

Noncommutative analog and generalization of $SU(n)$ principal fiber bundles

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Outline of the lectures

- Introduce a particular associative algebra related naturally to some constructions in ordinary fiber bundle theory.
- Equip this algebra with its derivation-based differential calculus. Study this differential calculus, and show how it is related to some usual differential structures in fiber theory.
- Compare the two notions of (ordinary) connections and noncommutative connections. Show how one is the generalisation of the other. Use it to propose Yang-Mills-Higgs type models.
- Make some algebraic constructions based on this algebra to recover ordinary characteristic classes.
- Generalize and study the notion of invariant connections by the action of a Lie group to this noncommutative context.

1 A brief review of ordinary fiber bundle theory

1.1 Principal and associated fiber bundles

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Principal fiber bundles

M smooth manifold, G Lie group. $G \rightarrow P \xrightarrow{\pi} M$ principal fiber bundle for the right action of G on P denoted by $p \mapsto p \cdot g = \tilde{R}_g p$.

- For any $p \in P$, define $V_p = \text{Ker}(T_p \pi : T_p P \rightarrow T_{\pi(p)} M)$ (vertical subspace of $T_p P$). For any $X \in \mathfrak{g}$, let

$$X^v(p) = \left(\frac{d}{dt} p \cdot \exp(tX) \right)_{t=0}$$

Then $V_p = \{X^v(p) / X \in \mathfrak{g}\}$. One has $\tilde{R}_{g*} V_p = V_{p \cdot g}$.

➔ define vertical vector fields, horizontal differential forms (vanish when one of its arguments is vertical).

- (U, ϕ) local trivialisation of P over $U \subset M$ if U open subset, $\phi : U \times G \xrightarrow{\cong} \pi^{-1}(U)$, $\pi(\phi(x, h)) = x$ and $\phi(x, hg) = \phi(x, h) \cdot g$ for any $x \in U$ and $g, h \in G$.
- (U_i, ϕ_i) and (U_j, ϕ_j) two local trivialisations such that $U_i \cap U_j \neq \emptyset$. There exists a differentiable map $g_{ij} : U_i \cap U_j \rightarrow G$ such that if $\phi_i(x, h_i) = \phi_j(x, h_j)$ for $h_i, h_j \in G$ then $h_i = g_{ij}(x)h_j$ for any $x \in U_i \cap U_j$. The g_{ij} are the transition functions. They satisfy $g_{ij}(x) = g_{ji}^{-1}(x)$ for any $x \in U_i \cap U_j$ and the cocycle condition $g_{ij}(x)g_{jk}(x)g_{ki}(x) = e$ for any $x \in U_i \cap U_j \cap U_k \neq \emptyset$.

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Associated fiber bundles

$G \rightarrow P \xrightarrow{\pi} M$ principal fiber bundle as before.

- F manifold on which G acts on the left: $\varphi \mapsto \ell_g \varphi$.
- On $P \times F$ we define the right action $(p, \varphi) \mapsto (p \cdot g, \ell_{g^{-1}} \varphi)$.
- $E = (P \times F) / G$ is the orbit space for this action.

This is the associated fiber bundle to P for the couple (F, ℓ) .

- Notations: $E = P \times_{\ell} F$ and $[p, \varphi] \in E$ the projection of (p, φ) in the quotient $P \times F \rightarrow (P \times F) / G$. Then $[p \cdot g, \varphi] = [p, \ell_g \varphi]$.

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Sections

$G \rightarrow P \xrightarrow{\pi} M$ principal fiber bundle, $E \rightarrow M$ associated fiber bundle.

- A section is a map $s : M \rightarrow E$ such that $\pi \circ s(x) = x$ for any $x \in M$.

Notation: $\Gamma(E)$ space of sections.

- $\Gamma(E) \simeq \mathcal{F}_G(P, F) = \{\Phi : P \rightarrow F / \Phi(p \cdot g) = \ell_{g^{-1}} \Phi(p)\}$

(equivariant maps $P \rightarrow F$).

- (U, ϕ) local trivialisation of P over U .

$s_U : U \rightarrow \pi^{-1}(U)$ given by $s_U(x) = \phi(x, e)$ is a local section of P .

Any section s of P is locally given by a local map $h : U \rightarrow G$ such that $s(x) = s_U(x) \cdot h(x) = \phi(x, h(x))$.

Any section s of E is locally given by a local map $\varphi : U \rightarrow F$ such that $s(x) = [s_U(x), \varphi(x)]$.

- (U_i, ϕ_i) and (U_j, ϕ_j) two local trivialisations such that $U_i \cap U_j \neq \emptyset$. One has

$$s_j(x) = \phi_j(x, e) = \phi_i(x, g_{ij}(x)) = \phi_i(x, e) \cdot g_{ij}(x) = s_i(x) \cdot g_{ij}(x)$$

so that on P , if $s(x) = s_i(x) \cdot h_i(x) = s_j(x) \cdot h_j(x)$, then

$$h_i(x) = g_{ij}(x) h_j(x)$$

On E , if $s(x) = [s_i(x), \varphi_i(x)] = [s_j(x), \varphi_j(x)]$ for $x \in U_i \cap U_j \neq \emptyset$, then

$$\varphi_i(x) = \ell_{g_{ij}(x)} \varphi_j(x)$$

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Vector bundles

Let F be a vector space and ℓ a representation (linear action).

- The associated fiber bundle E is called a vector bundle.
- $\Gamma(E)$ is a $C^\infty(M)$ -module for the pointwise multiplication: $f(x)s(x)$ for any $f \in C^\infty(M)$, $s \in \Gamma(E)$ and $x \in M$.
- If E and E' are vector bundles then E^* (dual), $E \oplus E'$ (Whitney sum) and $E \otimes E'$ (tensor product), $\bigwedge^\bullet E$ (exterior product) are defined.
Use: (F^*, ℓ^*) , $(F \oplus F', \ell \oplus \ell')$, $(F \otimes F', \ell \otimes \ell')$ and $(\bigwedge^\bullet F, \bigwedge \ell)$.

Example 1 (The endomorphism bundle)

Consider the case where F is finite dimensional.

$E^* \otimes E$ is associated to P for the couple $(F^* \otimes F, \ell^* \otimes \ell)$.

One has $F^* \otimes F \simeq \text{End}(F)$ by $(\alpha \otimes \varphi)(\varphi') = \alpha(\varphi')\varphi$.

We use the notation $\text{End}(E) = E^* \otimes E$. It is called the endomorphism bundle of E .

There is a pairing $\Gamma(E^*) \otimes \Gamma(E) \rightarrow C^\infty(M)$ denoted by $x \mapsto \langle \alpha(x), s(x) \rangle$.

