connections on A

❺ Relations with the principal fiber bundle

❻ Cohomology and characteristic classes

❼ Invariant noncommutative connections

Principal fiber bundles

M smooth manifold, G Lie group. $G \longrightarrow P \xrightarrow{\pi} M$ principal fiber bundle for the right action of G on P denoted by $p \mapsto p{\cdot}g = \widetilde{R}_g p$.

- For any $p \in P$, define $V_p = \mathrm{Ker}(T_p\pi : T_pP \to T_{\pi(p)}M)$ (vertical subspace of T_pP). For any $X \in \mathfrak{g}$, let
$$X^v(p) = \left(\frac{d}{dt} p{\cdot}\exp(tX)\right)_{t=0}$$
Then $V_p = \{X^v(p) / X \in \mathfrak{g}\}$. One has $\widetilde{R}_{g*} V_p = V_{p\cdot g}$.
➙ define vertical vector fields, horizontal differential forms (vanish when one of its arguments is vertical).
- (U, ϕ) local trivialisation of P over $U \subset M$ if U open subset, $\phi : U \times G \xrightarrow{\simeq} \pi^{-1}(U)$, $\pi(\phi(x,h)) = x$ and $\phi(x, hg) = \phi(x,h){\cdot}g$ for any $x \in U$ and $g, h \in G$.
- (U_i, ϕ_i) and (U_j, ϕ_j) two local trivialisations such that $U_i \cap U_j \neq \varnothing$. There exists a differentiable map $g_{ij} : U_i \cap U_j \to G$ such that if $\phi_i(x, h_i) = \phi_j(x, h_j)$ for $h_i, h_j \in G$ then $h_i = g_{ij}(x) h_j$ for any $x \in U_i \cap U_j$. The g_{ij} are the transition functions. They satisfy $g_{ij}(x) = g_{ji}^{-1}(x)$ for any $x \in U_i \cap U_j$ and the cocycle condition $g_{ij}(x) g_{jk}(x) g_{ki}(x) = e$ for any $x \in U_i \cap U_j \cap U_k \neq \varnothing$.

Associated fiber bundles

$G \to P \xrightarrow{\pi} M$ principal fiber bundle as before.

- F manifold on which G acts on the left: $\varphi \mapsto \ell_g \varphi$.
- On $P \times F$ we define the right action $(p, \varphi) \mapsto (p \cdot g, \ell_{g^{-1}} \varphi)$.
- $E = (P \times F)/G$ is the orbit space for this action.
  This is the associated fiber bundle to P for the couple (F, ℓ).
- Notations: $E = P \times_\ell F$ and $[p, \varphi] \in E$ the projection of (p, φ) in the quotient $P \times F \to (P \times F)/G$.
  Then $[p \cdot g, \varphi] = [p, \ell_g \varphi]$.

Sections

$G \twoheadrightarrow P \overset{\pi}{\twoheadrightarrow} M$ principal fiber bundle, $E \to M$ associated fiber bundle.

- A section is a map $s : M \to E$ such that $\pi \circ s(x) = x$ for any $x \in M$.
  Notation: $\Gamma(E)$ space of sections.
- $\Gamma(E) \simeq \mathcal{F}_G(P, F) = \{\Phi : P \to F \ / \ \Phi(p{\cdot}g) = \ell_{g^{-1}}\Phi(p)\}$
  (equivariant maps $P \to F$).
- (U, ϕ) local trivialisation of P over U.
  $s_U : U \to \pi^{-1}(U)$ given by $s_U(x) = \phi(x, e)$ is a local section of P.
  Any section s of P is locally given by a local map $h : U \to G$ such that $s(x) = s_U(x){\cdot}h(x) = \phi(x, h(x))$.
  Any section s of E is locally given by a local map $\varphi : U \to F$ such that $s(x) = [s_U(x), \varphi(x)]$.
- (U_i, ϕ_i) and (U_j, ϕ_j) two local trivialisations such that $U_i \cap U_j \neq \varnothing$. One has
  $$s_j(x) = \phi_j(x, e) = \phi_i(x, g_{ij}(x)) = \phi_i(x, e){\cdot}g_{ij}(x) = s_i(x){\cdot}g_{ij}(x)$$
  so that on P, if $s(x) = s_i(x){\cdot}h_i(x) = s_j(x){\cdot}h_j(x)$, then
  $$h_i(x) = g_{ij}(x)h_j(x)$$
  On E, if $s(x) = [s_i(x), \varphi_i(x)] = [s_j(x), \varphi_j(x)]$ for $x \in U_i \cap U_j \neq \varnothing$, then
  $$\varphi_i(x) = \ell_{g_{ij}(x)}\varphi_j(x)$$

Vector bundles

Let F be a vector space and ℓ a representation (linear action).

- The associated fiber bundle E is called a vector bundle.
- $\Gamma(E)$ is a $C^\infty(M)$-module for the pointwise multiplication: $f(x)s(x)$ for any $f \in C^\infty(M)$, $s \in \Gamma(E)$ and $x \in M$.
- If E and E' are vector bundles then E^* (dual), $E \oplus E'$ (Whitney sum) and $E \otimes E'$ (tensor product), $\bigwedge^\bullet E$ (exterior product) are defined.
  Use: (F^*, ℓ^*), $(\mathsf{F} \oplus \mathsf{F}', \ell \oplus \ell')$, $(\mathsf{F} \otimes \mathsf{F}', \ell \otimes \ell')$ and $(\bigwedge^\bullet \mathsf{F}, \bigwedge \ell)$.

Example 1 (The endomorphism bundle)

Consider the case where F is finite dimensional.
$E^* \otimes E$ is associated to P for the couple $(\mathsf{F}^* \otimes \mathsf{F}, \ell^* \otimes \ell)$.
One has $\mathsf{F}^* \otimes \mathsf{F} \simeq \mathrm{End}(\mathsf{F})$ by $(\alpha \otimes \varphi)(\varphi') = \alpha(\varphi')\varphi$.
We use the notation $\mathrm{End}(E) = E^* \otimes E$. It is called the endomorphism bundle of E.
There is a pairing $\Gamma(E^*) \otimes \Gamma(E) \to C^\infty(M)$ denoted by $x \mapsto \langle \alpha(x), s(x) \rangle$.
$\Gamma(E^* \otimes E) = \Gamma(\mathrm{End}(E))$ is an algebra, which identifies with $\Gamma(E^*) \otimes_{C^\infty(M)} \Gamma(E)$ and with the space of $C^\infty(M)$-module maps $\Gamma(E) \to \Gamma(E)$ by $(\alpha \otimes s)(s')(x) = \langle \alpha(x), s'(x) \rangle s(x)$.

Other useful examples

Examples 2 (Tangent and cotangent spaces)

$TM \to M$, $T^*M \to M$ are vector bundles over M.
$\Gamma(TM)$, also denoted by $\Gamma(M)$, is the $C^\infty(M)$-module of vector fields on M.
It is also a Lie algebra for the bracket $[X, Y]\cdot f = X\cdot Y\cdot f - Y\cdot X\cdot f$ for any $f \in C^\infty(M)$.
$\Gamma(\bigwedge^\bullet T^*M) = \Omega^\bullet(M)$ is the algebra of differential forms on M.

Examples 3 (The gauge group and its Lie algebra)

G acts on itself by conjugaison: $\alpha_g(h) = ghg^{-1}$. The associated fiber bundle $P \times_\alpha G$ has G as fiber but is not a principal bundle.
$\mathcal{G} = \Gamma(P \times_\alpha G)$ is a group, called the gauge group of P: it is the sub-group of $\text{Aut}(P)$ of vertical automorphisms.
$\Phi : P \to G$, G-equivariant: $p \mapsto p\cdot\Phi(p)$.
One has $p\cdot g \mapsto (p\cdot g)\cdot\Phi(p\cdot g) = (p\cdot g)\cdot(g^{-1}\Phi(p)g) = (p\cdot\Phi(p))\cdot g$.
G acts on the vector space $\mathfrak{g}$ by the adjoint action Ad.
Denote by $\text{Ad}P = P \times_{\text{Ad}} \mathfrak{g}$ the associated vector bundle. $\Gamma(\text{Ad}P)$ is the Lie algebra of the gauge group $\mathcal{G}$, denoted Lie$\mathcal{G}$.

Connections

$G \to P \xrightarrow{\pi} M$ principal fiber bundle, $E \to M$ associated vector bundle.
A connection is:

Geometrical definition: A smooth distribution H in TP such that for any $p \in P$ and $g \in G$:

$$T_p P = V_p \oplus H_p \quad \text{and} \quad \widetilde{R}_{g*} H_p = H_{p \cdot g}$$

➡ horizontal vector fields, vertical differential forms (vanish when one of its arguments is horizontal).
➡ horizontal lifting of vector fields on M: $\Gamma(M) \ni X \mapsto X^h \in \Gamma(P)$.

Algebraic definition: A 1-form on P taking values in the Lie algebra $\mathfrak{g}$, $\omega \in \Omega^1(P) \otimes \mathfrak{g}$, such that for any $g \in G$ and $X \in \mathfrak{g}$:

$$\widetilde{R}_g^* \omega = \mathrm{Ad}_{g^{-1}} \omega \text{ (equivariance)} \quad \text{and} \quad \omega(X^v) = X \text{ (vertical condition)}$$

➡ $H_p = \mathrm{Ker}\, \omega_{|p}$.

Analytic definition: A linear map $\nabla^E_X : \Gamma(E) \to \Gamma(E)$ defined for any $X \in \Gamma(M)$, such that for any $f \in C^\infty(M)$, $s \in \Gamma(E)$, $X, Y \in \Gamma(M)$:

$$\nabla^E_X(fs) = (X \cdot f)s + f\nabla^E_X s \qquad \nabla^E_{fX} s = f\nabla^E_X s \qquad \nabla^E_{X+Y} s = \nabla^E_X s + \nabla^E_Y s$$

➡ $\Gamma(E) \ni s \leftrightarrow \Phi \in \mathcal{F}_G(P, F)$ corresponds to $\nabla^E_X s \leftrightarrow X^h \cdot \Phi$.

Curvature of a connection

Geometrical definition: There exists a geometrical interpretation of the curvature which will not be explained here (obstruction to the closure of horizontal lifts of "infinitesimal" closed paths on M).

Algebraic definition: The curvature is the equivariant horizontal 2-form $\Omega \in \Omega^2(P) \otimes \mathfrak{g}$ defined for any $\mathcal{X}, \mathcal{Y} \in \Gamma(P)$ by

$$\Omega(\mathcal{X}, \mathcal{Y}) = d\omega(\mathcal{X}, \mathcal{Y}) + [\omega(\mathcal{X}), \omega(\mathcal{Y})]$$

It satisfies the Bianchi identity

$$d\Omega + [\omega, \Omega] = 0$$

Analytic definition: Given $\nabla^E_X : \Gamma(E) \to \Gamma(E)$, the curvature $R^E(X, Y)$ is the map defined for any $X, Y \in \Gamma(M)$ by

$$R^E(X, Y) = \nabla^E_X \nabla^E_Y - \nabla^E_Y \nabla^E_X - \nabla^E_{[X,Y]} : \Gamma(E) \to \Gamma(E)$$

It is a $C^\infty(M)$-module map.

η the representation of $\mathfrak{g}$ on F induced by the representation ℓ of G.
Then $\Gamma(E) \ni s \leftrightarrow \Phi \in \mathcal{F}_G(P, F)$ corresponds to $R^E(X, Y)s \leftrightarrow \eta(\Omega(\mathcal{X}, \mathcal{Y}))\cdot\Phi$ for any $\mathcal{X}, \mathcal{Y}$ s.t. $\pi_*\mathcal{X} = X$ and $\pi_*\mathcal{Y} = Y$.

Connections: local expressions

$\omega \in \Omega^1(P) \otimes \mathfrak{g}$ connection 1-form on P, Ω its curvature.
(U, ϕ) local trivialisation of P, s its local section.
Local expression of the connection and the curvature:

$$A = s^*\omega \in \Omega^1(U) \otimes \mathfrak{g} \qquad F = s^*\Omega \in \Omega^2(U) \otimes \mathfrak{g}$$

(U_i, ϕ_i) and (U_j, ϕ_j) two local trivialisations. On $U_i \cap U_j \neq \varnothing$ one has

$$A_j = g_{ij}^{-1} A_i g_{ij} + g_{ij}^{-1} \mathrm{d} g_{ij} \qquad F_j = g_{ij}^{-1} F_i g_{ij}$$

A family of 1-forms $\{A_i\}_i$ satisfying these gluing relations defines a connection 1-form on P.

Globally on P	**Locally on M**
$\omega \in \Omega^1(P) \otimes \mathfrak{g}$, equivariant, vertical condition.	Family of local 1-forms $\{A_i\}_i$, $A_i \in \Omega^1(U_i) \otimes \mathfrak{g}$, satisfying gluing non homogeneous relations.
$\Omega \in \Omega^2(P) \otimes \mathfrak{g}$, equivariant and horizontal.	Family of local 2-forms $\{F_i\}_i$, $F_i \in \Omega^2(U_i) \otimes \mathfrak{g}$, satisfying gluing homogeneous relations.

Remark 4 (Intermediate construction)
The curvature is a section of the associated vector bundle $\bigwedge^2 T^*M \otimes \mathrm{Ad}P$.
The connection cannot be the section of such an "intermediate" construction between forms on P and local forms on the U_i's.

Gauge transformations

$\mathcal{G} = \Gamma(P \times_\alpha G)$ acts on P.
$a \in \mathcal{G} \;\; \longleftrightarrow \Phi : P \to G$, G-equivariant.
a induces a vertical diffeomorphism $P \to P$, denoted by a.
$\omega \in \Omega^1(P) \otimes \mathfrak{g}$ connection.

- $a^*\omega$ is a connection and $a^*\Omega$ is its curvature.
- $a^*\omega = \Phi^{-1}\omega\Phi + \Phi^{-1}\mathrm{d}\Phi$.
- $a^*\Omega = \Phi^{-1}\Omega\Phi$.

Infinitesimal version: $\Phi = \exp(\xi)$ with $\xi : P \to \mathfrak{g}$, G-equivariant.
➡ ξ defines an element in $\mathrm{Lie}\mathcal{G} = \Gamma(\mathrm{Ad}P)$.
Infinitesimal action on connections and curvatures:

$$\omega \mapsto \mathrm{d}\xi + [\omega, \xi] \qquad\qquad \Omega \mapsto [\Omega, \xi]$$

Derivations of an associative algebra

A associative algebra with unit.
$\mathcal{Z}(\mathbf{A})$ center of **A**: commutative subalgebra.