$\Gamma(E^* \otimes E) = \Gamma(\text{End}(E))$ is an algebra, which identifies with $\Gamma(E^*) \otimes_{C^\infty(M)} \Gamma(E)$ and with the space of $C^\infty(M)$ -module maps $\Gamma(E) \rightarrow \Gamma(E)$ by $(\alpha \otimes s)(s')(x) = \langle \alpha(x), s'(x) \rangle s(x)$.

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Other useful examples

Examples 2 (Tangent and cotangent spaces)

$TM \rightarrow M, T^*M \rightarrow M$ are vector bundles over M .

$\Gamma(TM)$, also denoted by $\Gamma(M)$, is the $C^\infty(M)$ -module of vector fields on M .

It is also a Lie algebra for the bracket $[X, Y] \cdot f = X \cdot Y \cdot f - Y \cdot X \cdot f$ for any $f \in C^\infty(M)$.

$\Gamma(\bigwedge^\bullet T^*M) = \Omega^\bullet(M)$ is the algebra of differential forms on M .

Examples 3 (The gauge group and its Lie algebra)

G acts on itself by conjugaison: $\alpha_g(h) = ghg^{-1}$. The associated fiber bundle $P \times_\alpha G$ has G as fiber but is not a principal bundle.

$\mathcal{G} = \Gamma(P \times_\alpha G)$ is a group, called the gauge group of P : it is the sub-group of $\text{Aut}(P)$ of vertical automorphisms.

$\Phi : P \rightarrow G$, G -equivariant: $p \mapsto p \cdot \Phi(p)$.

One has $p \cdot g \mapsto (p \cdot g) \cdot \Phi(p \cdot g) = (p \cdot g) \cdot (g^{-1} \Phi(p) g) = (p \cdot \Phi(p)) \cdot g$.

G acts on the vector space \mathfrak{g} by the adjoint action Ad .

Denote by $\text{Ad}P = P \times_{\text{Ad}} \mathfrak{g}$ the associated vector bundle. $\Gamma(\text{Ad}P)$ is the Lie algebra of the gauge group \mathcal{G} , denoted $\text{Lie}\mathcal{G}$.

1.2 Connections

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Connections

$G \rightarrow P \xrightarrow{\pi} M$ principal fiber bundle, $E \rightarrow M$ associated vector bundle.

A connection is:

Geometrical definition: A smooth distribution H in TP such that for any $p \in P$ and $g \in G$:

$$T_p P = V_p \oplus H_p \quad \text{and} \quad \tilde{R}_{g*} H_p = H_{p \cdot g}$$

- ➔ horizontal vector fields, vertical differential forms (vanish when one of its arguments is horizontal).
- ➔ horizontal lifting of vector fields on M : $\Gamma(M) \ni X \mapsto X^h \in \Gamma(P)$.

Algebraic definition: A 1-form on P taking values in the Lie algebra \mathfrak{g} , $\omega \in \Omega^1(P) \otimes \mathfrak{g}$, such that for any $g \in G$ and $X \in \mathfrak{g}$:

$$\tilde{R}_g^* \omega = \text{Ad}_{g^{-1}} \omega \quad (\text{equivariance}) \quad \text{and} \quad \omega(X^v) = X \quad (\text{vertical condition})$$

➔ $H_p = \text{Ker } \omega|_p$.

Analytic definition: A linear map $\nabla_X^E : \Gamma(E) \rightarrow \Gamma(E)$ defined for any $X \in \Gamma(M)$, such that for any $f \in C^\infty(M)$, $s \in \Gamma(E)$, $X, Y \in \Gamma(M)$:

$$\nabla_X^E(f s) = (X \cdot f) s + f \nabla_X^E s \qquad \nabla_{f X}^E s = f \nabla_X^E s \qquad \nabla_{X+Y}^E s = \nabla_X^E s + \nabla_Y^E s$$

➔ $\Gamma(E) \ni s \leftrightarrow \Phi \in \mathcal{F}_G(P, F)$ corresponds to $\nabla_X^E s \leftrightarrow X^h \cdot \Phi$.

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Curvature of a connection

Geometrical definition: There exists a geometrical interpretation of the curvature which will not be explained here (obstruction to the closure of horizontal lifts of “infinitesimal” closed paths on M).

Algebraic definition: The curvature is the equivariant horizontal 2-form $\Omega \in \Omega^2(P) \otimes \mathfrak{g}$ defined for any $\mathcal{X}, \mathcal{Y} \in \Gamma(P)$ by

$$\Omega(\mathcal{X}, \mathcal{Y}) = d\omega(\mathcal{X}, \mathcal{Y}) + [\omega(\mathcal{X}), \omega(\mathcal{Y})]$$

It satisfies the Bianchi identity

$$d\Omega + [\omega, \Omega] = 0$$

Analytic definition: Given $\nabla_X^E : \Gamma(E) \rightarrow \Gamma(E)$, the curvature $R^E(X, Y)$ is the map defined for any $X, Y \in \Gamma(M)$ by

$$R^E(X, Y) = \nabla_X^E \nabla_Y^E - \nabla_Y^E \nabla_X^E - \nabla_{[X, Y]}^E : \Gamma(E) \rightarrow \Gamma(E)$$

It is a $C^\infty(M)$ -module map.

η the representation of \mathfrak{g} on F induced by the representation ℓ of G .

Then $\Gamma(E) \ni s \leftrightarrow \Phi \in \mathcal{F}_G(P, F)$ corresponds to $R^E(X, Y)s \leftrightarrow \eta(\Omega(\mathcal{X}, \mathcal{Y})) \cdot \Phi$ for any \mathcal{X}, \mathcal{Y} s.t. $\pi_* \mathcal{X} = X$ and $\pi_* \mathcal{Y} = Y$.

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Connections: local expressions

$\omega \in \Omega^1(P) \otimes \mathfrak{g}$ connection 1-form on P , Ω its curvature.

(U, ϕ) local trivialisation of P , s its local section.

Local expression of the connection and the curvature:

$$A = s^* \omega \in \Omega^1(U) \otimes \mathfrak{g} \qquad F = s^* \Omega \in \Omega^2(U) \otimes \mathfrak{g}$$

(U_i, ϕ_i) and (U_j, ϕ_j) two local trivialisations. On $U_i \cap U_j \neq \emptyset$ one has

$$A_j = g_{ij}^{-1} A_i g_{ij} + g_{ij}^{-1} dg_{ij} \qquad F_j = g_{ij}^{-1} F_i g_{ij}$$

A family of 1-forms $\{A_i\}_i$ satisfying these gluing relations defines a connection 1-form on P .