Definition 5 (Vector space of derivations of A)
$\text{Der}(\mathbf{A}) = \{\mathfrak{X} : \mathbf{A} \to \mathbf{A} \ / \ \mathfrak{X} \text{ linear}, \mathfrak{X}(ab) = \mathfrak{X}(a)b + a\mathfrak{X}(b), \forall a, b \in \mathbf{A}\}$

Proposition 6 (Structure of Der(A))
$\text{Der}(\mathbf{A})$ *is a Lie algebra for the bracket* $[\mathfrak{X}, \mathfrak{Y}]a = \mathfrak{X}\mathfrak{Y}a - \mathfrak{Y}\mathfrak{X}a$ *(*$\forall \mathfrak{X}, \mathfrak{Y} \in \text{Der}(\mathbf{A})$*),*
is a $\mathcal{Z}(\mathbf{A})$*-module for the product* $(f\mathfrak{X})a = f(\mathfrak{X}a)$ *(*$\forall f \in \mathcal{Z}(\mathbf{A})$, $\forall \mathfrak{X} \in \text{Der}(\mathbf{A})$*).*

$\text{Int}(\mathbf{A}) = \{\text{ad}_a : b \mapsto [a, b] \ / \ a \in \mathbf{A}\} \subset \text{Der}(\mathbf{A})$*, vector space of inner derivations, is a Lie ideal and a* $\mathcal{Z}(\mathbf{A})$*-submodule.*
With $\text{Out}(\mathbf{A}) = \text{Der}(\mathbf{A})/\text{Int}(\mathbf{A})$*, there is a s.e.s. of Lie algebras and* $\mathcal{Z}(\mathbf{A})$*-modules*

$$0 \longrightarrow \text{Int}(\mathbf{A}) \longrightarrow \text{Der}(\mathbf{A}) \longrightarrow \text{Out}(\mathbf{A}) \longrightarrow 0$$

Definition 7 (Real derivations for involutive algebras)
If **A** is an involutive algebra (notation: $a \mapsto a^*$), $\mathfrak{X} \in \text{Der}(\mathbf{A})$ is real if $(\mathfrak{X}a)^* = \mathfrak{X}a^*$ for any $a \in \mathbf{A}$. Notation: $\text{Der}_{\mathbb{R}}(\mathbf{A})$.

Derivation-based differential calculus

A associative algebra with unit.

Definition 8 (The graded differential algebra $\underline{\Omega}^\bullet_{\mathrm{Der}}(\mathbf{A})$)
Let $\underline{\Omega}^n_{\mathrm{Der}}(\mathbf{A})$ be the set of $\mathcal{Z}(\mathbf{A})$-multilinear antisymmetric maps from $\mathrm{Der}(\mathbf{A})^n$ to $\mathbf{A}$, with $\underline{\Omega}^0_{\mathrm{Der}}(\mathbf{A}) = \mathbf{A}$. Define

$$\underline{\Omega}^\bullet_{\mathrm{Der}}(\mathbf{A}) = \textstyle\bigoplus_{n\geq 0} \underline{\Omega}^n_{\mathrm{Der}}(\mathbf{A})$$

- $\mathbb{N}$-graded algebra for the product

$$(\omega\eta)(\mathfrak{X}_1, \dots, \mathfrak{X}_{p+q}) = \frac{1}{p!q!} \sum_{\sigma\in\mathfrak{S}_{p+q}} (-1)^{\mathrm{sign}(\sigma)} \omega(\mathfrak{X}_{\sigma(1)}, \dots, \mathfrak{X}_{\sigma(p)})\eta(\mathfrak{X}_{\sigma(p+1)}, \dots, \mathfrak{X}_{\sigma(p+q)})$$

- differential graded algebra for the differential d (of degree 1) defined by (Koszul formula)

$$\begin{aligned} \mathrm{d}\omega(\mathfrak{X}_1, \dots, \mathfrak{X}_{n+1}) = & \sum_{i=1}^{n+1} (-1)^{i+1} \mathfrak{X}_i \omega(\mathfrak{X}_1, \dots \overset{i}{\vee} \dots, \mathfrak{X}_{n+1}) \\ & + \sum_{1\leq i<j\leq n+1} (-1)^{i+j} \omega([\mathfrak{X}_i, \mathfrak{X}_j], \dots \overset{i}{\vee} \dots \overset{j}{\vee} \dots, \mathfrak{X}_{n+1}) \end{aligned}$$

Derivation-based differential calculus

Definition 9 (The graded differential algebra $\Omega^\bullet_{\mathrm{Der}}(\mathbf{A})$)

$\Omega^\bullet_{\mathrm{Der}}(\mathbf{A}) \subset \underline{\Omega}^\bullet_{\mathrm{Der}}(\mathbf{A})$ the sub differential graded algebra generated in degree 0 by $\mathbf{A}$.

Every element in $\Omega^n_{\mathrm{Der}}(\mathbf{A})$ is a sum of terms of the form $a_0 \mathrm{d}a_1 \cdots \mathrm{d}a_n$ for $a_0, \dots, a_n \in \mathbf{A}$.

Example 10 (The algebra $\mathbf{A} = C^\infty(M)$)

Let M be a M and let $\mathbf{A} = C^\infty(M)$.

- $\mathcal{Z}(\mathbf{A}) = C^\infty(M)$.
- $\mathrm{Der}(\mathbf{A}) = \Gamma(M)$, $\mathrm{Int}(\mathbf{A}) = 0$, $\mathrm{Out}(\mathbf{A}) = \Gamma(M)$.
- $\Omega^\bullet_{\mathrm{Der}}(\mathbf{A}) = \underline{\Omega}^\bullet_{\mathrm{Der}}(\mathbf{A}) = \Omega^\bullet(M)$, gr. diff. alg. of de Rham forms on M.

Definition 11 (Restricted derivation-based differential calculus)

Let $\mathfrak{g} \subset \mathrm{Der}(\mathbf{A})$ be a sub Lie algebra and a sub $\mathcal{Z}(\mathbf{A})$-module. The restricted derivation-based differential calculus $\underline{\Omega}^\bullet_{\mathfrak{g}}(\mathbf{A})$ is defined as the set of $\mathcal{Z}(\mathbf{A})$-multilinear antisymmetric maps from $\mathfrak{g}^n$ to $\mathbf{A}$ for $n \geq 0$, using the previous formulae for the product and the differential.

Cartan operations

Let $\mathfrak{g}$ be a Lie subalgebra of $\text{Der}(\mathbf{A})$.
$\mathfrak{g}$ defines a natural operation in the sense of H. Cartan on $(\underline{\Omega}^\bullet_{\text{Der}}(\mathbf{A}), \text{d})$.

- Interior product: graded derivation of degree -1 on $\underline{\Omega}^\bullet_{\text{Der}}(\mathbf{A})$.
  $$i_{\mathfrak{X}} : \underline{\Omega}^n_{\text{Der}}(\mathbf{A}) \to \underline{\Omega}^{n-1}_{\text{Der}}(\mathbf{A}) \qquad (i_{\mathfrak{X}}\omega)(\mathfrak{X}_1, \dots, \mathfrak{X}_{n-1}) = \omega(\mathfrak{X}, \mathfrak{X}_1, \dots, \mathfrak{X}_{n-1})$$
  $\forall \mathfrak{X} \in \mathfrak{g}$, $\forall \omega \in \underline{\Omega}^n_{\text{Der}}(\mathbf{A})$ and $\forall \mathfrak{X}_i \in \text{Der}(\mathbf{A})$.
  $i_{\mathfrak{X}}$ is 0 on $\underline{\Omega}^0_{\text{Der}}(\mathbf{A}) = \mathbf{A}$.
- Lie derivative: graded derivation of degree 0 on $\underline{\Omega}^\bullet_{\text{Der}}(\mathbf{A})$.
  $$L_{\mathfrak{X}} = i_{\mathfrak{X}}\text{d} + \text{d}i_{\mathfrak{X}} : \underline{\Omega}^n_{\text{Der}}(\mathbf{A}) \to \underline{\Omega}^n_{\text{Der}}(\mathbf{A})$$
- One has:
  $$i_{\mathfrak{X}} i_{\mathfrak{Y}} + i_{\mathfrak{Y}} i_{\mathfrak{X}} = 0 \qquad L_{\mathfrak{X}} i_{\mathfrak{Y}} - i_{\mathfrak{Y}} L_{\mathfrak{X}} = i_{[\mathfrak{X},\mathfrak{Y}]}$$
  $$L_{\mathfrak{X}} L_{\mathfrak{Y}} - L_{\mathfrak{Y}} L_{\mathfrak{X}} = L_{[\mathfrak{X},\mathfrak{Y}]} \qquad L_{\mathfrak{X}}\text{d} - \text{d}L_{\mathfrak{X}} = 0$$
- Subspaces of $\underline{\Omega}^\bullet_{\text{Der}}(\mathbf{A})$ associated to this operation:
  - Horizontal subspace: kernel of all the $i_{\mathfrak{X}}$ for $\mathfrak{X} \in \mathfrak{g}$. Gr. Alg.
  - Invariant subspace: kernel of all the $L_{\mathfrak{X}}$ for $\mathfrak{X} \in \mathfrak{g}$. Gr. Diff. Alg.
  - Basic subspace: kernel of all the $i_{\mathfrak{X}}$ and $L_{\mathfrak{X}}$ for $\mathfrak{X} \in \mathfrak{g}$. Gr. Diff. Alg.

Example: $\mathfrak{g} = \text{Int}(\mathbf{A})$ defines an operation.

Noncommutative connections

Let $\boldsymbol{M}$ be a right $\mathbf{A}$-module.

Definition 12 (Noncommutative connection, curvature)
A noncommutative connection is a linear map $\widehat{\nabla}_{\mathfrak{X}} : \boldsymbol{M} \to \boldsymbol{M}$, defined for any $\mathfrak{X} \in \mathrm{Der}(\mathbf{A})$, such that $\forall \mathfrak{X}, \mathfrak{Y} \in \mathrm{Der}(\mathbf{A})$, $\forall a \in \mathbf{A}$, $\forall m \in \boldsymbol{M}$, $\forall f \in \mathcal{Z}(\mathbf{A})$:

$$\widehat{\nabla}_{\mathfrak{X}}(ma) = m(\mathfrak{X}a) + (\widehat{\nabla}_{\mathfrak{X}} m)a, \quad \widehat{\nabla}_{f\mathfrak{X}} m = f \widehat{\nabla}_{\mathfrak{X}} m, \quad \widehat{\nabla}_{\mathfrak{X}+\mathfrak{Y}} m = \widehat{\nabla}_{\mathfrak{X}} m + \widehat{\nabla}_{\mathfrak{Y}} m$$

The curvature of $\widehat{\nabla}$ is the linear map $\widehat{R}(\mathfrak{X}, \mathfrak{Y}) : \boldsymbol{M} \to \boldsymbol{M}$ defined for any $\mathfrak{X}, \mathfrak{Y} \in \mathrm{Der}(\mathbf{A})$ by

$$\widehat{R}(\mathfrak{X}, \mathfrak{Y})m = [\widehat{\nabla}_{\mathfrak{X}}, \widehat{\nabla}_{\mathfrak{Y}}]m - \widehat{\nabla}_{[\mathfrak{X}, \mathfrak{Y}]} m$$

Proposition 13 (General properties)

- *The space of connections is an affine space modeled over the vector space* $\mathrm{Hom}^{\mathbf{A}}(\underline{\Omega}^1_{\mathrm{Der}}(\mathbf{A}), \boldsymbol{M})$ *(right* $\mathbf{A}$*-module homomorphisms).*
- $\widehat{R}(\mathfrak{X}, \mathfrak{Y}) : \boldsymbol{M} \to \boldsymbol{M}$ *is a right* $\mathbf{A}$*-module homomorphism.*

Gauge transformations

$\boldsymbol{M}$ a right $\mathbf{A}$-module.

Definition 14 (Gauge group)

The gauge group of $\boldsymbol{M}$ is the group of automorphisms of $\boldsymbol{M}$ as a right $\mathbf{A}$-module.

Proposition 15 (Gauge transformations)

For any Φ in the gauge group of $\boldsymbol{M}$ and any n.c. connection $\widehat{\nabla}$, the map $\widehat{\nabla}^{\Phi}_{\mathfrak{X}} = \Phi^{-1} \circ \widehat{\nabla}_{\mathfrak{X}} \circ \Phi : \boldsymbol{M} \to \boldsymbol{M}$ is a n.c. connection.
This defines the action of the gauge group on the space of n.c. connections.

Proof of the derivation rule:

$$\begin{aligned}
\widehat{\nabla}^{\Phi}_{\mathfrak{X}}(ma) &= \Phi^{-1}\left(\widehat{\nabla}_{\mathfrak{X}}\Phi(ma)\right) = \Phi^{-1}\left(\widehat{\nabla}_{\mathfrak{X}}(\Phi(m)a)\right) \\
&= \Phi^{-1}\left(\Phi(m)(\mathfrak{X}a) + (\widehat{\nabla}_{\mathfrak{X}}\Phi(m))a\right) \\
&= m(\mathfrak{X}a) + (\Phi^{-1}\widehat{\nabla}_{\mathfrak{X}}\Phi)(m)a = m(\mathfrak{X}a) + (\widehat{\nabla}^{\Phi}_{\mathfrak{X}}m)a
\end{aligned}$$

□

Hermitean structures

Suppose now that **A** is an involutive algebra.
Let $\boldsymbol{M}$ be a right **A**-module.

Definition 16 (Hermitean structure)

A Hermitean structure on $\boldsymbol{M}$ is a sesquilinear form $\langle -,-\rangle : \boldsymbol{M} \times \boldsymbol{M} \to \mathbf{A}$ such that, $\forall m_1, m_2 \in \boldsymbol{M}$, $\forall a_1, a_2 \in \mathbf{A}$,

$$\langle m_1, m_2\rangle^* = \langle m_2, m_1\rangle \qquad \langle m_1 a_1, m_2 a_2\rangle = a_1^* \langle m_1, m_2\rangle a_2$$

A n.c. connection $\widehat{\nabla}$ is compatible with $\langle -,-\rangle$ if, $\forall m_1, m_2 \in \boldsymbol{M}$, $\forall \mathfrak{X} \in \mathrm{Der}_{\mathbb{R}}(\mathbf{A})$,

$$\mathfrak{X}\langle m_1, m_2\rangle = \langle \widehat{\nabla}_{\mathfrak{X}} m_1, m_2\rangle + \langle m_1, \widehat{\nabla}_{\mathfrak{X}} m_2\rangle$$

Definition 17 (“Unitary” gauge transformations)

An element Φ in the gauge group is compatible with the Hermitean structure if $\langle \Phi(m_1), \Phi(m_2)\rangle = \langle m_1, m_2\rangle$.

Lemma 18

The space of compatible n.c. connections with $\langle -,-\rangle$ is stable under “unitary” gauge transformations.

Noncommutative connections: $M = \mathbf{A}$

A is a unital algebra.
As a special case we consider the right **A**-module $M = \mathbf{A}$.
Let $\widehat{\nabla}_{\mathfrak{X}} : \mathbf{A} \to \mathbf{A}$ be a noncommutative connection.