Globally on P

$\omega \in \Omega^1(P) \otimes \mathfrak{g}$, equivariant, vertical condition.

$\Omega \in \Omega^2(P) \otimes \mathfrak{g}$, equivariant and horizontal.

Locally on M

Family of local 1-forms $\{A_i\}_i$, $A_i \in \Omega^1(U_i) \otimes \mathfrak{g}$, satisfying gluing non homogeneous relations.

Family of local 2-forms $\{F_i\}_i$, $F_i \in \Omega^2(U_i) \otimes \mathfrak{g}$, satisfying gluing homogeneous relations.

Remark 4 (Intermediate construction)

The curvature is a section of the associated vector bundle $\wedge^2 T^* M \otimes \text{Ad} P$.

The connection cannot be the section of such an “intermediate” construction between forms on P and local forms on the U_i 's.

1.3 Gauge transformations

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Gauge transformations

$\mathcal{G} = \Gamma(P \times_{\alpha} G)$ acts on P .
 $a \in \mathcal{G} \leftrightarrow \Phi : P \rightarrow G$, G -equivariant.
 a induces a vertical diffeomorphism $P \rightarrow P$, denoted by a .
 $\omega \in \Omega^1(P) \otimes \mathfrak{g}$ connection.

- $a^*\omega$ is a connection and $a^*\Omega$ is its curvature.
- $a^*\omega = \Phi^{-1}\omega\Phi + \Phi^{-1}d\Phi$.
- $a^*\Omega = \Phi^{-1}\Omega\Phi$.

Infinitesimal version: $\Phi = \exp(\xi)$ with $\xi : P \rightarrow \mathfrak{g}$, G -equivariant.
 $\rightarrow \xi$ defines an element in $\text{Lie}\mathcal{G} = \Gamma(\text{Ad}P)$.
 Infinitesimal action on connections and curvatures:

$$\omega \mapsto d\xi + [\omega, \xi] \qquad \Omega \mapsto [\Omega, \xi]$$

2 Derivation-based noncommutative geometry

2.1 Derivation-based differential calculus

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Derivations of an associative algebra

\mathbf{A} associative algebra with unit.
 $\mathcal{Z}(\mathbf{A})$ center of \mathbf{A} : commutative subalgebra.

Definition 5 (Vector space of derivations of \mathbf{A})
 $\text{Der}(\mathbf{A}) = \{\mathfrak{X} : \mathbf{A} \rightarrow \mathbf{A} / \mathfrak{X} \text{ linear, } \mathfrak{X}(ab) = \mathfrak{X}(a)b + a\mathfrak{X}(b), \forall a, b \in \mathbf{A}\}$

Proposition 6 (Structure of $\text{Der}(\mathbf{A})$)
 $\text{Der}(\mathbf{A})$ is a Lie algebra for the bracket $[\mathfrak{X}, \mathfrak{Y}]a = \mathfrak{X}\mathfrak{Y}a - \mathfrak{Y}\mathfrak{X}a$ ($\forall \mathfrak{X}, \mathfrak{Y} \in \text{Der}(\mathbf{A})$), is a $\mathcal{Z}(\mathbf{A})$ -module for the product $(f\mathfrak{X})a = f(\mathfrak{X}a)$ ($\forall f \in \mathcal{Z}(\mathbf{A}), \forall \mathfrak{X} \in \text{Der}(\mathbf{A})$).

$\text{Int}(\mathbf{A}) = \{\text{ad}_a : b \mapsto [a, b] / a \in \mathbf{A}\} \subset \text{Der}(\mathbf{A})$, vector space of inner derivations, is a Lie ideal and a $\mathcal{Z}(\mathbf{A})$ -submodule.

With $\text{Out}(\mathbf{A}) = \text{Der}(\mathbf{A})/\text{Int}(\mathbf{A})$, there is a s.e.s. of Lie algebras and $\mathcal{Z}(\mathbf{A})$ -modules

$$0 \rightarrow \text{Int}(\mathbf{A}) \rightarrow \text{Der}(\mathbf{A}) \rightarrow \text{Out}(\mathbf{A}) \rightarrow 0$$

Definition 7 (Real derivations for involutive algebras)
 If \mathbf{A} is an involutive algebra (notation: $a \mapsto a^*$), $\mathfrak{X} \in \text{Der}(\mathbf{A})$ is real if $(\mathfrak{X}a)^* = \mathfrak{X}a^*$ for any $a \in \mathbf{A}$. Notation: $\text{Der}_{\mathbb{R}}(\mathbf{A})$.

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Derivation-based differential calculus

\mathbf{A} associative algebra with unit.

Definition 8 (The graded differential algebra $\underline{\Omega}_{\text{Der}}^{\bullet}(\mathbf{A})$)

Let $\underline{\Omega}_{\text{Der}}^n(\mathbf{A})$ be the set of $\mathcal{Z}(\mathbf{A})$ -multilinear antisymmetric maps from $\text{Der}(\mathbf{A})^n$ to \mathbf{A} , with $\underline{\Omega}_{\text{Der}}^0(\mathbf{A}) = \mathbf{A}$. Define

$$\underline{\Omega}_{\text{Der}}^{\bullet}(\mathbf{A}) = \bigoplus_{n \geq 0} \underline{\Omega}_{\text{Der}}^n(\mathbf{A})$$

- \mathbb{N} -graded algebra for the product

$$(\omega\eta)(\mathfrak{X}_1, \dots, \mathfrak{X}_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in \mathfrak{S}_{p+q}} (-1)^{\text{sign}(\sigma)} \omega(\mathfrak{X}_{\sigma(1)}, \dots, \mathfrak{X}_{\sigma(p)}) \eta(\mathfrak{X}_{\sigma(p+1)}, \dots, \mathfrak{X}_{\sigma(p+q)})$$

- differential graded algebra for the differential d (of degree 1) defined by (Koszul formula)

$$d\omega(\mathfrak{X}_1, \dots, \mathfrak{X}_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} \mathfrak{X}_i \omega(\mathfrak{X}_1, \dots, \overset{\cdot}{\mathfrak{X}}_i, \dots, \mathfrak{X}_{n+1}) \\ + \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} \omega([\mathfrak{X}_i, \mathfrak{X}_j], \dots, \overset{\cdot}{\mathfrak{X}}_i, \dots, \overset{\cdot}{\mathfrak{X}}_j, \dots, \mathfrak{X}_{n+1})$$

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Derivation-based differential calculus

Definition 9 (The graded differential algebra $\Omega_{\text{Der}}^{\bullet}(\mathbf{A})$)

$\Omega_{\text{Der}}^{\bullet}(\mathbf{A}) \subset \underline{\Omega}_{\text{Der}}^{\bullet}(\mathbf{A})$ the sub differential graded algebra generated in degree 0 by \mathbf{A} .

Every element in $\Omega_{\text{Der}}^n(\mathbf{A})$ is a sum of terms of the form $a_0 da_1 \cdots da_n$ for $a_0, \dots, a_n \in \mathbf{A}$.

Example 10 (The algebra $\mathbf{A} = C^{\infty}(M)$)

Let M be a M and let $\mathbf{A} = C^{\infty}(M)$.

- $\mathcal{Z}(\mathbf{A}) = C^{\infty}(M)$.
- $\text{Der}(\mathbf{A}) = \Gamma(M)$, $\text{Int}(\mathbf{A}) = 0$, $\text{Out}(\mathbf{A}) = \Gamma(M)$.
- $\Omega_{\text{Der}}^{\bullet}(\mathbf{A}) = \underline{\Omega}_{\text{Der}}^{\bullet}(\mathbf{A}) = \Omega^{\bullet}(M)$, gr. diff. alg. of de Rham forms on M .

Definition 11 (Restricted derivation-based differential calculus)

Let $\mathfrak{g} \subset \text{Der}(\mathbf{A})$ be a sub Lie algebra and a sub $\mathcal{Z}(\mathbf{A})$ -module. The restricted derivation-based differential calculus $\underline{\Omega}_{\mathfrak{g}}^{\bullet}(\mathbf{A})$ is defined as the set of $\mathcal{Z}(\mathbf{A})$ -multilinear antisymmetric maps from \mathfrak{g}^n to \mathbf{A} for $n \geq 0$, using the previous formulae for the product and the differential.

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Cartan operations

Let \mathfrak{g} be a Lie subalgebra of $\text{Der}(\mathbf{A})$.

\mathfrak{g} defines a natural operation in the sense of H. Cartan on $(\underline{\Omega}_{\text{Der}}^{\bullet}(\mathbf{A}), d)$.

- Interior product: graded derivation of degree -1 on $\underline{\Omega}_{\text{Der}}^{\bullet}(\mathbf{A})$.

$$i_{\mathfrak{X}} : \underline{\Omega}_{\text{Der}}^n(\mathbf{A}) \rightarrow \underline{\Omega}_{\text{Der}}^{n-1}(\mathbf{A}) \quad (i_{\mathfrak{X}}\omega)(\mathfrak{X}_1, \dots, \mathfrak{X}_{n-1}) = \omega(\mathfrak{X}, \mathfrak{X}_1, \dots, \mathfrak{X}_{n-1})$$

$\forall \mathfrak{X} \in \mathfrak{g}$, $\forall \omega \in \underline{\Omega}_{\text{Der}}^n(\mathbf{A})$ and $\forall \mathfrak{X}_i \in \text{Der}(\mathbf{A})$.

$i_{\mathfrak{X}}$ is 0 on $\underline{\Omega}_{\text{Der}}^0(\mathbf{A}) = \mathbf{A}$.

- Lie derivative: graded derivation of degree 0 on $\underline{\Omega}_{\text{Der}}^{\bullet}(\mathbf{A})$.

$$L_{\mathfrak{X}} = i_{\mathfrak{X}}d + di_{\mathfrak{X}} : \underline{\Omega}_{\text{Der}}^n(\mathbf{A}) \rightarrow \underline{\Omega}_{\text{Der}}^n(\mathbf{A})$$

- One has:

$$\begin{aligned} i_{\mathfrak{X}}i_{\mathfrak{Y}} + i_{\mathfrak{Y}}i_{\mathfrak{X}} &= 0 & L_{\mathfrak{X}}i_{\mathfrak{Y}} - i_{\mathfrak{Y}}L_{\mathfrak{X}} &= i_{[\mathfrak{X}, \mathfrak{Y}]} \\ L_{\mathfrak{X}}L_{\mathfrak{Y}} - L_{\mathfrak{Y}}L_{\mathfrak{X}} &= L_{[\mathfrak{X}, \mathfrak{Y}]} & L_{\mathfrak{X}}d - dL_{\mathfrak{X}} &= 0 \end{aligned}$$

- Subspaces of $\underline{\Omega}_{\text{Der}}^{\bullet}(\mathbf{A})$ associated to this operation:

- Horizontal subspace: kernel of all the $i_{\mathfrak{X}}$ for $\mathfrak{X} \in \mathfrak{g}$. Gr. Alg.
- Invariant subspace: kernel of all the $L_{\mathfrak{X}}$ for $\mathfrak{X} \in \mathfrak{g}$. Gr. Diff. Alg.
- Basic subspace: kernel of all the $i_{\mathfrak{X}}$ and $L_{\mathfrak{X}}$ for $\mathfrak{X} \in \mathfrak{g}$. Gr. Diff. Alg.

Example: $\mathfrak{g} = \text{Int}(\mathbf{A})$ defines an operation.

2.2 Noncommutative connections and their properties

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Noncommutative connections

Let M be a right \mathbf{A} -module.

Definition 12 (Noncommutative connection, curvature)

A noncommutative connection is a linear map $\widehat{\nabla}_{\mathfrak{X}} : M \rightarrow M$, defined for any $\mathfrak{X} \in \text{Der}(\mathbf{A})$, such that $\forall \mathfrak{X}, \mathfrak{Y} \in \text{Der}(\mathbf{A}), \forall a \in \mathbf{A}, \forall m \in M, \forall f \in \mathcal{Z}(\mathbf{A})$:

$$\widehat{\nabla}_{\mathfrak{X}}(ma) = m(\mathfrak{X}a) + (\widehat{\nabla}_{\mathfrak{X}}m)a, \quad \widehat{\nabla}_{f\mathfrak{X}}m = f\widehat{\nabla}_{\mathfrak{X}}m, \quad \widehat{\nabla}_{\mathfrak{X}+\mathfrak{Y}}m = \widehat{\nabla}_{\mathfrak{X}}m + \widehat{\nabla}_{\mathfrak{Y}}m$$

The curvature of $\widehat{\nabla}$ is the linear map $\widehat{R}(\mathfrak{X}, \mathfrak{Y}) : M \rightarrow M$ defined for any $\mathfrak{X}, \mathfrak{Y} \in \text{Der}(\mathbf{A})$ by

$$\widehat{R}(\mathfrak{X}, \mathfrak{Y})m = [\widehat{\nabla}_{\mathfrak{X}}, \widehat{\nabla}_{\mathfrak{Y}}]m - \widehat{\nabla}_{[\mathfrak{X}, \mathfrak{Y}]}m$$

Proposition 13 (General properties)

One has:

- The space of connections is an affine space modeled over the vector space $\text{Hom}^{\mathbf{A}}(\underline{\Omega}_{\text{Der}}^1(\mathbf{A}), M)$ (right \mathbf{A} -module homomorphisms).
- $\widehat{R}(\mathfrak{X}, \mathfrak{Y}) : M \rightarrow M$ is a right \mathbf{A} -module homomorphism.