Proposition 19 (Noncommutative connections on $M = \mathbf{A}$)
$\widehat{\nabla}$ *is completely given by* $\widehat{\nabla}_{\mathfrak{X}} \mathbb{1} = \omega(\mathfrak{X})$*, with* $\omega \in \underline{\Omega}^1_{\text{Der}}(\mathbf{A})$*, by the relation*

$$\widehat{\nabla}_{\mathfrak{X}} a = \mathfrak{X} a + \omega(\mathfrak{X}) a$$

The curvature of $\widehat{\nabla}$ *is the multiplication on the left on* **A** *by the n.c. 2-form*

$$\Omega(\mathfrak{X}, \mathfrak{Y}) = \mathrm{d}\omega(\mathfrak{X}, \mathfrak{Y}) + [\omega(\mathfrak{X}), \omega(\mathfrak{Y})]$$

The gauge group is identified with the invertible elements $g \in \mathbf{A}$ *by* $\Phi_g(a) = ga$*.*
The gauge transformations on $\widehat{\nabla}$ *take the following form on* ω *and* Ω*:*

$$\omega \mapsto \omega^g = g^{-1} \omega g + g^{-1} \mathrm{d}g \qquad \Omega \mapsto \Omega^g = g^{-1} \Omega g$$

$\widehat{\nabla}^0_{\mathfrak{X}}$ *defined by* $a \mapsto \mathfrak{X} a$ *is a n.c. connection on* **A***.*

Particular case: **A** involutive. Hermitean structure: $\langle a, b \rangle = a^* b$.
$U(\mathbf{A}) = \{ u \in \mathbf{A} \ / \ u^* u = u u^* = \mathbb{1} \}$, the group of unitary elements of **A**, is the unitary gauge group.

Noncommutative connections: $M = \mathbf{A}$

Remark 20 (Vector space *versus* gauge transformations)

In ordinary differential geometry, the space of connections is an affine space. Here it looks like the vector space $\underline{\Omega}^1_{\mathrm{Der}}(\mathbf{A})$.
Gauge transformations are not compatible with this linear structure:

$$(\lambda_1\omega_1 + \lambda_2\omega_2)^u = u^{-1}(\lambda_1\omega_1 + \lambda_2\omega_2)u + u^{-1}\mathrm{d}u$$

$$\lambda_1\omega_1^u + \lambda_2\omega_2^u = \lambda_1(u^{-1}\omega_1 u + u^{-1}\mathrm{d}u) + \lambda_2(u^{-1}\omega_2 u + u^{-1}\mathrm{d}u)$$

are not equal except for $\lambda_1 + \lambda_2 = 1$.

Proposition 21 (Canonical gauge invariant n.c. connection)

If there exists a n.c. 1-form $\xi \in \underline{\Omega}^1_{\mathrm{Der}}(\mathbf{A})$ such that $\mathrm{d}a = [\xi, a]$ for any $a \in \mathbf{A}$, then the canonical n.c. connection defined by $\widehat{\nabla}^{-\xi}_{\mathfrak{X}} a = \mathfrak{X}a - \xi(\mathfrak{X})a$ can be written as $\widehat{\nabla}^{-\xi}_{\mathfrak{X}} a = -a\xi(\mathfrak{X})$ and it is gauge invariant.

Proof.

One has $\mathfrak{X}a = [\xi(\mathfrak{X}), a]$, so $\widehat{\nabla}^{-\xi}_{\mathfrak{X}} a = [\xi(\mathfrak{X}), a] - \xi(\mathfrak{X})a = -a\xi(\mathfrak{X})$.
Let $u \in U(\mathbf{A})$ be a unitary gauge transformation. Its action on the n.c. 1-form $-\xi$ is $(-\xi)^u = -u^{-1}\xi u + u^{-1}\mathrm{d}u = u^{-1}(-\xi u + [\xi, u]) = u^{-1}(-u\xi) = -\xi$. □

Noncommutative connections: $M = \mathbf{A}^N$

As a special case we consider the right $\mathbf{A}$-module $\boldsymbol{M} = \mathbf{A}^N$.
Let $\widehat{\nabla}_{\mathfrak{X}} : \mathbf{A}^N \to \mathbf{A}^N$ be a noncommutative connection. Let $e_i = (0, \dots, \mathbb{1}, \dots, 0)$, for $i = 1, \dots, N$, (basis of the right module $\mathbf{A}^N$). Look at $m = e_i a^i$ as a column vector for the a^i's, use the notation $\mathfrak{X}m = e_i(\mathfrak{X}a^i)$ and the matrix product.

Proposition 22 (Noncommutative connections on $\boldsymbol{M} = \mathbf{A}^N$)
*$\widehat{\nabla}$ is completely determined by N^2 n.c. 1-forms $\omega_i^j \in \underline{\Omega}^1_{\mathrm{Der}}(\mathbf{A})$ defined by $\widehat{\nabla}_{\mathfrak{X}} e_i = e_j \omega_i^j(\mathfrak{X})$, through the relation $\widehat{\nabla}_{\mathfrak{X}} m = \mathfrak{X}m + \omega(\mathfrak{X})m$, with $\omega = (\omega_i^j) \in M_N(\underline{\Omega}^1_{\mathrm{Der}}(\mathbf{A}))$. The curvature of $\widehat{\nabla}$ is the multiplication on the left on $\mathbf{A}^N$ by the matrix of n.c. 2-forms $\Omega = \mathrm{d}\omega + [\omega, \omega] \in M_N(\underline{\Omega}^2_{\mathrm{Der}}(\mathbf{A}))$.
The gauge group of $\mathbf{A}^N$ is $GL_N(\mathbf{A})$ (invertibles in $M_N(\mathbf{A})$), which acts by left (matrix) multiplication. The gauge transformations take the forms $\omega^g = g^{-1}\omega g + g^{-1}\mathrm{d}g$ and $\Omega^g = g^{-1}\Omega g$ in matrix notations.
$\widehat{\nabla}^0_{\mathfrak{X}}$ defined by $m \mapsto \mathfrak{X}m$ is a n.c. connection on $\mathbf{A}^N$.*

Particular case: $\mathbf{A}$ involutive. Hermitean structure: $\langle (a^i), (b^j) \rangle = \sum_{i=1}^N (a^i)^* b^i$.
$U_N(\mathbf{A}) = \{u \in M_N(\mathbf{A}) \ / \ u^* u = u u^* = \mathbb{1}_N\}$, the group of unitary elements of $M_N(\mathbf{A})$, is the unitary gauge group.

Noncommutative connections: projective finitely generated modules

$\boldsymbol{M}$ is a direct summand in $\mathbf{A}^N$. There exists a projection $p \in M_N(\mathbf{A})$ such that $\boldsymbol{M} = p\mathbf{A}^N$.

Proposition 23 (Noncommutative connections on p.f.g.m.)

If $\widehat{\nabla}$ is a n.c. connection on $\mathbf{A}^N$, then $m \mapsto p\widehat{\nabla}_{\mathfrak{X}} m$ defines a n.c. connection on $\boldsymbol{M}$, where $m \in \boldsymbol{M} \subset \mathbf{A}^N$.
The curvature of the n.c. connection obtained this way from $\widehat{\nabla}^0_{\mathfrak{X}}$ is the multiplication on the left on $\boldsymbol{M} \subset \mathbf{A}^N$ by the matrix of n.c. 2-forms $p\mathrm{d}p\mathrm{d}p$.

Example 24 (The algebra $\mathbf{A} = C^\infty(M)$)

We saw that the n.c. derivation-based differential calculus is the ordinary de Rham calculus.
By the theorem by Serre and Swan, projective finitely generated modules over $C^\infty(M)$ are sections of vector bundles.
The definitions of connections and n.c. connections coincide.

$\mathbf{A} = M_n(\mathbb{C})$: derivations, differential calculus

Consider the case $\mathbf{A} = M_n(\mathbb{C}) = M_n$, the finite dimensional algebra of $n \times n$ complex matrices. This is an involutive algebra for the adjointness of matrices.

Proposition 25 (General properties of the differential calculus)

- *$\mathcal{Z}(M_n) = \mathbb{C}$.*
- $\mathrm{Der}(M_n) = \mathrm{Int}(M_n) \simeq \mathfrak{sl}_n = \mathfrak{sl}(n, \mathbb{C})$ *(traceless matrices) by the isomorphism which associates to any $\gamma \in \mathfrak{sl}_n(\mathbb{C})$ the derivation* $\mathrm{ad}_\gamma : a \mapsto [\gamma, a]$. $\mathrm{Der}_{\mathbb{R}}(M_n) = \mathfrak{su}(n)$. $\mathrm{Out}(M_n) = 0$.
- $\underline{\Omega}^\bullet_{\mathrm{Der}}(M_n) = \Omega^\bullet_{\mathrm{Der}}(M_n) \simeq M_n \otimes \bigwedge^\bullet \mathfrak{sl}_n^*$*, with the differential* d' *coming from the Lie algebra differential complex for $\mathfrak{sl}_n$ represented on M_n by the commutator.*
- *There exits a canonical 1-form $i\theta \in \Omega^1_{\mathrm{Der}}(M_n)$ such that for any $\gamma \in M_n(\mathbb{C})$*
$$i\theta(\mathrm{ad}_\gamma) = \gamma - \frac{1}{n}\,\mathrm{Tr}(\gamma)\mathbb{1}$$
$i\theta$ makes the explicit isomorphism $\mathrm{Int}(M_n(\mathbb{C})) \xrightarrow{\simeq} \mathfrak{sl}_n$.
- *$i\theta$ satisfies the relation $\mathrm{d}'(i\theta) - (i\theta)^2 = 0$. This makes $i\theta$ look very much like the Maurer-Cartan form in the geometry of Lie groups (here $SL_n(\mathbb{C})$).*
- *For any $a \in M_n$, one has $\mathrm{d}'a = [i\theta, a] \in \Omega^1_{\mathrm{Der}}(M_n)$. False in higher degrees.*

$A = M_n(\mathbb{C})$: explicit decompositions, cohomology

- $\{E_k\}_{k=1,\dots,n^2-1}$ basis for $\mathfrak{sl}_n$ of hermitean matrices.
  ➞ basis of $\text{Der}(M_n) \simeq \mathfrak{sl}_n$: $n^2 - 1$ derivations $\partial_k = ad_{iE_k}$ (real derivations).
  ➞ The E_k's and $\mathbb{1}$ form a basis of M_n.
- θ^ℓ's defined by duality: $\theta^\ell(\partial_k) = \delta^\ell_k$.
  ➞ $\{\theta^\ell\}$ basis of 1-forms in $\bigwedge^\bullet \mathfrak{sl}_n^*$, they anticommute: $\theta^\ell \theta^k = -\theta^k \theta^\ell$.
- Let $[E_k, E_\ell] = C^m_{k\ell} E_m$, then:

$$\mathrm{d}'\mathbb{1} = 0 \qquad \mathrm{d}' E_k = -C^m_{k\ell} E_m \theta^\ell \qquad \mathrm{d}'\theta^k = -\frac{1}{2} C^k_{\ell m} \theta^\ell \theta^m$$

- $i\theta = iE_k\theta^k \in M_n \otimes \bigwedge^1 \mathfrak{sl}_n^*$.

Proposition 26 (The cohomology of the differential calculus)
The cohomology of the differential algebra $(\Omega^\bullet_{\text{Der}}(M_n), \mathrm{d}')$ *is*

$$H^\bullet(\Omega^\bullet_{\text{Der}}(M_n), d') = \mathcal{I}(\textstyle\bigwedge^\bullet \mathfrak{sl}_n^*)$$

the invariant elements for the natural Lie derivative.
The algebra $\mathcal{I}(\bigwedge^\bullet \mathfrak{sl}_n^*)$ *is the graded commutative algebra generated by elements* c^n_{2r-1} *in degree* $2r-1$ *for* $r \in \{2, 3, \dots, n\}$.

$\mathbf{A} = M_n(\mathbb{C})$: metric, integration

Let us introduce $g_{k\ell} = \frac{1}{n}\,\mathrm{Tr}(E_k E_\ell)$.
Then the $g_{k\ell}$'s define a natural metric (scalar product) on $\mathrm{Der}(M_n)$ if one lets $g(\partial_k, \partial_\ell) = g_{k\ell}$.
Every differential form of maximal degree $\omega \in \Omega^{n^2-1}_{\mathrm{Der}}(M_n)$ can be written uniquely in the form

$$\omega = a\sqrt{|g|}\theta^1 \cdots \theta^{n^2-1}$$

where $a \in M_n$ and where $|g|$ is the determinant of the matrix $(g_{k\ell})$.

Definition 27 (Noncommutative integration)

One defines a noncommutative integration

$$\int_{\text{n.c.}} : \Omega^{\bullet}_{\mathrm{Der}}(M_n) \to \mathbb{C}$$

by $\int_{\text{n.c.}} \omega = \frac{1}{n}\,\mathrm{Tr}(a)$ if $\omega \in \Omega^{n^2-1}_{\mathrm{Der}}(M_n)$ written as above, and 0 otherwise.
This integration satisfies the closure relation

$$\int_{\text{n.c.}} \mathrm{d}'\omega = 0$$

$\mathbf{A} = M_n(\mathbb{C})$: canonical n.c. connection

Let us consider the right **A**-module $M = \mathbf{A}$.
The n.c. 1-form $-i\theta$ defines a canonical n.c. connection by the relation $\widehat{\nabla}^{-i\theta}_{\mathfrak{X}} a = \mathfrak{X}a - i\theta(\mathfrak{X})a$ for any $a \in \mathbf{A}$.

Proposition 28 (Properties of $\widehat{\nabla}^{-i\theta}$)

For any $a \in M_n$ and $\mathfrak{X} = \mathrm{ad}_\gamma \in \mathrm{Der}(M_n)$ (with $\mathrm{Tr}\,\gamma = 0$), one has

$$\widehat{\nabla}^{-i\theta}_{\mathfrak{X}} a = -ai\theta(\mathfrak{X}) = -a\gamma$$

$\widehat{\nabla}^{-i\theta}$ is gauge invariant.
The curvature of $\widehat{\nabla}^{-i\theta}$ is zero.

Proof.
This is a consequence of the existence of the canonical gauge invariant n.c. connection implied by the relation $\mathrm{d}'a = [i\theta, a]$.
The curvature is the n.c. 2-form
$\Omega(\mathfrak{X}, \mathfrak{Y}) = \mathrm{d}' i\theta(\mathfrak{X}, \mathfrak{Y}) + [i\theta(\mathfrak{X}), i\theta(\mathfrak{Y})] = \mathrm{d}' i\theta(\mathfrak{X}, \mathfrak{Y}) + (i\theta)^2(\mathfrak{X}, \mathfrak{Y}) = 0.$ □

$\mathbf{A} = M_n(\mathbb{C})$: flat connections

Let us consider the right **A**-module $\boldsymbol{M} = M_{r,n}$, the vector space of $r \times n$ complex matrices with the obvious right module structure and the Hermitean structure $\langle m_1, m_2 \rangle = m_1^* m_2 \in M_n$.