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Gauge transformations

M a right \mathbf{A} -module.

Definition 14 (Gauge group)

The gauge group of M is the group of automorphisms of M as a right \mathbf{A} -module.

Proposition 15 (Gauge transformations)

For any Φ in the gauge group of M and any n.c. connection $\widehat{\nabla}$, the map $\widehat{\nabla}_{\mathfrak{X}}^{\Phi} = \Phi^{-1} \circ \widehat{\nabla}_{\mathfrak{X}} \circ \Phi : M \rightarrow M$ is a n.c. connection.

This defines the action of the gauge group on the space of n.c. connections.

Proof of the derivation rule:

$$\begin{aligned} \widehat{\nabla}_{\mathfrak{X}}^{\Phi}(ma) &= \Phi^{-1} \left(\widehat{\nabla}_{\mathfrak{X}}\Phi(ma) \right) = \Phi^{-1} \left(\widehat{\nabla}_{\mathfrak{X}}(\Phi(m)a) \right) \\ &= \Phi^{-1} \left(\Phi(m)(\mathfrak{X}a) + (\widehat{\nabla}_{\mathfrak{X}}\Phi(m))a \right) \\ &= m(\mathfrak{X}a) + (\Phi^{-1}\widehat{\nabla}_{\mathfrak{X}}\Phi)(m)a = m(\mathfrak{X}a) + (\widehat{\nabla}_{\mathfrak{X}}^{\Phi}m)a \end{aligned}$$

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Hermitean structures

Suppose now that \mathbf{A} is an involutive algebra.

Let M be a right \mathbf{A} -module.

Definition 16 (Hermitean structure)

A Hermitean structure on M is a sesquilinear form $\langle -, - \rangle : M \times M \rightarrow \mathbf{A}$ such that, $\forall m_1, m_2 \in M, \forall a_1, a_2 \in \mathbf{A}$,

$$\langle m_1, m_2 \rangle^* = \langle m_2, m_1 \rangle \quad \langle m_1 a_1, m_2 a_2 \rangle = a_1^* \langle m_1, m_2 \rangle a_2$$

A n.c. connection $\widehat{\nabla}$ is compatible with $\langle -, - \rangle$ if, $\forall m_1, m_2 \in M, \forall \mathfrak{X} \in \text{Der}_{\mathbb{R}}(\mathbf{A})$,

$$\mathfrak{X}\langle m_1, m_2 \rangle = \langle \widehat{\nabla}_{\mathfrak{X}}m_1, m_2 \rangle + \langle m_1, \widehat{\nabla}_{\mathfrak{X}}m_2 \rangle$$

Definition 17 (“Unitary” gauge transformations)

An element Φ in the gauge group is compatible with the Hermitean structure if $\langle \Phi(m_1), \Phi(m_2) \rangle = \langle m_1, m_2 \rangle$.

Lemma 18

The space of compatible n.c. connections with $\langle -, - \rangle$ is stable under “unitary” gauge transformations.

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Noncommutative connections: $M = \mathbf{A}$

\mathbf{A} is a unital algebra.

As a special case we consider the right \mathbf{A} -module $M = \mathbf{A}$.

Let $\widehat{\nabla}_{\mathfrak{X}} : \mathbf{A} \rightarrow \mathbf{A}$ be a noncommutative connection.

Proposition 19 (Noncommutative connections on $M = \mathbf{A}$)

$\widehat{\nabla}$ is completely given by $\widehat{\nabla}_{\mathfrak{X}}\mathbb{1} = \omega(\mathfrak{X})$, with $\omega \in \Omega_{\text{Der}}^1(\mathbf{A})$, by the relation

$$\widehat{\nabla}_{\mathfrak{X}}a = \mathfrak{X}a + \omega(\mathfrak{X})a$$

The curvature of $\widehat{\nabla}$ is the multiplication on the left on \mathbf{A} by the n.c. 2-form

$$\Omega(\mathfrak{X}, \mathfrak{Y}) = d\omega(\mathfrak{X}, \mathfrak{Y}) + [\omega(\mathfrak{X}), \omega(\mathfrak{Y})]$$

The gauge group is identified with the invertible elements $g \in \mathbf{A}$ by $\Phi_g(a) = ga$. The gauge transformations on $\widehat{\nabla}$ take the following form on ω and Ω :

$$\omega \mapsto \omega^g = g^{-1}\omega g + g^{-1}dg \qquad \Omega \mapsto \Omega^g = g^{-1}\Omega g$$

$\widehat{\nabla}_{\mathfrak{X}}^0$ defined by $a \mapsto \mathfrak{X}a$ is a n.c. connection on \mathbf{A} .

Particular case: \mathbf{A} involutive. Hermitean structure: $\langle a, b \rangle = a^*b$.

$U(\mathbf{A}) = \{u \in \mathbf{A} / u^*u = uu^* = \mathbb{1}\}$, the group of unitary elements of \mathbf{A} , is the unitary gauge group.

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Noncommutative connections: $M = \mathbf{A}$

Remark 20 (Vector space *versus* gauge transformations)

In ordinary differential geometry, the space of connections is an affine space.

Here it looks like the vector space $\Omega_{\text{Der}}^1(\mathbf{A})$.

Gauge transformations are not compatible with this linear structure:

$$\begin{aligned} (\lambda_1\omega_1 + \lambda_2\omega_2)^u &= u^{-1}(\lambda_1\omega_1 + \lambda_2\omega_2)u + u^{-1}du \\ \lambda_1\omega_1^u + \lambda_2\omega_2^u &= \lambda_1(u^{-1}\omega_1u + u^{-1}du) + \lambda_2(u^{-1}\omega_2u + u^{-1}du) \end{aligned}$$

are not equal except for $\lambda_1 + \lambda_2 = 1$.