Proposition 29 ($\widehat{\nabla}^{-i\theta}$, flat n.c. connections)

The n.c. connection $\widehat{\nabla}^{-i\theta}_{\mathfrak{X}} m = -mi\theta(\mathfrak{X})$ *is well defined, it is compatible with the Hermitean structure and its curvature is zero.*

Any n.c. connection can be written $\widehat{\nabla}_{\mathfrak{X}} a = \widehat{\nabla}^{-i\theta}_{\mathfrak{X}} a + A(\mathfrak{X}) a$ *for* $A = A_k \theta^k$ *with* $A_k \in M_r$. *The curvature of* $\widehat{\nabla}$ *is the multiplication on the left by the* M_r*-valued n.c. 2-form*

$$F = \frac{1}{2}([A_k, A_\ell] - C^m_{k\ell} A_m)\theta^k \theta^\ell$$

This curvature vanishes if and only if $A : \mathfrak{sl}_n \to M_r$ *is a representation of the Lie algebra* $\mathfrak{sl}_n$.

Two flat connections are in the same gauge orbit if and only if the corresponding Lie algebra representations are equivalent.

$\mathbf{A} = C^\infty(M) \otimes M_n(\mathbb{C})$: generalities

Consider the mixed of the two algebras $C^\infty(M)$ and $M_n(\mathbb{C})$ studied before, in the form of matrix valued functions on M ($\dim M = m$).

Proposition 30 (General properties of the differential calculus)

- $\mathcal{Z}(\mathbf{A}) = C^\infty(M)$.
- $\mathrm{Der}(\mathbf{A}) = [\mathrm{Der}(C^\infty(M)) \otimes \mathbb{1}] \oplus [C^\infty(M) \otimes \mathrm{Der}(M_n)] = \Gamma(M) \oplus [C^\infty(M) \otimes \mathfrak{sl}_n]$ *as Lie algebras and $C^\infty(M)$-modules. Notations:* $\mathfrak{X} = X + \mathrm{ad}_\gamma$ *with* $X \in \Gamma(M)$ *and* $\gamma \in C^\infty(M) \otimes \mathfrak{sl}_n = \mathbf{A}_0$ *(traceless elements in* $\mathbf{A}$*).*
  ➞ $\mathrm{Int}(\mathbf{A}) = \mathbf{A}_0$ *and* $\mathrm{Out}(\mathbf{A}) = \Gamma(M)$.
- $\underline{\Omega}^\bullet_{\mathrm{Der}}(\mathbf{A}) = \Omega^\bullet_{\mathrm{Der}}(\mathbf{A}) = \Omega^\bullet(M) \otimes \Omega^\bullet_{\mathrm{Der}}(M_n)$ *with the differential* $\widehat{\mathrm{d}} = \mathrm{d} + \mathrm{d}'$.
- *The n.c. 1-form $i\theta$ is defined as $i\theta(X + \mathrm{ad}_\gamma) = \gamma$. It splits the s.e.s.*

$$0 \longrightarrow \mathbf{A}_0 \underset{\overset{i\theta}{\longleftarrow}}{\longrightarrow} \mathrm{Der}(\mathbf{A}) \longrightarrow \Gamma(M) \longrightarrow 0$$

- *N.C. integration is a well-defined map of differential complexes*

$$\int_{n.c.} : \Omega^\bullet_{\mathrm{Der}}(\mathbf{A}) \to \Omega^{\bullet - (n^2-1)}(M) \qquad\qquad \int_{n.c.} \widehat{\mathrm{d}}\omega = \mathrm{d} \int_{n.c.} \omega$$

$\mathbf{A} = C^\infty(M) \otimes M_n(\mathbb{C})$: other structures

Using a metric h on M and the metric $g_{k\ell} = \frac{1}{n}\operatorname{Tr}(E_k E_\ell)$ on the matrix part, one can define a metric on $\operatorname{Der}(\mathbf{A})$:

$$\widehat{g}(X + \operatorname{ad}_\gamma, Y + \operatorname{ad}_\eta) = h(X, Y) + \frac{1}{m^2} g(\gamma\eta)$$

where m is a positive constant which measures the relative "weight" of the two "spaces". In physical natural units, it has the dimension of a mass.

Consider the right $\mathbf{A}$-module $\boldsymbol{M} = \mathbf{A}$. As for the algebra M_n, the n.c. 1-form $-i\theta$ defines a canonical n.c. connection by the relation $\widehat{\nabla}^{-i\theta}_{\mathfrak{X}} a = \mathfrak{X}a - i\theta(\mathfrak{X})a$ for any $a \in \mathbf{A}$.

Proposition 31 (Properties of $\widehat{\nabla}^{-i\theta}$)

For any $a \in \mathbf{A}$ and $\mathfrak{X} = X + \operatorname{ad}_\gamma \in \operatorname{Der}(\mathbf{A})$, one has $\widehat{\nabla}^{-i\theta}_{\mathfrak{X}} a = X{\cdot}a - a\gamma$.
The curvature of $\widehat{\nabla}^{-i\theta}$ is zero.
The gauge transformed connection $\widehat{\nabla}^{-i\theta g}$ by $g \in C^\infty(M) \otimes GL_n(\mathbb{C})$ is associated to the n.c. 1-form $\mathfrak{X} \mapsto -i\theta(\mathfrak{X}) + g^{-1}(X{\cdot}g) = -\gamma + g^{-1}(X{\cdot}g)$.

The endomorphism algebra

E a $SU(n)$-vector bundle over M (fiber $\mathbb{C}^n$). $\text{End}(E)$ fiber bundle of endomorphisms of E, $\mathbf{A}$ the algebra of sections of $\text{End}(E)$.
For later references, the trivial case is the situation where $E = M \times \mathbb{C}^n$ is the trivial bundle ➡ $\mathbf{A} = C^\infty(M) \otimes M_n$. But in general, $\mathbf{A}$ is (globally) more complicated.

Remark 32 (Relation to ordinary geometry)
The endomorphism fiber bundle $\text{End}(E)$ is associated to a $SU(n)$ principal fibre bundle P for the couple (M_n, Ad).
Because $G = SU(n) \subset M_n(\mathbb{C})$ and $\mathfrak{g} = \mathfrak{su}(n) \subset M_n(\mathbb{C})$, one has $P \times_\alpha G \subset \text{End}(E)$ and $\text{Ad}P \subset \text{End}(E)$.
The gauge group $\mathcal{G}$ and its Lie algebra $\text{Lie}\mathcal{G}$ are subspaces of $\mathbf{A}$.

Locally, using trivialisations of E, the algebra $\mathbf{A}$ looks like $C^\infty(U) \otimes M_n$.

Proposition 33 (Basic properties)
$\mathcal{Z}(\mathbf{A}) = C^\infty(M)$.
Involution, trace map and determinant ($\text{Tr}, \det : \mathbf{A} \to C^\infty(M)$*), are well defined fiberwise. Define* $SU(\mathbf{A})$ *as the unitaries in* $\mathbf{A}$ *of determinant* 1*, and* $\mathfrak{su}(\mathbf{A})$ *as the traceless antihermitean elements. Then* $\mathcal{G} = SU(\mathbf{A})$ *and* $\text{Lie}\mathcal{G} = \mathfrak{su}(\mathbf{A})$.

Derivations

Let $\rho : \mathrm{Der}(\mathbf{A}) \to \mathrm{Der}(\mathbf{A})/\mathrm{Int}(\mathbf{A}) = \mathrm{Out}(\mathbf{A})$ be the projection.

Proposition 34 (The derivations of A)

ρ is the restriction of derivations $\mathfrak{X} \in \mathrm{Der}(\mathbf{A})$ to $\mathcal{Z}(\mathbf{A})$.
$\mathrm{Out}(\mathbf{A}) \simeq \mathrm{Der}(C^\infty(M)) = \Gamma(M)$.
$\mathrm{Int}(\mathbf{A})$ is isomorphic to $\mathbf{A}_0$, the traceless elements in $\mathbf{A}$.
The s.e.s. of Lie algebras and $C^\infty(M)$-modules of derivations looks like

$$0 \longrightarrow \mathrm{Int}(\mathbf{A}) \longrightarrow \mathrm{Der}(\mathbf{A}) \xrightarrow{\rho} \Gamma(M) \longrightarrow 0$$
$$\mathfrak{X} \longmapsto X$$

- This generalizes the decomposition obtained in the trivial case.
- There is no *a priori* canonical splitting in the non trivial case.
- The "n.c. 1-form" $i\theta$ cannot be defined here. But one can define a map of $C^\infty(M)$-modules:

$$i\theta : \mathrm{Int}(\mathbf{A}) \to \mathbf{A}_0 \qquad \mathrm{ad}_\gamma \mapsto \gamma - \tfrac{1}{n}\,\mathrm{Tr}(\gamma)\mathbb{1}$$

Proposition 35

$$\underline{\Omega}^\bullet_{\mathrm{Der}}(\mathbf{A}) = \Omega^\bullet_{\mathrm{Der}}(\mathbf{A})$$

Connections

Let ∇^E be any (usual) connection on E.
Use notations: $X \in \Gamma(M)$, $\alpha \in \Gamma(E^*)$ and $s \in \Gamma(E)$.
Define the connections ∇^{E^*} on E^* and then ∇ on $\mathrm{End}(E)$ by the relations

$$X \cdot \langle \alpha, s \rangle = \langle \nabla^{E^*}_X \alpha, s \rangle + \langle \alpha, \nabla^E_X s \rangle \qquad \nabla_X(\alpha \otimes s) = (\nabla^{E^*}_X \alpha) \otimes s + \alpha \otimes (\nabla^E_X s)$$

Use the notation $X = \rho(\mathfrak{X}) \in \Gamma(M)$ for any $\mathfrak{X} \in \mathrm{Der}(\mathbf{A})$.

Proposition 36

For any $X \in \Gamma(M)$, ∇_X is a derivation of $\mathbf{A}$. For any $\mathfrak{X} \in \mathrm{Der}(\mathbf{A})$, $\mathfrak{X} - \nabla_X \in \mathrm{Int}(\mathbf{A})$. $\mathfrak{X} \mapsto \alpha(\mathfrak{X}) = -i\theta(\mathfrak{X} - \nabla_X)$ defines a n.c. 1-form $\alpha \in \Omega^1_{\mathrm{Der}}(\mathbf{A})$. One has $\mathfrak{X} = \nabla_X - \mathrm{ad}_{\alpha(\mathfrak{X})}$.
$\forall \gamma \in \mathbf{A}_0$, $\alpha(\mathrm{ad}_\gamma) = -\gamma$, $\forall \mathfrak{X} \in \mathrm{Der}(\mathbf{A})$, $\mathrm{Tr}\, \alpha(\mathfrak{X}) = 0$ and $\forall \mathfrak{X} \in \mathrm{Der}_{\mathbb{R}}(\mathbf{A})$, $\alpha(\mathfrak{X})^* + \alpha(\mathfrak{X}) = 0$.

- $X \mapsto \nabla_X$ is a splitting as $C^\infty(M)$-modules of the s.e.s.

$$0 \longrightarrow \mathbf{A}_0 \longrightarrow \mathrm{Der}(\mathbf{A}) \underset{\overset{\nabla}{\longleftarrow}}{\longrightarrow} \Gamma(M) \longrightarrow 0$$

  The obstruction to be a splitting of Lie algebras is the curvature of ∇: $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$.
- The relation $\alpha(\mathrm{ad}_\gamma) = -\gamma$ shows that α extends $-i\theta : \mathrm{Int}(\mathbf{A}) \to \mathbf{A}_0$.

Connections (continued)

Proposition 37 (Ordinary connections and n.c. forms)

The map $\nabla^E \mapsto \alpha$ is an isomorphism between the affine spaces

$$SU(n)\text{-connections on } E$$

$$\text{Traceless antihermitean n.c. 1-forms on } \mathbf{A} \text{ s.t. } \alpha(\mathrm{ad}_\gamma) = -\gamma$$

The n.c. 2-form $(\mathfrak{X}, \mathfrak{Y}) \mapsto \Omega(\mathfrak{X}, \mathfrak{Y}) = \widehat{\mathrm{d}}\alpha(\mathfrak{X}, \mathfrak{Y}) + [\alpha(\mathfrak{X}), \alpha(\mathfrak{Y})]$ depends only on the projections X and Y of $\mathfrak{X}$ and $\mathfrak{Y}$. This means that it is a horizontal 2-form for the operation of $\mathrm{Int}(\mathbf{A})$ on $\Omega^\bullet_{\mathrm{Der}}(\mathbf{A})$.

$\Gamma(\bigwedge^\bullet T^*M \otimes \mathrm{End}(E))$ is the graded algebra of horizontal forms in $\Omega^\bullet_{\mathrm{Der}}(\mathbf{A})$ for this operation.

The curvature R^E of ∇^E, considered in the intermediate construction as a section of $\bigwedge^2 T^*M \otimes \mathrm{Ad}P \subset \bigwedge^2 T^*M \otimes \mathrm{End}(E)$, is exactly Ω.

Remark 38 (Intermediate construction)

This proposition shows that the intermediate construction can be completed using noncommutative geometry in order to take into account the connection 1-form, which is now a n.c. 1-form. The vertical and equivariant conditions at the level of P are replaced by a unique condition on inner derivations at the level of $\mathbf{A}$.

Connections (continued)

Let $u \in \mathcal{G} = SU(\mathbf{A})$ and $\xi \in \text{Lie}\mathcal{G} = \mathfrak{su}(\mathbf{A})$.
The n.c. 1-form α^u corresponding to the gauge transformed connection ∇^{Eu} is

$$\alpha^u = u^* \alpha u + u^* \widehat{\mathrm{d}} u$$

Proposition 39 (Infinitesimal gauge transformations)

The infinitesimal gauge transformation induced by ξ is

$$\alpha \mapsto -\widehat{\mathrm{d}}\xi - [\alpha, \xi] = L_{\mathrm{ad}_\xi} \alpha$$

This means that

Infinitesimal gauge transformations on connections on E

$=$

Lie derivative of real inner derivations on $\mathbf{A}$.

Derivations of A in local trivialisations

Let $U_i \subset M$ be a local trivialisation of E, and so of $\mathrm{End}(E)$. We denote by $a_i^{\mathrm{loc}} : U_i \to M_n$ the restriction of the global section $a \in \mathbf{A}$.
Over $U_i \cap U_j \neq \varnothing$, one has $a_j^{\mathrm{loc}} = \mathrm{Ad}_{g_{ij}^{-1}} a_i^{\mathrm{loc}} = g_{ij}^{-1} a_i^{\mathrm{loc}} g_{ij}$.
Locally a derivation $\mathfrak{X} \in \mathrm{Der}(\mathbf{A})$ can be written $\mathfrak{X}_i^{\mathrm{loc}} = X_i + \mathrm{ad}_{\gamma_i}$ with $\gamma_i : U_i \to M_n$ (traceless) and X_i a vector fields on U. Using the map ρ, one gets that X_i is the restriction of $X = \rho(\mathfrak{X})$ to U_i ➡ notation $X = X_i$.
Using compatibility with the gluing relations for sections, one finds

$$\gamma_j = g_{ij}^{-1} \gamma_i g_{ij} + g_{ij}^{-1} X{\cdot}g_{ij}$$

The n.c. 1-form α is locally given by the expressions

$$\alpha_i^{\mathrm{loc}}(X + \mathrm{ad}_{\gamma_i}) = A_i(X) - \gamma_i$$

where the A_i's form the family of local trivialisation of the connection 1-form.

$$\begin{aligned}
\alpha_j^{\mathrm{loc}}(X + \mathrm{ad}_{\gamma_j}) &= A_j(X) - \gamma_j \\
&= (g_{ij}^{-1} A_i(X) g_{ij} + g_{ij}^{-1} X{\cdot}g_{ij}) - (g_{ij}^{-1} \gamma_i g_{ij} + g_{ij}^{-1} X{\cdot}g_{ij}) \\
&= g_{ij}^{-1}(A_i(X) - \gamma_i) g_{ij} = g_{ij}^{-1} \alpha_i^{\mathrm{loc}}(X + \mathrm{ad}_{\gamma_i}) g_{ij}
\end{aligned}$$

So that these expressions indeed define a global section in $\mathbf{A}$.

Some general facts

The right **A**-module is taken to be $\boldsymbol{M} = \mathbf{A}$ with Hermitean structure $(a, b) \mapsto a^* b$. A n.c. connection $\widehat{\nabla}$ is then given by a n.c. 1-form $\omega \in \Omega^1_{\mathrm{Der}}(\mathbf{A})$ by the relation $\widehat{\nabla}_{\mathfrak{X}} a = \mathfrak{X} a + \omega(\mathfrak{X}) a$ ➞ studying $\widehat{\nabla}$ is studying ω.

Proposition 40 (The n.c. connection associated to α)

Let ∇^E be a connection on E, α its associated n.c. 1-form.

- *The n.c. connection $\widehat{\nabla}^\alpha$ defined by the n.c. 1-form α is given by*
$$\widehat{\nabla}^\alpha_{\mathfrak{X}} a = \nabla_X a + a\alpha(\mathfrak{X})$$
- *$\widehat{\nabla}^\alpha$ is compatible with the Hermitean structure.*
- *The curvature of $\widehat{\nabla}^\alpha$ is $\widehat{R}^\alpha(\mathfrak{X}, \mathfrak{Y}) = R^E(X, Y)$*
- *A gauge transformation induced by $u \in \mathcal{G} = SU(\mathbf{A})$ on the connection ∇^E induces a (n.c.) gauge transformation on $\widehat{\nabla}^\alpha$.*

- Recall that $\mathfrak{X} = \nabla_X - \mathrm{ad}_{\alpha(\mathfrak{X})}$ and $\widehat{\nabla}^\alpha_{\mathfrak{X}} a = \mathfrak{X} a + \alpha(\mathfrak{X}) a$.
- Curvature of $\widehat{\nabla}^\alpha = \widehat{\mathrm{d}}\alpha(\mathfrak{X}, \mathfrak{Y}) + [\alpha(\mathfrak{X}), \alpha(\mathfrak{Y})] =$ curvature of ∇.
- In a gauge transformation, one has $\alpha^u = u^* \alpha u + u^* \widehat{\mathrm{d}} u$, which is also the n.c. gauge transformation applied to $\widehat{\nabla}^\alpha$.

The space of n.c. connections

We arrive at the main point of this lecture:

Theorem 41 (Ordinary connections as n.c. connections)
The space of n.c. connections on the right **A***-module* **A** *compatible with the Hermitean structure* $(a, b) \mapsto a^* b$ *contains the space of ordinary connections on* E*.*
This inclusion is compatible with the corresponding definitions of curvature and gauge transformations.

- From now on one can consider that an ordinary connection is a n.c. connection.
- This point of view generalizes the notion of connection through the intermediate construction.
- A natural question is: what are n.c. connections from a physical point of view?

Decomposition of n.c. connections

Let us fix a connection ∇^E on E, and denote by α its associated n.c. 1-form.
Any n.c. connection $\widehat{\nabla}$ can be decomposed as

$$\widehat{\nabla}_{\mathfrak{X}} a = \widehat{\nabla}^{\alpha}_{\mathfrak{X}} a + \mathcal{A}(\mathfrak{X}) a$$

with $\mathcal{A} \in \Omega^1_{\mathrm{Der}}(\mathbf{A})$, so that $\omega = \alpha + \mathcal{A}$ is the n.c. 1-form for $\widehat{\nabla}$.
Using the relation $\mathfrak{X} = \nabla_X - \mathrm{ad}_{\alpha(\mathfrak{X})}$, one decomposes $\mathcal{A}$ as
$\mathcal{A}(\mathfrak{X}) = \mathfrak{a}(X) - \mathfrak{b}(\alpha(\mathfrak{X}))$, where $\mathfrak{b} : \mathbf{A}_0 \to \mathbf{A}$ is defined by $\mathfrak{b}(\gamma) = \mathcal{A}(\mathrm{ad}_\gamma)$.
The curvature of $\widehat{\nabla}$ can be written

$$\begin{aligned}\widehat{R}(\mathfrak{X}, \mathfrak{Y}) &= R^E(X, Y) + \nabla_X \mathcal{A}(\mathfrak{Y}) - \nabla_Y \mathcal{A}(\mathfrak{X}) - \mathcal{A}([\mathfrak{X}, \mathfrak{Y}]) + [\mathcal{A}(\mathfrak{X}), \mathcal{A}(\mathfrak{Y})] \\ &= R^{E,\mathfrak{a}}(X, Y) - \nabla^{\mathfrak{a}}_X \mathfrak{b}(\alpha(\mathfrak{Y})) + \nabla^{\mathfrak{a}}_Y \mathfrak{b}(\alpha(\mathfrak{X})) \\ &\qquad + [\mathfrak{b}(\alpha(\mathfrak{X})), \mathfrak{b}(\alpha(\mathfrak{Y}))] + \mathfrak{b}(\alpha([\mathfrak{X}, \mathfrak{Y}]))\end{aligned}$$

where $R^{E,\mathfrak{a}}$ is the curvature of the connection $\nabla^{E,\mathfrak{a}}_X s = \nabla^E_X s + \mathfrak{a}(X) s$ on E and $\nabla^{\mathfrak{a}}$ is its associated connection on $\mathrm{End}(E)$.
Performing a gauge transformation with $u \in \mathcal{G} = SU(\mathbf{A})$, one has

$$\mathcal{A}^u = u^* \mathcal{A} u + u^*(\nabla u) \qquad \mathfrak{a}^u = u^* \mathfrak{a} u + u^*(\nabla u) \qquad \mathfrak{b}^u = u^* \mathfrak{b} u$$

Notice the replacement of the differential by ∇ in many of these expressions.

The trivial case

Consider the situation $E = M \times \mathbb{C}^n$ and $\mathbf{A} = C^\infty(M) \otimes M_n$.
As a reference (ordinary) connection, take $\nabla^E_X s = X{\cdot}s$, so that, using the local expression of α, one has

$$\alpha(\mathfrak{X}) = \alpha(X + \mathrm{ad}_\gamma) = -\gamma = -i\theta(\mathfrak{X})$$

with $\mathrm{Tr}\,\gamma = 0$. Then $\widehat{\nabla}^\alpha = \widehat{\nabla}^{-i\theta}$. Moreover, $\mathfrak{b}(\alpha(\mathfrak{X})) = \mathfrak{b}(-\gamma) = -\mathfrak{b}(\gamma)$, so that

$$\widehat{\nabla}_{\mathfrak{X}} a = X{\cdot}a + \mathfrak{a}(X)a + \mathfrak{b}(\gamma)a = \overline{\nabla}^{\mathfrak{a}}_X a + \mathfrak{b}(\gamma)a$$

where $\overline{\nabla}^{\mathfrak{a}}$ is an ordinary connection on $\mathrm{End}(E)$, but is not $\nabla^{\mathfrak{a}}$.
$X \mapsto \mathfrak{a}(X)$ behave like a gauge potential with respect to gauge transformations (here $\nabla = \mathrm{d}$). The difference between ordinary connections and n.c. connections is the presence of $\mathfrak{b}$, which represents some additional fields in physics. It has homogeneous gauge transformations.
The curvature can be written, for $\mathfrak{X} = X + \mathrm{ad}_\gamma$ and $\mathfrak{Y} = Y + \mathrm{ad}_\eta$,

$$\widehat{R}(\mathfrak{X}, \mathfrak{Y}) = R^{E,\mathfrak{a}}(X, Y) + (\widetilde{\nabla}^{\mathfrak{a}}_X \mathfrak{b})(\eta) - (\widetilde{\nabla}^{\mathfrak{a}}_Y \mathfrak{b})(\gamma) + [\mathfrak{b}(\gamma), \mathfrak{b}(\eta)] - \mathfrak{b}([\gamma, \eta])$$

where $\widetilde{\nabla}^{\mathfrak{a}}$ is the connection $(\widetilde{\nabla}^{\mathfrak{a}}_X \mathfrak{b})(\eta) = X{\cdot}\mathfrak{b}(\eta) - \mathfrak{b}(X{\cdot}\eta) + [\mathfrak{a}(X), \mathfrak{b}(\eta)]$ on the space of $C^\infty(M)$-linear maps $\mathbf{A}_0 \to \mathbf{A}$.

Yang-Mills-Higgs Lagrangian

Consider the trivial case $\mathbf{A} = C^\infty(M) \otimes M_n$ and the right $\mathbf{A}$-module $\mathbf{A}$.
Let $\mathfrak{a} = \mathfrak{a}_\mu dx^\mu$ and $\mathfrak{b} = \mathfrak{b}_k \theta^k$, with $\mathfrak{a}_\mu, \mathfrak{b}_k \in C^\infty(M) \otimes M_n$.
The curvature is then the n.c. 2-form

$$\widehat{R} = \tfrac{1}{2}(\partial_\mu \mathfrak{a}_\nu - \partial_\nu \mathfrak{a}_\mu + [\mathfrak{a}_\mu, \mathfrak{a}_\nu]) dx^\mu dx^\nu + (\partial_\mu \mathfrak{b}_k + [\mathfrak{a}_\mu, \mathfrak{b}_k]) dx^\mu \theta^k + \tfrac{1}{2}([\mathfrak{b}_k, \mathfrak{b}_\ell] - C^m_{k\ell} \mathfrak{b}_m) \theta^k \theta^\ell$$

Using a metric (here euclidean) on $\mathrm{Der}(\mathbf{A})$ and an associated Hodge star operation, one can define a Lagrangian. Using ordinary and n.c. integration, one then defines the action:

$$S(\widehat{R}) = \int dx \, \mathrm{Tr} \left\{ \sum_{\mu,\nu} \tfrac{1}{4}(\partial_\mu \mathfrak{a}_\nu - \partial_\nu \mathfrak{a}_\mu + [\mathfrak{a}_\mu, \mathfrak{a}_\nu])^2 + m^2 \sum_{\mu,k} (\partial_\mu \mathfrak{b}_k + [\mathfrak{a}_\mu, \mathfrak{b}_k])^2 + m^4 \sum_{k,\ell} \tfrac{1}{4}([\mathfrak{b}_k, \mathfrak{b}_\ell] - C^m_{k\ell} \mathfrak{b}_m)^2 \right\}$$

The integrand is zero when

$\mathfrak{a}$ gauge equivalent to 0 $\qquad$ $d\mathfrak{b} = 0$ $\qquad$ $[\mathfrak{b}_k, \mathfrak{b}_\ell] = C^m_{k\ell} \mathfrak{b}_m$

so that the $\mathfrak{b}_k$'s are constant and induce a representation of $\mathfrak{sl}_n$ in M_n.
For the right $\mathbf{A}$-module $\boldsymbol{M} = C^\infty(M) \otimes M_{r,n}$, one would get similar results: flat connections are classified by inequivalent representations of $\mathfrak{sl}_n$ in M_r.

Yang-Mills-Higgs Lagrangian

From a fields theory point of view:

- The $\mathfrak{a}_\mu$ fields behave like ordinary Yang-Mills fields, for a $SU(n)$ gauge theory.
- The $\mathfrak{b}_k$ fields behave as Higgs fields: the vacuum states can be non trivial and the Higgs mechanism of mass generation is possible.
- The coupling between these fields is a covariant derivative in the adjoint representation.

For a more general situation where **A** is not the trivial case, one can proceed in the same line:

- One has to use a reference connection on E to help to decompose n.c. connections.
- The curvature looks similar except for the presence of the reference connection.
- The Hodge star operator is defined.
- The action splits into three terms, and the vacuum states are related to the global structure of the vector fiber bundle E.

The algebra $\mathbf{B} = C^\infty(P) \otimes M_n$

Let P the $SU(n)$-principal fiber bundle for which E is associated.
Consider the associative algebra $\mathbf{B} = C^\infty(P) \otimes M_n$. Then

- $\mathcal{Z}(\mathbf{B}) = C^\infty(P)$, $\text{Der}(\mathbf{B}) = \Gamma(P) \oplus [C^\infty(P) \otimes \mathfrak{sl}_n]$ and $\Omega^\bullet_{\text{Der}}(\mathbf{B}) = \Omega^\bullet(P) \otimes \Omega^\bullet_{\text{Der}}(M_n)$ with the differential $\widehat{\text{d}} = \text{d} + \text{d}'$.
- $\mathfrak{su}(n)$ is a real Lie subalgebra of $\text{Der}(\mathbf{B})$ for two inclusions:
  $\xi \mapsto \xi^v$ vertical vector field on P $\qquad$ $\xi \mapsto \text{ad}_\xi$ inner derivation
- $\mathfrak{g}_{\text{ad}} = \{\text{ad}_\xi \ / \ \xi \in \mathfrak{su}(n)\}$ and $\mathfrak{g}_{\text{equ}} = \{\xi^v + \text{ad}_\xi \ / \ \xi \in \mathfrak{su}(n)\}$ are Lie subalgebras of $\text{Der}(\mathbf{B})$.

Proposition 42
The algebra $C^\infty(P)$ (resp. $\mathbf{A}$) is the invariants of $\mathfrak{g}_{\text{ad}}$ (resp. $\mathfrak{g}_{\text{equ}}$) in $\mathbf{B}$.