Proposition 21 (Canonical gauge invariant n.c. connection)

If there exists a n.c. 1-form $\xi \in \Omega_{\text{Der}}^1(\mathbf{A})$ such that $da = [\xi, a]$ for any $a \in \mathbf{A}$, then the canonical n.c. connection defined by $\widehat{\nabla}_{\mathfrak{X}}^{-\xi}a = \mathfrak{X}a - \xi(\mathfrak{X})a$ can be written as $\widehat{\nabla}_{\mathfrak{X}}^{-\xi}a = -a\xi(\mathfrak{X})$ and it is gauge invariant.

Proof. One has $\mathfrak{X}a = [\xi(\mathfrak{X}), a]$, so $\widehat{\nabla}_{\mathfrak{X}}^{-\xi}a = [\xi(\mathfrak{X}), a] - \xi(\mathfrak{X})a = -a\xi(\mathfrak{X})$.

Let $u \in U(\mathbf{A})$ be a unitary gauge transformation. Its action on the n.c. 1-form $-\xi$ is $(-\xi)^u = -u^{-1}\xi u + u^{-1}du = u^{-1}(-\xi u + [\xi, u]) = u^{-1}(-u\xi) = -\xi$. \square

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Noncommutative connections: $M = \mathbf{A}^N$

As a special case we consider the right \mathbf{A} -module $M = \mathbf{A}^N$.

Let $\widehat{\nabla}_{\mathfrak{X}} : \mathbf{A}^N \rightarrow \mathbf{A}^N$ be a noncommutative connection. Let $e_i = (0, \dots, \mathbb{1}, \dots, 0)$, for $i = 1, \dots, N$, (basis of the right module \mathbf{A}^N). Look at $m = e_i a^i$ as a column vector for the a^i 's, use the notation $\mathfrak{X}m = e_i(\mathfrak{X}a^i)$ and the matrix product.

Proposition 22 (Noncommutative connections on $M = \mathbf{A}^N$)

$\widehat{\nabla}$ is completely determined by N^2 n.c. 1-forms $\omega_i^j \in \Omega_{\text{Der}}^1(\mathbf{A})$ defined by $\widehat{\nabla}_{\mathfrak{X}}e_i = e_j\omega_i^j(\mathfrak{X})$, through the relation $\widehat{\nabla}_{\mathfrak{X}}m = \mathfrak{X}m + \omega(\mathfrak{X})m$, with $\omega = (\omega_i^j) \in M_N(\Omega_{\text{Der}}^1(\mathbf{A}))$. The curvature of $\widehat{\nabla}$ is the multiplication on the left on \mathbf{A}^N by the matrix of n.c. 2-forms $\Omega = d\omega + [\omega, \omega] \in M_N(\Omega_{\text{Der}}^2(\mathbf{A}))$.

The gauge group of \mathbf{A}^N is $GL_N(\mathbf{A})$ (invertibles in $M_N(\mathbf{A})$), which acts by left (matrix) multiplication. The gauge transformations take the forms $\omega^g = g^{-1}\omega g + g^{-1}dg$ and $\Omega^g = g^{-1}\Omega g$ in matrix notations.

$\widehat{\nabla}_{\mathfrak{X}}^0$ defined by $m \mapsto \mathfrak{X}m$ is a n.c. connection on \mathbf{A}^N .

Particular case: \mathbf{A} involutive. Hermitean structure: $\langle (a^i), (b^j) \rangle = \sum_{i=1}^N (a^i)^*b^i$.

$U_N(\mathbf{A}) = \{u \in M_N(\mathbf{A}) / u^*u = uu^* = \mathbb{1}_N\}$, the group of unitary elements of $M_N(\mathbf{A})$, is the unitary gauge group.

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Noncommutative connections: projective finitely generated modules

M is a direct summand in \mathbf{A}^N . There exists a projection $p \in M_N(\mathbf{A})$ such that $M = p\mathbf{A}^N$.

Proposition 23 (Noncommutative connections on p.f.g.m.)

If $\widehat{\nabla}$ is a n.c. connection on \mathbf{A}^N , then $m \mapsto p\widehat{\nabla}_x m$ defines a n.c. connection on M , where $m \in M \subset \mathbf{A}^N$. The curvature of the n.c. connection obtained this way from $\widehat{\nabla}_x^0$ is the multiplication on the left on $M \subset \mathbf{A}^N$ by the matrix of n.c. 2-forms pdp .

Example 24 (The algebra $\mathbf{A} = C^\infty(M)$)

We saw that the n.c. derivation-based differential calculus is the ordinary de Rham calculus.

By the theorem by Serre and Swan, projective finitely generated modules over $C^\infty(M)$ are sections of vector bundles.

The definitions of connections and n.c. connections coincide.

2.3 Two important examples

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$\mathbf{A} = M_n(\mathbb{C})$: derivations, differential calculus

Consider the case $\mathbf{A} = M_n(\mathbb{C}) = M_n$, the finite dimensional algebra of $n \times n$ complex matrices. This is an involutive algebra for the adjointness of matrices.

Proposition 25 (General properties of the differential calculus)

One has the following results:

- $\mathcal{Z}(M_n) = \mathbb{C}$.
- $\text{Der}(M_n) = \text{Int}(M_n) \simeq \mathfrak{sl}_n = \mathfrak{sl}(n, \mathbb{C})$ (traceless matrices) by the isomorphism which associates to any $\gamma \in \mathfrak{sl}_n(\mathbb{C})$ the derivation $\text{ad}_\gamma : a \mapsto [\gamma, a]$.
- $\text{Der}_{\mathbb{R}}(M_n) = \mathfrak{su}(n)$. $\text{Out}(M_n) = 0$.
- $\Omega_{\text{Der}}^\bullet(M_n) = \Omega_{\text{Der}}^\bullet(M_n) \simeq M_n \otimes \bigwedge^\bullet \mathfrak{sl}_n^*$, with the differential d' coming from the Lie algebra differential complex for \mathfrak{sl}_n represented on M_n by the commutator.

- There exists a canonical 1-form $i\theta \in \Omega_{\text{Der}}^1(M_n)$ such that for any $\gamma \in M_n(\mathbb{C})$

$$i\theta(\text{ad}_\gamma) = \gamma - \frac{1}{n} \text{Tr}(\gamma)\mathbb{1}$$

$i\theta$ makes the explicit isomorphism $\text{Int}(M_n(\mathbb{C})) \xrightarrow{\cong} \mathfrak{sl}_n$.

- $i\theta$ satisfies the relation $d'(i\theta) - (i\theta)^2 = 0$. This makes $i\theta$ look very much like the Maurer-Cartan form in the geometry of Lie groups (here $SL_n(\mathbb{C})$).
- For any $a \in M_n$, one has $d'a = [i\theta, a] \in \Omega_{\text{Der}}^1(M_n)$. False in higher degrees.

.....