Proof.
$C^\infty(P)$ is the invariants of $\mathfrak{g}_{\text{ad}}$ because $\text{ad}_\xi b = 0$ for any $\xi \in \mathfrak{su}(n)$ implies $b \in \mathcal{Z}(\mathbf{B})$. $\mathbf{A}$ is the invariants of $\mathfrak{g}_{\text{equ}}$ because $\mathbf{A}$ is the set of sections of $\text{End}(E)$, which is $\mathcal{F}_{SU(n)}(P, M_n)$. The relation $\xi^v \cdot b + \text{ad}_\xi b = 0$ for any $\xi \in \mathfrak{su}(n)$ is the infinitesimal version of the equivariance. □

Cartan operations

The Lie subalgebras $\mathfrak{g}_{\text{ad}}$ and $\mathfrak{g}_{\text{equ}}$ define Cartan operations on $(\Omega^\bullet_{\text{Der}}(\mathbf{B}), \widehat{\mathrm{d}})$.
The previous proposition tells us that the algebras $\mathbf{B}$, $C^\infty(P)$ and $\mathbf{A}$ are related by these operations.
Moreover, $C^\infty(M)$ is itself the invariant elements for $\xi \mapsto \xi^v$ in $C^\infty(P)$ and the invariants in $\mathbf{A}$ for the operation of $\text{Int}(\mathbf{A})$.

Proposition 43 (Relations between the differential calculi)

At the level of differential calculi, all these relations generalize in the following structure:

$$\begin{array}{ccc}
\Omega^\bullet(P) \otimes \Omega^\bullet_{\text{Der}}(M_n) & \xleftarrow[\mathfrak{su}(n) \ni \xi \mapsto \text{ad}_\xi]{\textit{basic elements}} & \Omega^\bullet(P) \\
\uparrow {\scriptstyle \textit{basic elements}\ \mathfrak{su}(n) \ni \xi \mapsto \xi^{v} + \text{ad}_\xi} & & \uparrow {\scriptstyle \textit{basic elements}\ \mathfrak{su}(n) \ni \xi \mapsto \xi^{v}} \\
\Omega^\bullet_{\text{Der}}(\mathbf{A}) & \xleftarrow[\text{Int}(\mathbf{A})]{\textit{basic elements}} & \Omega^\bullet(M)
\end{array}$$

Derivations of A and B

There exist some strong relations between the derivations of **A**, some derivations of **B**, and some vector fields on P and M.

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \mathcal{Z}_{\mathrm{Der}}(\mathbf{A}) & \longrightarrow & \Gamma(VP) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Int}(\mathbf{A}) & \longrightarrow & \mathcal{N}_{\mathrm{Der}}(\mathbf{A}) & \xrightarrow{\rho_{\mathbf{P}}} & \Gamma_M(P) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\pi_*} & & \\
0 & \longrightarrow & \mathrm{Int}(\mathbf{A}) & \longrightarrow & \mathrm{Der}(\mathbf{A}) & \xrightarrow{\rho} & \Gamma(M) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

This exact commutative diagram illustrated these relations.

Derivations of A and B

There exist some strong relations between the derivations of **A**, some derivations of **B**, and some vector fields on P and M.

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \mathcal{Z}_{\mathrm{Der}}(\mathbf{A}) & \longrightarrow & \Gamma(VP) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Int}(\mathbf{A}) & \longrightarrow & \mathcal{N}_{\mathrm{Der}}(\mathbf{A}) & \xrightarrow{\rho_{\mathbf{P}}} & \Gamma_M(P) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \pi_*} & & \\
0 & \longrightarrow & \mathrm{Int}(\mathbf{A}) & \longrightarrow & \mathrm{Der}(\mathbf{A}) & \xrightarrow{\rho} & \Gamma(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Short exact sequence which relates vector fields on M, derivations on **A** and inner derivations on **A**.

Derivations of A and B

There exist some strong relations between the derivations of $\mathbf{A}$, some derivations of $\mathbf{B}$, and some vector fields on P and M.

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \mathcal{Z}_{\text{Der}}(\mathbf{A}) & \longrightarrow & \Gamma(VP) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{Int}(\mathbf{A}) & \longrightarrow & \mathcal{N}_{\text{Der}}(\mathbf{A}) & \xrightarrow{\rho_{\mathbf{P}}} & \Gamma_M(P) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \pi_*} & & \\
0 & \longrightarrow & \text{Int}(\mathbf{A}) & \longrightarrow & \text{Der}(\mathbf{A}) & \xrightarrow{\rho} & \Gamma(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

$\mathcal{N}_{\text{Der}}(\mathbf{A}) \subset \text{Der}(\mathbf{B})$ subset of derivations on $\mathbf{B}$ which preserve $\mathbf{A} \subset \mathbf{B}$.
$\mathcal{Z}_{\text{Der}}(\mathbf{A}) \subset \text{Der}(\mathbf{B})$ subset of derivations on $\mathbf{B}$ which vanish on $\mathbf{A}$.
The Lie algebra $\mathcal{Z}_{\text{Der}}(\mathbf{A})$ is generated as a $C^\infty(P)$-module by the elements $\xi^v + \text{ad}_\xi$ for any $\xi \in \mathfrak{su}(n)$.

Derivations of A and B

There exist some strong relations between the derivations of **A**, some derivations of **B**, and some vector fields on P and M.

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \mathcal{Z}_{\text{Der}}(\mathbf{A}) & \longrightarrow & \Gamma(VP) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{Int}(\mathbf{A}) & \longrightarrow & \mathcal{N}_{\text{Der}}(\mathbf{A}) & \xrightarrow{\rho_{\mathbf{P}}} & \Gamma_M(P) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \tau & & \downarrow \pi_* & & \\
0 & \longrightarrow & \text{Int}(\mathbf{A}) & \longrightarrow & \text{Der}(\mathbf{A}) & \xrightarrow{\rho} & \Gamma(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Geometrical objects. $\Gamma(VP)$ vertical vector fields on P.
$\Gamma_M(P) = \{\mathcal{X} \in \Gamma(P) / \pi_*\mathcal{X}(p) = \pi_*\mathcal{X}(p') \ \forall p, p' \in P \text{ s.t. } \pi(p) = \pi(p')\}$
Lie algebra of vector fields on P which can be mapped to vector fields on M using the tangent maps $\pi_* : T_pP \to T_{\pi(p)}M$.

Derivations of A and B

There exist some strong relations between the derivations of **A**, some derivations of **B**, and some vector fields on P and M.

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \mathcal{Z}_{\mathrm{Der}}(\mathbf{A}) & \longrightarrow & \Gamma(VP) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Int}(\mathbf{A}) & \longrightarrow & \mathcal{N}_{\mathrm{Der}}(\mathbf{A}) & \xrightarrow{\rho_{\mathbf{P}}} & \Gamma_M(P) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \pi_*} & & \\
0 & \longrightarrow & \mathrm{Int}(\mathbf{A}) & \longrightarrow & \mathrm{Der}(\mathbf{A}) & \xrightarrow{\rho} & \Gamma(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Elements in $\mathrm{Int}(\mathbf{A})$ considered as ad_γ for $\gamma \in \mathbf{A}_0 \subset \mathbf{B}$.
ρ_P restriction to $\mathcal{N}_{\mathrm{Der}}(\mathbf{A})$ of the projection on the first term in $\mathrm{Der}(\mathbf{B}) = \Gamma(P) \oplus [C^\infty(P) \otimes \mathrm{Der}(M_n)]$.

Integration

The n.c. integration induces a map

$$\int_{\text{n.c.}} : \Omega^r_{\text{Der}}(\mathbf{B}) \to \Omega^{r-(n^2-1)}(P)$$

Proposition 44 (Integration)
If $\omega \in \Omega^r_{\text{Der}}(\mathbf{B})$ is a horizontal (resp. basic) n.c. form for one of the operation of $\mathfrak{g}_{\text{ad}}$ or $\mathfrak{g}_{\text{equ}}$, then $\int_{n.c.} \omega \in \Omega^{r-(n^2-1)}(P)$ is horizontal (resp. basic) for the corresponding operation restricted to $\Omega^\bullet(P) \subset \Omega^\bullet_{\text{Der}}(\mathbf{B})$.
This n.c. integration then restricts to a "n.c. integration along the n.c. fiber" $\Omega^\bullet_{\text{Der}}(\mathbf{A}) \to \Omega^{\bullet-(n^2-1)}(M)$.
This n.c. integration is compatible with the differentials, and it induces maps in cohomologies

$$\int_{n.c.} : H^\bullet(\Omega^\bullet_{\text{Der}}(\mathbf{B}), \widehat{\text{d}}) \to H_{dR}^{\bullet-(n^2-1)}(P)$$

$$\int_{n.c.} : H^\bullet(\Omega^\bullet_{\text{Der}}(\mathbf{A}), \widehat{\text{d}}) \to H_{dR}^{\bullet-(n^2-1)}(M)$$

Ordinary and n.c. connections

Let $\widehat{\nabla}$ be a n.c. connection on the right **A**-module **A**, and denote by $\alpha \in \Omega^1_{\mathrm{Der}}(\mathbf{A})$ it associated n.c. 1-form.
As a basic n.c. 1-form in $\Omega^1_{\mathrm{Der}}(\mathbf{B})$ for the operation of $\mathfrak{g}_{\mathrm{equ}}$, one can write

$$\alpha = \omega - \phi \in [\Omega^1(P) \otimes M_n] \oplus [C^\infty(P) \otimes M_n \otimes \mathfrak{sl}_n^*]$$

The basic condition implies the relations

$$(L_{\xi^\mathsf{v}} + L_{\mathrm{ad}_\xi})\omega = 0 \qquad (L_{\xi^\mathsf{v}} + L_{\mathrm{ad}_\xi})\phi = 0 \qquad i_{\xi^\mathsf{v}}\omega - i_{\mathrm{ad}_\xi}\phi = 0$$

for any $\xi \in \mathfrak{su}(n)$.

Proposition 45 (Ordinary connection)
Let ∇^E be an ordinary connection on E and $\alpha \in \Omega^1_{\mathrm{Der}}(\mathbf{A})$ its associated n.c. 1-form. Then, as a basic element in $\Omega^1_{\mathrm{Der}}(\mathbf{B})$, one has

$$\alpha = \omega - i\theta$$

where $\omega \in \Omega^1(P) \otimes \mathfrak{su}(n) \subset \Omega^1(P) \otimes M_n$ is the connection 1-form on P and $i\theta$ the canonical n.c. 1-form defined in $\Omega^1_{\mathrm{Der}}(\mathbf{B})$.

In order to prove this formula, one has to use the equivariance and the vertical condition for ω, and some of the properties listed before on $i\theta$.

Connection as splittings

$$
\begin{array}{ccc}
\mathcal{N}_{\mathrm{Der}}(\mathbf{A}) & \xrightarrow{\ \rho_{\mathbf{P}}\ } & \Gamma_M(P) \\
\downarrow \tau & & \downarrow \pi_* \\
\mathrm{Der}(\mathbf{A}) & \xrightarrow[\ \rho\ ]{} & \Gamma(M)
\end{array}
$$

$$
(\pi_* \mathcal{X})^h + \omega(\mathcal{X})^v + \mathrm{ad}_{\omega(\mathcal{X})} \longleftarrow\!\mapsto \mathcal{X}
$$

$$
\begin{array}{ccc}
\rho(\mathfrak{X})^h - \mathrm{ad}_{\alpha(\mathfrak{X})^{\mathbf{B}}} & & X^h \\
\uparrow & & \uparrow \\
\mathfrak{X} & & X
\end{array}
$$

$$
\nabla_X \longleftarrow\!\mapsto X
$$

Notice that

$$
(\pi_* \mathcal{X})^h + \omega(\mathcal{X})^v + \mathrm{ad}_{\omega(\mathcal{X})} \neq \rho(\mathfrak{X})^h - \mathrm{ad}_{\alpha(\mathfrak{X})^{\mathbf{B}}}
$$

Indeed one has:

$$
\mathrm{Der}(\mathbf{B}) \ni \mathfrak{X} = X^h + \underbrace{\mathrm{ad}_Z}_{\in \mathrm{Int}(\mathbf{A})} + \underbrace{\omega(\mathcal{X})^v + \mathrm{ad}_{\omega(\mathcal{X})}}_{\in \mathcal{Z}_{\mathbf{Der}}(\mathbf{A})}
$$

About the cohomology of a fiber bundle

Let us recall the Leray theorem in ordinary differential geometry.

Theorem 46 (Leray)

For any fiber bundle $F \longrightarrow E \xrightarrow{\pi} M$, *there exists a spectral sequence* $\{\boldsymbol{E}_r\}$ *converging to the cohomology of the total space* $H^\bullet_{dR}(E)$ *with*

$$\boldsymbol{E}_2^{p,q} = H^p(\mathfrak{U}; \mathcal{H}^q)$$

where $\mathcal{H}^q(U) = H^q_{dR}(\pi^{-1}U)$ *is a locally constant presheaf on the good covering* $\mathfrak{U}$ *of* M.

If M *is simply connected and* $H^q_{dR}(F)$ *is finite dimensional, then*

$$\boldsymbol{E}_2^{p,q} = H^p_{dR}(M) \otimes H^q_{dR}(F)$$

One of the proofs of this theorem relies on the construction of a Čech-de Rham bicomplex:

$$\boldsymbol{K}^{p,q} = \prod_{\alpha_0 < \cdots < \alpha_p} \Omega^q(E_{U_{\alpha_0 \dots \alpha_p}}) = \prod_{\alpha_0 < \cdots < \alpha_p} \Omega^q(\pi^{-1} U_{\alpha_0 \dots \alpha_p})$$

where $U_{\alpha_0 \dots \alpha_p} = U_{\alpha_0} \cap \cdots U_{\alpha_p}$ for $U_{\alpha_i} \in \mathfrak{U}$, $\mathrm{d} : \boldsymbol{K}^{p,q} \to \boldsymbol{K}^{p,q+1}$ is the ordinary de Rham differential on the spaces $\Omega^\bullet(E_{U_{\alpha_0 \dots \alpha_p}})$, and $\delta : \boldsymbol{K}^{p,q} \to \boldsymbol{K}^{p+1,q}$ is the Čech differential $(\delta\omega_p)_{\alpha_0 \dots \alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \omega_{\alpha_0 \dots \widehat{\alpha_i} \dots \alpha_{p+1} | U_{\alpha_0 \dots \alpha_{p+1}}}$.

The noncommutative Čech-de Rham bicomplex

One can introduce a noncommutative Čech-de Rham bicomplex for $\mathbf{A}$.
Denote by $\mathbf{A}(U) \simeq C^\infty(U) \otimes M_n$ the sections of $\text{End}(E)$ restricted over a local trivialisation $U \subset M$. Denote by $g_{UV} : U \cap V \to SU(n)$ the transition functions for E. Let $\mathfrak{U}$ be a good cover of M.
For any n.c. p-form $\omega = a_0 \widehat{\text{d}} a_1 \cdots \widehat{\text{d}} a_p \in \Omega^p_{\text{Der}}(\mathbf{A}(U))$ and any differential function $g : U \to SU(n)$, define the action of g by $\omega^g = (g^{-1} a_0 g) \widehat{\text{d}}(g^{-1} a_1 g) \cdots \widehat{\text{d}}(g^{-1} a_p g)$.