$\mathbf{A} = M_n(\mathbb{C})$: explicit decompositions, cohomology

- $\{E_k\}_{k=1, \dots, n^2-1}$ basis for \mathfrak{sl}_n of hermitean matrices.
 - basis of $\text{Der}(M_n) \simeq \mathfrak{sl}_n$: $n^2 - 1$ derivations $\partial_k = \text{ad}_{iE_k}$ (real derivations).
 - The E_k 's and $\mathbb{1}$ form a basis of M_n .
- θ^ℓ 's defined by duality: $\theta^\ell(\partial_k) = \delta_k^\ell$.
 - $\{\theta^\ell\}$ basis of 1-forms in $\bigwedge^\bullet \mathfrak{sl}_n^*$, they anticommute: $\theta^\ell \theta^k = -\theta^k \theta^\ell$.

- Let $[E_k, E_\ell] = C_{k\ell}^m E_m$, then:

$$d'\mathbb{1} = 0 \qquad d'E_k = -C_{k\ell}^m E_m \theta^\ell \qquad d'\theta^k = -\frac{1}{2} C_{\ell m}^k \theta^\ell \theta^m$$

- $i\theta = iE_k \theta^k \in M_n \otimes \bigwedge^1 \mathfrak{sl}_n^*$.

Proposition 26 (The cohomology of the differential calculus)

The cohomology of the differential algebra $(\Omega_{\text{Der}}^\bullet(M_n), d')$ is

$$H^\bullet(\Omega_{\text{Der}}^\bullet(M_n), d') = \mathcal{I}(\wedge^\bullet \mathfrak{sl}_n^*)$$

the invariant elements for the natural Lie derivative.

The algebra $\mathcal{I}(\wedge^\bullet \mathfrak{sl}_n^*)$ is the graded commutative algebra generated by elements c_{2r-1}^n in degree $2r - 1$ for $r \in \{2, 3, \dots, n\}$.

.....
A = M_n(C): metric, integration

Let us introduce $g_{k\ell} = \frac{1}{n} \text{Tr}(E_k E_\ell)$.

Then the $g_{k\ell}$'s define a natural metric (scalar product) on $\text{Der}(M_n)$ if one lets $g(\partial_k, \partial_\ell) = g_{k\ell}$.

Every differential form of maximal degree $\omega \in \Omega_{\text{Der}}^{n^2-1}(M_n)$ can be written uniquely in the form

$$\omega = a\sqrt{|g|}\theta^1 \dots \theta^{n^2-1}$$

where $a \in M_n$ and where $|g|$ is the determinant of the matrix $(g_{k\ell})$.

Definition 27 (Noncommutative integration)

One defines a noncommutative integration

$$\int_{\text{n.c.}} : \Omega_{\text{Der}}^\bullet(M_n) \rightarrow \mathbb{C}$$

by $\int_{\text{n.c.}} \omega = \frac{1}{n} \text{Tr}(a)$ if $\omega \in \Omega_{\text{Der}}^{n^2-1}(M_n)$ written as above, and 0 otherwise.

This integration satisfies the closure relation

$$\int_{\text{n.c.}} d'\omega = 0$$

.....
A = M_n(C): canonical n.c. connection

Let us consider the right **A**-module $\mathbf{M} = \mathbf{A}$.

The n.c. 1-form $-i\theta$ defines a canonical n.c. connection by the relation $\widehat{\nabla}_{\mathfrak{X}}^{-i\theta} a = \mathfrak{X}a - i\theta(\mathfrak{X})a$ for any $a \in \mathbf{A}$.

Proposition 28 (Properties of $\widehat{\nabla}^{-i\theta}$)

For any $a \in M_n$ and $\mathfrak{X} = \text{ad}_\gamma \in \text{Der}(M_n)$ (with $\text{Tr } \gamma = 0$), one has

$$\widehat{\nabla}_{\mathfrak{X}}^{-i\theta} a = -ai\theta(\mathfrak{X}) = -a\gamma$$

$\widehat{\nabla}^{-i\theta}$ is gauge invariant.

The curvature of $\widehat{\nabla}^{-i\theta}$ is zero.

Proof. This is a consequence of the existence of the canonical gauge invariant n.c. connection implied by the relation $d'a = [i\theta, a]$.

The curvature is the n.c. 2-form $\Omega(\mathfrak{X}, \mathfrak{Y}) = d'i\theta(\mathfrak{X}, \mathfrak{Y}) + [i\theta(\mathfrak{X}), i\theta(\mathfrak{Y})] = d'i\theta(\mathfrak{X}, \mathfrak{Y}) + (i\theta)^2(\mathfrak{X}, \mathfrak{Y}) = 0$. □

.....
A = M_n(C): flat connections

Let us consider the right **A**-module $\mathbf{M} = M_{r,n}$, the vector space of $r \times n$ complex matrices with the obvious right module structure and the Hermitean structure $\langle m_1, m_2 \rangle = m_1^* m_2 \in M_n$.

Proposition 29 ($\widehat{\nabla}^{-i\theta}$, flat n.c. connections)

The n.c. connection $\widehat{\nabla}_{\mathfrak{X}}^{-i\theta} m = -mi\theta(\mathfrak{X})$ is well defined, it is compatible with the Hermitean structure and its curvature is zero.

Any n.c. connection can be written $\widehat{\nabla}_{\mathfrak{X}} a = \widehat{\nabla}_{\mathfrak{X}}^{-i\theta} a + A(\mathfrak{X})a$ for $A = A_k \theta^k$ with $A_k \in M_r$. The curvature of $\widehat{\nabla}$ is the multiplication on the left by the M_r -valued n.c. 2-form

$$F = \frac{1}{2}([A_k, A_\ell] - C_{k\ell}^m A_m)\theta^k \theta^\ell$$

This curvature vanishes if and only if $A : \mathfrak{sl}_n \rightarrow M_r$ is a representation of the Lie algebra \mathfrak{sl}_n .

Two flat connections are in the same gauge orbit if and only if the corresponding Lie algebra representations are equivalent.

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A = C[∞](M) ⊗ M_n(C): generalities

Consider the mixed of the two algebras C[∞](M) and M_n(C) studied before, in the form of matrix valued functions on M (dim M = m).