Lemma 47 (The presheaf $\Omega^\bullet_{\text{Der}}(\mathbf{A}(U))$)
For any $V \subset U$, the maps $i^V_U : \Omega^\bullet_{\text{Der}}(\mathbf{A}(U)) \to \Omega^\bullet_{\text{Der}}(\mathbf{A}(V))$ given by $\omega \mapsto (\omega_{|V})^{g_{UV}}$ (restriction to V and action of g_{UV}) give to $U \mapsto \Omega^\bullet_{\text{Der}}(\mathbf{A}(U))$ a structure of presheaf on M, which we denote by $\mathcal{F}$.

Let $\boldsymbol{C}^{p,q}(\mathfrak{U}; \mathcal{F}) = \prod_{\alpha_0 < \cdots < \alpha_p} \Omega^q_{\text{Der}}(\mathbf{A}(U_{\alpha_0 \ldots \alpha_p}))$, where by convention the trivialisation over $U_{\alpha_0 \ldots \alpha_p}$ is the one over U_{α_p}. Let $\widehat{\text{d}} : \boldsymbol{C}^{p,q} \to \boldsymbol{C}^{p,q+1}$ be the n.c. differential, and define $\delta : \boldsymbol{C}^{p,q}(\mathfrak{U}; \mathcal{F}) \to \boldsymbol{C}^{p+1,q}(\mathfrak{U}; \mathcal{F})$ by ($g_{\alpha\beta} = g_{U_\alpha U_\beta}$)

$$(\delta\omega)_{\alpha_0 \ldots \alpha_{p+1}} = \sum_{i=0}^{p} (-1)^i (\omega_{\alpha_0 \ldots \widehat{\alpha}_i \ldots \alpha_{p+1}})_{|U_{\alpha_0 \cdots \alpha_{p+1}}} + (-1)^{p+1} (\omega_{\alpha_0 \ldots \alpha_p})^{g_{\alpha_p \alpha_{p+1}}}_{|U_{\alpha_0 \cdots \alpha_{p+1}}}$$

The cohomology of $\Omega^\bullet_{\mathrm{Der}}(\mathbf{A})$

Denote by $\boldsymbol{C}^{-1,q}(\mathfrak{U};\mathcal{F}) = \Omega^q_{\mathrm{Der}}(\mathbf{A})$ and define $\delta : \boldsymbol{C}^{-1,q}(\mathfrak{U};\mathcal{F}) \to \boldsymbol{C}^{0,q}(\mathfrak{U};\mathcal{F})$ as the restrictions to the trivialisations.
One has the following results about the cohomology of $\Omega^\bullet_{\mathrm{Der}}(\mathbf{A})$:

Proposition 48 (Noncommutative Leray theorem)
The cohomology of the total complex of the bicomplex $(\boldsymbol{C}^{\bullet,\bullet}(\mathfrak{U};\mathcal{F}), \widehat{\mathrm{d}}, \delta)$ *is the cohomology of* $\Omega^\bullet_{\mathrm{Der}}(\mathbf{A})$.
The spectral sequence $\{\boldsymbol{E}_r\}$ *associated to the filtration*
$$F^p\boldsymbol{C}(\mathfrak{U};\mathcal{F}) = \bigoplus_{s\geq p}\bigoplus_{q\geq 0}\boldsymbol{C}^{s,q}(\mathfrak{U};\mathcal{F})$$
converges to the cohomology of $\Omega^\bullet_{\mathrm{Der}}(\mathbf{A})$ *and satisfies*
$$\boldsymbol{E}_2 = H^\bullet_{dR}(M)\otimes\mathcal{I}(\textstyle\bigwedge^\bullet\mathfrak{sl}_n^*)$$

Recall that the structure of $\mathcal{I}(\bigwedge^\bullet\mathfrak{sl}_n^*)$ is known.

Splitting of short exact sequences of Lie algebras

Let $0 \longrightarrow \mathfrak{i} \longrightarrow \mathfrak{g} \xrightarrow{\pi} \mathfrak{h} \longrightarrow 0$ be a s.e.s. of Lie algebras. Let $\varphi : \mathfrak{h} \to \mathfrak{g}$ be a morphism which splits it as vector spaces. Define $R_\varphi = \mathrm{d}_\mathfrak{h}\varphi + \frac{1}{2}[\varphi, \varphi] : \bigwedge^2 \mathfrak{h}^* \otimes \mathfrak{g}$ with $\mathrm{d}_\mathfrak{h}$ the differential on $\bigwedge^\bullet \mathfrak{h}^* \otimes \mathfrak{g}$ for the trivial representation of $\mathfrak{h}$ on $\mathfrak{g}$. For any $x, y \in \mathfrak{h}$, one has $R_\varphi(x, y) = -\varphi([x, y]) + [\varphi(x), \varphi(y)]$ (obstruction to be a Lie algebra morphism).

➥ R_φ looks like a curvature: R_φ belongs to $\bigwedge^2 \mathfrak{h}^* \otimes \mathfrak{i}$ and satisfies the Bianchi identity $d_\mathfrak{h} R_\varphi + [\varphi, R_\varphi] = 0$.

Let V be a vector space and ρ a representation of $\mathfrak{h}$ on V. Denote by $\mathsf{S}^q_\rho(\mathfrak{i}, \mathsf{V})$ the space of linear symmetric maps $\otimes^q \mathfrak{i} \to \mathsf{V}$ which intertwine the adjoint representation $\mathrm{ad}^{\otimes^q}$ of $\mathfrak{g}$ on $\otimes^q \mathfrak{i}$ and the representation $\rho \circ \pi$ of $\mathfrak{g}$ on V. Let ϵ be the antisymmetrisation map $\otimes^\bullet \mathfrak{h}^* \to \bigwedge^\bullet \mathfrak{h}^*$.

Proposition 49 (Lecomte)

For any $\alpha \in \mathsf{S}^q_\rho(\mathfrak{i}, \mathsf{V})$, let $\alpha_\varphi = \epsilon \circ \alpha(R_\varphi \otimes \cdots \otimes R_\varphi) \in \bigwedge^{2q} \mathfrak{h}^* \otimes \mathsf{V}$. Then one has $\mathrm{d}\alpha_\varphi = 0$ where d is the differential of the complex $\bigwedge^\bullet \mathfrak{h}^* \otimes \mathsf{V}$.
The cohomology class of α_φ in $H^{2q}(\mathfrak{h}; \mathsf{V})$ does not depends on the choice of φ.
If the s.e.s. is split exact as a Lie algebra s.e.s. then this cohomology class is zero.

Characteristic classes constructed from A

Consider the s.e.s. $0 \longrightarrow \mathrm{Int}(\mathbf{A}) \longrightarrow \mathrm{Der}(\mathbf{A}) \overset{\rho}{\longrightarrow} \Gamma(M) \longrightarrow 0$.
It is possible to generalise the previous construction in order to take into account the structures of $\mathcal{Z}(\mathbf{A})$-modules.
We identify $\mathrm{Int}(\mathbf{A})$ with $\mathbf{A}_0$. The adjoint representation of $\mathrm{Der}(\mathbf{A})$ on $\mathrm{Int}(\mathbf{A})$ is given by $\mathrm{ad}_{\mathfrak{X}}(\mathrm{ad}_a) = [\mathfrak{X}, \mathrm{ad}_a] = \mathrm{ad}_{\mathfrak{X}(a)}$ so that it is $(\mathfrak{X}, a) \mapsto \mathfrak{X}(a)$ on $\mathbf{A}_0$.
The vector space (and $\mathcal{Z}(\mathbf{A})$-module) we consider is $\mathcal{Z}(\mathbf{A})$ itself, on which the representation ρ is $(\mathfrak{X}, f) \mapsto \rho(\mathfrak{X}) \cdot f$.
Let $\mathrm{S}^q_{\mathcal{Z}(\mathbf{A})}(\mathbf{A}_0, \mathcal{Z}(\mathbf{A}))$ be the space of $\mathcal{Z}(\mathbf{A})$-linear symmetric maps $\otimes^q_{\mathcal{Z}(\mathbf{A})} \mathbf{A}_0 \to \mathcal{Z}(\mathbf{A})$ which intertwine the adjoint representation $\mathrm{ad}^{\otimes^q}$ of $\mathrm{Der}(\mathbf{A})$ on $\otimes^q_{\mathcal{Z}(\mathbf{A})} \mathrm{Int}(\mathbf{A}) = \otimes^q_{\mathcal{Z}(\mathbf{A})} \mathbf{A}_0$ and the representation ρ of $\mathrm{Der}(\mathbf{A})$ on $\mathcal{Z}(\mathbf{A})$.
Thanks to the $\mathcal{Z}(\mathbf{A})$-linearity, maps in $\mathrm{S}^q_{\mathcal{Z}(\mathbf{A})}(\mathbf{A}_0, \mathcal{Z}(\mathbf{A}))$ are local on M
➡ look at them in local trivialisation of E.
The intertwining relations can be written, with $\mathfrak{X}^{\mathrm{loc}} = X + \mathrm{ad}_\gamma$:

$$\sum_{i=1}^q \phi(a_1 \otimes \cdots \otimes X \cdot a_i \otimes \cdots \otimes a_q) = X \cdot \phi(a_1 \otimes \cdots \otimes a_q)$$
$$\sum_{i=1}^q \phi(a_1 \otimes \cdots \otimes [\gamma, a_i] \otimes \cdots \otimes a_q) = 0$$

Characteristic classes constructed from A

Proposition 50 (Characteristic classes)
The space $\mathsf{S}^q_{\mathcal{Z}(\mathbf{A})}(\mathbf{A}_0, \mathcal{Z}(\mathbf{A}))$ is well defined ($\mathcal{Z}(\mathbf{A})$-linearity and intertwiners are compatible) and

$$\mathsf{S}^q_{\mathcal{Z}(\mathbf{A})}(\mathbf{A}_0, \mathcal{Z}(\mathbf{A})) = \mathcal{P}^q_I(\mathfrak{sl}_n)$$

the space of invariant polynomials on the Lie algebra $\mathfrak{sl}_n$.
The differential complex in which the characteristic classes for the splitting are defined is $\mathrm{Hom}_{\mathcal{Z}(\mathbf{A})}(\bigwedge^\bullet_{\mathcal{Z}(\mathbf{A})}\Gamma(M), \mathcal{Z}(\mathbf{A}))$, which is the de Rham complex of differential forms on M.
The characteristic classes one computes in this way are precisely the ordinary characteristic classes of the principal fiber bundle P or of the vector bundle E.

The last statement relies on the fact that any ordinary connection ∇^E on E gives rise to a splitting of the s.e.s. whose curvature is exactly the obstruction to be a morphism of Lie algebras. The construction based on $\mathsf{S}^q_{\mathcal{Z}(\mathbf{A})}(\mathbf{A}_0, \mathcal{Z}(\mathbf{A})) = \mathcal{P}^q_I(\mathfrak{sl}_n)$ is then the Chern-Weil morphism.

One then concludes that the s.e.s. $0 \longrightarrow \mathrm{Int}(\mathbf{A}) \longrightarrow \mathrm{Der}(\mathbf{A}) \xrightarrow{\rho} \Gamma(M) \longrightarrow 0$ contains the characteristic classes of E.

Action of a Lie group

$G \to P \xrightarrow{\pi} M$ principal fiber bundle. H Lie group acting on the left on P, such that the action commutes with the right action of G.

- H induces a left action on M. Assumption: this action is simple, which means M admits the fiber bundle structure $H/H_0 \to M \to M/H$ where H_0 is an isotropy subgroup: $H_0 = H_{x_0} = \{h \in H \ / \ h{\cdot}x_0 = x_0\}$.
- Fix H_0 an isotropy subgroup.
  - $N = \{x \in M \ / \ H_x = H_0\}$.
  - $\mathcal{N}_H(H_0) = \{h \in H \ / \ hH_0 = H_0 h\}$ normalizer of H_0 in H.
  - H_0 normal subgroup of $\mathcal{N}_H(H_0)$, one has the principal fiber bundle

$$\mathcal{N}_H(H_0)/H_0 \to N \to M/H$$

    $H/H_0 \to M \to M/H$ is associated to this bundle for the natural action of $\mathcal{N}_H(H_0)/H_0$ on H/H_0 (left multiplication).
- $S = H \times G$ acts on the right on P: $(h, g){\cdot}p = h^{-1}{\cdot}p{\cdot}g$.
  - For $p \in P$, let $\lambda_p : H_{\pi(p)} \to G$ such that $h{\cdot}p = p{\cdot}\lambda_p(h)$.
  - $S_p = \{(h, \lambda_p(h)) \ / \ h \in H_{\pi(p)}\}$ is the isotropy subgroup of $p \in P$ ➡ the action of S on P is simple.
  - Fix S_0 an isotropy subgroup, let $Q = \{p \in P \ / \ S_p = S_0\}$, then

$$S/S_0 \to P \to M/H \text{ is associated to } \mathcal{N}_S(S_0)/S_0 \to Q \to M/H$$

The reduced fiber bundle Q

Proposition 51 (Some properties of Q)
The map $\lambda_p : H_{\pi(p)} \to G$ such that $h{\cdot}p = p{\cdot}\lambda_p(h)$ satisfies

$$\lambda_{p\cdot g}(h) = g^{-1}\lambda_p(h)g$$

For any $q \in Q$, λ_q depends only on $\pi(q) \in M$: $\lambda_q(h) = \lambda_{q\cdot g}(h)$.
For a fixed x_0 in M, we denote it by $\lambda : H_0 \to G$ with $H_0 = H_{x_0}$.
The projection $\pi : P \to M$ induces the fiber bundle structure

$$\mathcal{Z}_G(\lambda(H_0)) \longrightarrow Q \xrightarrow{\pi_Q} \pi(Q)$$

with $\mathcal{Z}_G(\lambda(H_0)) = \{g \in G \ / \ g\lambda(h_0) = \lambda(h_0)g, \forall h_0 \in H_0\}$, the centralizer of $\lambda(H_0)$ in G, and $\pi(Q) \subset N$.

In the following, we will need the diagram of fibrations:

$$\begin{array}{ccccc}
\mathcal{N}_S(S_0)/S_0 & \longrightarrow & Q & \longrightarrow & M/H \\
\big\downarrow & & \big\downarrow & & \| \\
S/S_0 & \longrightarrow & P & \longrightarrow & M/H
\end{array}$$

Lie algebras decompositions

We are interested in the Lie algebras of different groups introduced before:

- Over the group H: $\mathfrak{h}$ Lie algebra of H.
  - $\mathfrak{h}_0$ Lie algebra of H_0, the fixed isotropy group.
  - $\mathfrak{k}$ Lie algebra of the quotient group $\mathcal{N}_H(H_0)/H_0$.
  - $\mathfrak{n}_0 = \mathfrak{h}_0 \oplus \mathfrak{k}$ Lie algebra of $\mathcal{N}_H(H_0)$, the normalizer of H_0 in H.
  - $\mathfrak{l}$ vector space in the orthogonal decomposition $\mathfrak{h} = \mathfrak{n}_0 \oplus \mathfrak{l}$ s. t. $[\mathfrak{n}_0, \mathfrak{l}] \subset \mathfrak{l}$.
- Over the group G: $\mathfrak{g}$ Lie algebra of G.
  - $\mathfrak{z}_0$ Lie algebra of $\mathcal{Z}_G(\lambda(H_0))$, the centralizer of $\lambda(H_0)$ in G.
  - $\mathfrak{m}$ vector space in the orthogonal decomposition $\mathfrak{g} = \mathfrak{z}_0 \oplus \mathfrak{m}$ s.t. $[\mathfrak{z}_0, \mathfrak{m}] \subset \mathfrak{m}$.
- Over the group $S = H \times G$: $\mathfrak{s} = \mathfrak{h} \oplus \mathfrak{g}$ Lie algebra of S.
  - $\mathfrak{s}_0 = \{(x_0, \lambda_* x_0) \ / \ x_0 \in \mathfrak{h}_0\}$ Lie algebra of S_0, the fixed isotropy group.
  - $\mathfrak{s}_0 \oplus \mathfrak{k} \oplus \mathfrak{z}_0$ Lie algebra of $\mathcal{N}_S(S_0)$, the normalizer of S_0 in S.
  - $\mathfrak{k} \oplus \mathfrak{z}_0$ Lie algebra of the quotient group $\mathcal{N}_S(S_0)/S_0$.

Proposition 52 (Decomposition of TP)
For any $q \in Q$, one has $\mathfrak{k}_q^Q \oplus (\mathfrak{z}_0)_q^Q \subset T_q Q$ and $T_q P = T_q Q \oplus \mathfrak{l}_q^P \oplus \mathfrak{m}_q^P$.

$\mathfrak{a}_q^R$ are the tangent vectors associated to $x \in \mathfrak{a}$ through the fundamental vector fields on R for the action of the corresponding group H or G.

Action of a Lie group on A

Let H be a compact connected Lie group acting on the $SL(n)$-principal fiber bundle P (so that $G = SL(n)$).

Proposition 53 (Operations of $\mathfrak{h}$ on $\Omega^\bullet_{\mathrm{Der}}(\mathbf{B})$ and $\Omega^\bullet_{\mathrm{Der}}(\mathbf{A})$)
The operation of $\mathfrak{h}$ on $\Omega^\bullet(P)$ induced by the action of H on P extends to an operation of $\mathfrak{h}$ on $\Omega^\bullet_{\mathrm{Der}}(\mathbf{B}) = \Omega^\bullet(P) \otimes \Omega^\bullet_{\mathrm{Der}}(M_n)$ by a trivial action on the second factor. This operation commutes with the operations of $\mathfrak{g}_{\mathrm{ad}}$ and $\mathfrak{g}_{\mathrm{equ}}$, and so reduces to the operation of $\mathfrak{h}$ on $\Omega^\bullet(P)$ and to an operation of $\mathfrak{h}$ on $\Omega^\bullet_{\mathrm{Der}}(\mathbf{A})$.

Definition 54 (Invariant n.c. connection)
A n.c. connection $\widehat{\nabla}$ on the right $\mathbf{A}$-module $\boldsymbol{M} = \mathbf{A}$ is $\mathfrak{h}$-invariant if, $\forall y \in \mathfrak{h}$, $\forall \mathfrak{X} \in \mathrm{Der}(\mathbf{A})$ and $\forall a \in \mathbf{A}$, $L_y(\widehat{\nabla}_{\mathfrak{X}} a) = \widehat{\nabla}_{[y,\mathfrak{X}]} a + \widehat{\nabla}_{\mathfrak{X}}(L_y a)$ for the action of $\mathfrak{h}$ on $\Omega^\bullet_{\mathrm{Der}}(\mathbf{A})$.

Proposition 55 (Invariance of the n.c. 1-form of a n.c. connection)
The n.c. connection $\widehat{\nabla}$ is $\mathfrak{h}$-invariant if and only if its n.c. 1-form α is invariant: $L_y\alpha = 0$ for all $y \in \mathfrak{h}$.

The space of invariant n.c. connections

The problem to solve: find all the n.c. 1-forms written as $\alpha = \omega - \phi \in [\Omega^1(P) \otimes M_n] \oplus [C^\infty(P) \otimes M_n \otimes \mathfrak{sl}_n^*]$ satisfying

$$(L_{\xi^v} + L_{\mathrm{ad}_\xi})\omega = 0 \quad (L_{\xi^v} + L_{\mathrm{ad}_\xi})\phi = 0 \quad i_{\xi^v}\omega - i_{\mathrm{ad}_\xi}\phi = 0 \quad L_y(\omega - \phi) = 0$$

for all $\xi \in \mathfrak{g} = \mathfrak{sl}_n$ and $y \in \mathfrak{h}$.

- $L_y\omega = 0$ and $L_y\phi = 0$ for all $y \in \mathfrak{h}$ (H-invariance): permits to study ω and ϕ over $Q \subset P$ only. For all $q \in Q$
$$\omega_q : T_qP = T_qQ \oplus \mathfrak{l}_q^P \oplus \mathfrak{m}_q^P \to M_n \qquad \phi_q : \mathfrak{g} \to M_n$$
- $i_{\xi^v}\omega - i_{\mathrm{ad}_\xi}\phi = 0$ for all $\xi \in \mathfrak{g}$ ($\mathfrak{g}$-horizontality): $\phi_q(\xi)$ is determined by $\omega_q(\xi_q^v)$.

Let us first consider the T_qQ part of T_qP. Denote by μ_q the restriction of ω_q to T_qQ. One has $(\mathfrak{z}_0)_q^Q \subset T_qQ$ ➞ μ and ϕ both defined on $\mathfrak{z}_0 \subset \mathfrak{g}$, $\mu(z^Q) = \phi(z)$, $\forall z \in \mathfrak{z}_0$. Denote by η_q the restriction of ϕ_q to $\mathfrak{z}_0$.

Proposition 56 (The algebra W)

Let $\mathbf{W} = \mathcal{Z}_{M_n}(\lambda_*\mathfrak{g}_0)$ be the centralizer of $\lambda_*\mathfrak{g}_0$ in M_n. It is an associative algebra and $\mathfrak{z}_0 \subset \mathrm{Der}(\mathbf{W})$. Let $\Omega^\bullet_{\mathfrak{z}_0}(\mathbf{W}) = \mathbf{W} \otimes \bigwedge^\bullet \mathfrak{z}_0^*$ be the restricted derivation-based differential calculus associated to it. There is a natural operation of $\mathfrak{z}_0$ on $\Omega^\bullet(Q) \otimes \Omega^\bullet_{\mathfrak{z}_0}(\mathbf{W})$, and $\mu - \eta \in \left(\Omega^\bullet(Q) \otimes \Omega^\bullet_{\mathfrak{z}_0}(\mathbf{W})\right)^1_{\mathfrak{z}_0\text{-basic}}$.

The space of invariant n.c. connections

Proposition 57 (The differential calculus $\Omega^\bullet_{\mathfrak{k}\oplus\mathfrak{z}_0}(M/H;\mathbf{W})$)

There are natural operations of $\mathfrak{k} \subset \mathfrak{h}$ and $\mathfrak{z}_0 \subset \mathfrak{g}$ on the differential algebra $\Omega^\bullet(Q) \otimes \Omega^\bullet_{\mathfrak{z}_0}(\mathbf{W}) \otimes \bigwedge^\bullet \mathfrak{k}^*$.
Define

$$\Omega^\bullet_{\mathfrak{k}\oplus\mathfrak{z}_0}(M/H;\mathbf{W}) = \left(\Omega^\bullet(Q) \otimes \Omega^\bullet_{\mathfrak{z}_0}(\mathbf{W}) \otimes \textstyle\bigwedge^\bullet \mathfrak{k}^*\right)_{\mathfrak{k}\,\oplus\,\mathfrak{z}_0\text{-basic}}$$

The algebra $\mathbf{C} = \Omega^0_{\mathfrak{k}\oplus\mathfrak{z}_0}(M/H;\mathbf{W}) = (C^\infty(Q) \otimes \mathbf{W})_{\mathfrak{k}\,\oplus\,\mathfrak{z}_0\text{-invariants}}$ is the algebra of sections of a $\mathbf{W}$-fiber bundle associated to the principal fiber bundle

$$\mathcal{N}_S(S_0)/S_0 \longrightarrow Q \longrightarrow M/H$$

Recall that $\mathfrak{k}^Q_q \subset T_q Q$. For any $k \in \mathfrak{k}$, define $\nu(k) = \mu(k^Q)$, so that $\nu \in C^\infty(Q) \otimes \mathbf{W} \otimes \mathfrak{k}^*$.

Proposition 58 (The $T_q Q$ part of $T_q P$)

One has $\mu - \eta - \nu \in \Omega^1_{\mathfrak{k}\oplus\mathfrak{z}_0}(M/H;\mathbf{W})$.
This expression contains all the information about the restriction of ω to TQ and about ϕ.

The space of invariant n.c. connections

Consider now the $\mathfrak{l}^P_q \oplus \mathfrak{m}^P_q$ part of $T_q P$.
Recall that $[\mathfrak{h}_0 \oplus \mathfrak{k}, \mathfrak{l}] \subset \mathfrak{l}$ and $[\mathfrak{z}_0, \mathfrak{m}] \subset \mathfrak{m}$, so that there are natural actions $[\mathfrak{h}_0, \mathfrak{l} \oplus \mathfrak{m}] \subset \mathfrak{l} \oplus \mathfrak{m}$ and $[\mathfrak{k} \oplus \mathfrak{z}_0, \mathfrak{l} \oplus \mathfrak{m}] \subset \mathfrak{l} \oplus \mathfrak{m}$.
Recall that $\mathfrak{s}_0 \oplus \mathfrak{k} \oplus \mathfrak{z}_0$ is the Lie algebra of $\mathcal{N}_S(S_0)$ and $\mathfrak{k} \oplus \mathfrak{z}_0$ is the Lie algebra of $\mathcal{N}_S(S_0)/S_0$, with $\mathfrak{s}_0 = \{(x_0, \lambda_* x_0) \ / \ x_0 \in \mathfrak{h}_0\}$.
For the restriction of ω to Q, the H-invariance and the G-invariance combine together into a $\mathcal{N}_S(S_0)$-invariance.
One can treat this invariance in two steps: one for S_0 and one for $\mathcal{N}_S(S_0)/S_0$.
Define the vector space of S_0-invariant linear maps $\mathfrak{l} \oplus \mathfrak{m} \to M_n$:

$$\mathsf{F} = \{ f : \mathfrak{l} \oplus \mathfrak{m} \to M_n \ / \ f([x_0, v]) - [\lambda_* x_0, f(v)] = 0, \ \forall x_0 \in \mathfrak{h}_0, \ \forall v \in \mathfrak{l} \oplus \mathfrak{m} \}$$

on which $\mathfrak{k} \oplus \mathfrak{z}_0$ acts naturally: $(L_{k+z} f)(v) = -f([k, v]) + [z, f(v)]$.
Let $\boldsymbol{M} = (C^\infty(Q) \otimes \mathsf{F})_{\mathfrak{k} \oplus \mathfrak{z}_0\text{-invariants}}$.

Proposition 59 (The $\mathfrak{l}^P_q \oplus \mathfrak{m}^P_q$ part of $T_q P$)

$\boldsymbol{M}$ is the space of sections of the F*-fiber bundle associated to the principal fiber bundle $\mathcal{N}_S(S_0)/S_0 \to Q \to M/H$. It is a* **C***-bimodule.*
The restriction of ω to the subspaces $\mathfrak{l}^P_q \oplus \mathfrak{m}^P_q$ is in $\boldsymbol{M}$.

The space of invariant n.c. connections

One get the final identification:

Theorem 60 (The space of H-invariant n.c. connections)
The space of H-invariant n.c. connections on $\mathbf{A}$ *is* $\Omega^1_{\mathfrak{k}\oplus\mathfrak{z}_0}(M/H;\mathbf{W}) \oplus \boldsymbol{M}$.

Remark 61 (Naturality of the spaces)
In this result, all the spaces are constructed from the principal fiber bundle $\mathcal{N}_S(S_0)/S_0 \twoheadrightarrow Q \twoheadrightarrow M/H$ with the help of geometrical or algebraic methods which are natural in this n.c. framework:

- $\mathbf{C} = (C^\infty(Q) \otimes \mathbf{W})_{\mathfrak{k}\,\oplus\,\mathfrak{z}_0\text{-invariants}}$ is modeled on the finite dimensional algebra $\mathbf{W} \subset M_n$. It looks like a "reduced algebra" of $\mathbf{A}$.
- $\Omega^\bullet_{\mathfrak{k}\oplus\mathfrak{z}_0}(M/H;\mathbf{W})$ is a natural differential calculus over $\mathbf{C}$.
- $\boldsymbol{M} = (C^\infty(Q) \otimes \mathrm{F})_{\mathfrak{k}\,\oplus\,\mathfrak{z}_0\text{-invariants}}$ is a natural $\mathbf{C}$-bimodule.
- The space $SU(\mathbf{C})$ acts naturally on the space $\Omega^1_{\mathfrak{k}\oplus\mathfrak{z}_0}(M/H;\mathbf{W}) \oplus \boldsymbol{M}$ as restriction of n.c. gauge transformations.
- All these spaces are sections of fiber bundles over the base space M/H.

Perspectives

- Look at other ordinary constructions about P using the noncommutative geometry of $\mathbf{A}$.
- Extend this procedure to other kinds of fiber bundles: Clifford algebra of a metric manifold, Dixmier-Douady bundle of algebras, ...
- Look at the physical consequences of the extra fields. Could they help to understand the "geometry" behind the Higgs fields?

Bibliography

- Dubois-Violette, M. (1988).
  Dérivations et calcul differentiel non commutatif.
  C.R. Acad. Sci. Paris, Série I, 307:403–408.
- Dubois-Violette, M., Kerner, R., and Madore, J. (1990a).
  Noncommutative differential geometry and new models of gauge theory.
  J. Math. Phys., 31:323.
- Dubois-Violette, M., Kerner, R., and Madore, J. (1990b).
  Noncommutative differential geometry of matrix algebras.
  J. Math. Phys., 31:316.
- Dubois-Violette, M. and Masson, T. (1998).
  $SU(n)$-connections and noncommutative differential geometry.
  Journal of Geometry and Physics, 25(1,2):104.
- Lecomte, P. (1985).
  Sur la suite exacte canonique associée à un fibré principal.
  Bull. Soc. Math. France, 113:259–271.

Bibliography (cont.)

- Masson, T. (1995).
  Géométrie non commutative et applications à la théorie des champs.
  PhD thesis, Université Paris XI.
- Masson, T. (1999).
  On the noncommutative geometry of the endomorphism algebra of a vector bundle.
  Journal of Geometry and Physics, 31:142.
- Masson, T. and Sérié, E. (2005).
  Invariant noncommutative connections.
  J. Math. Phys., 46:123503.
- Sérié, E. (2005).
  Gauge theories in noncommutative geometry and generalization of the Born-Infeld model.
  PhD thesis, Université Paris XI.