Proposition 30 (General properties of the differential calculus)

One has the following results:

- Z(A) = C[∞](M).
- Der(A) = [Der(C[∞](M)) ⊗ 1] ⊕ [C[∞](M) ⊗ Der(M_n)] = Γ(M) ⊕ [C[∞](M) ⊗ sl_n] as Lie algebras and C[∞](M)-modules. Notations: $\mathfrak{X} = X + \text{ad}_\gamma$ with $X \in \Gamma(M)$ and $\gamma \in C^\infty(M) \otimes \mathfrak{sl}_n = \mathbf{A}_0$ (traceless elements in A).
 ➔ Int(A) = A₀ and Out(A) = Γ(M).
- Ω[•]_{Der}(A) = Ω[•]_{Der}(A) = Ω[•](M) ⊗ Ω[•]_{Der}(M_n) with the differential $\widehat{d} = d + d'$.
- The n.c. 1-form $i\theta$ is defined as $i\theta(X + \text{ad}_\gamma) = \gamma$. It splits the s.e.s.

$$0 \longrightarrow \mathbf{A}_0 \xrightarrow{i\theta} \text{Der}(\mathbf{A}) \longrightarrow \Gamma(M) \longrightarrow 0$$

- N.C. integration is a well-defined map of differential complexes

$$\int_{\text{n.c.}} : \Omega_{\text{Der}}^{\bullet}(\mathbf{A}) \rightarrow \Omega^{\bullet-(n^2-1)}(M) \qquad \int_{\text{n.c.}} \widehat{d}\omega = d \int_{\text{n.c.}} \omega$$

.....

A = C[∞](M) ⊗ M_n(C): other structures

Using a metric h on M and the metric $g_{kl} = \frac{1}{n} \text{Tr}(E_k E_l)$ on the matrix part, one can define a metric on Der(A):

$$\widehat{g}(X + \text{ad}_\gamma, Y + \text{ad}_\eta) = h(X, Y) + \frac{1}{m^2} g(\gamma, \eta)$$

where m is a positive constant which measures the relative “weight” of the two “spaces”. In physical natural units, it has the dimension of a mass.

Consider the right A-module $\mathbf{M} = \mathbf{A}$. As for the algebra M_n, the n.c. 1-form $-i\theta$ defines a canonical n.c. connection by the relation $\widehat{\nabla}_{\mathfrak{X}}^{-i\theta} a = \mathfrak{X}a - i\theta(\mathfrak{X})a$ for any $a \in \mathbf{A}$.

Proposition 31 (Properties of $\widehat{\nabla}^{-i\theta}$)

For any $a \in \mathbf{A}$ and $\mathfrak{X} = X + \text{ad}_\gamma \in \text{Der}(\mathbf{A})$, one has $\widehat{\nabla}_{\mathfrak{X}}^{-i\theta} a = X \cdot a - a\gamma$.

The curvature of $\widehat{\nabla}^{-i\theta}$ is zero.

The gauge transformed connection $\widehat{\nabla}^{-i\theta g}$ by $g \in C^\infty(M) \otimes GL_n(\mathbb{C})$ is associated to the n.c. 1-form $\mathfrak{X} \mapsto -i\theta(\mathfrak{X}) + g^{-1}(X \cdot g) = -\gamma + g^{-1}(X \cdot g)$.

3 The endomorphism algebra of a vector bundle

3.1 The algebra and its derivations

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The endomorphism algebra

E a SU(n)-vector bundle over M (fiber Cⁿ). End(E) fiber bundle of endomorphisms of E, A the algebra of sections of End(E).

For later references, the trivial case is the situation where $E = M \times \mathbb{C}^n$ is the trivial bundle ➔ $\mathbf{A} = C^\infty(M) \otimes M_n$. But in general, A is (globally) more complicated.

Remark 32 (Relation to ordinary geometry)

The endomorphism fiber bundle End(E) is associated to a SU(n) principal fibre bundle P for the couple (M_n, Ad).

Because $G = SU(n) \subset M_n(\mathbb{C})$ and $\mathfrak{g} = \mathfrak{su}(n) \subset M_n(\mathbb{C})$, one has $P \times_\alpha G \subset \text{End}(E)$ and $\text{Ad}P \subset \text{End}(E)$.

The gauge group G and its Lie algebra LieG are subspaces of A.

Locally, using trivialisations of E , the algebra \mathbf{A} looks like $C^\infty(U) \otimes M_n$.

Proposition 33 (Basic properties)

$$\mathcal{Z}(\mathbf{A}) = C^\infty(M).$$

Involution, trace map and determinant ($\text{Tr}, \det : \mathbf{A} \rightarrow C^\infty(M)$), *are well defined fiberwise. Define* $SU(\mathbf{A})$ *as the unitaries in* \mathbf{A} *of determinant 1, and* $\mathfrak{su}(\mathbf{A})$ *as the traceless antihermitean elements. Then* $\mathcal{G} = SU(\mathbf{A})$ *and* $\text{Lie}\mathcal{G} = \mathfrak{su}(\mathbf{A})$.

.....

Derivations

Let $\rho : \text{Der}(\mathbf{A}) \rightarrow \text{Der}(\mathbf{A})/\text{Int}(\mathbf{A}) = \text{Out}(\mathbf{A})$ be the projection.

Proposition 34 (The derivations of \mathbf{A})

ρ is the restriction of derivations $\mathfrak{X} \in \text{Der}(\mathbf{A})$ to $\mathcal{Z}(\mathbf{A})$.

$$\text{Out}(\mathbf{A}) \simeq \text{Der}(C^\infty(M)) = \Gamma(M).$$

$\text{Int}(\mathbf{A})$ is isomorphic to \mathbf{A}_0 , the traceless elements in \mathbf{A} .

The s.e.s. of Lie algebras and $C^\infty(M)$ -modules of derivations looks like

$$0 \longrightarrow \text{Int}(\mathbf{A}) \longrightarrow \text{Der}(\mathbf{A}) \xrightarrow{\rho} \Gamma(M) \longrightarrow 0$$

$$\mathfrak{X} \longmapsto X$$

- This generalizes the decomposition obtained in the trivial case.
- There is no a priori canonical splitting in the non trivial case.
- The “n.c. 1-form” $i\theta$ cannot be defined here. But one can define a map of $C^\infty(M)$ -modules:

$$i\theta : \text{Int}(\mathbf{A}) \rightarrow \mathbf{A}_0 \qquad \text{ad}_\gamma \mapsto \gamma - \frac{1}{n} \text{Tr}(\gamma)\mathbb{1}$$

Proposition 35

$$\underline{\Omega}_{\text{Der}}^\bullet(\mathbf{A}) = \Omega_{\text{Der}}^\bullet(\mathbf{A})$$

3.2 Ordinary connections

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Connections

Let ∇^E be any (usual) connection on E .

Use notations: $X \in \Gamma(M)$, $\alpha \in \Gamma(E^*)$ and $s \in \Gamma(E)$.

Define the connections ∇^{E^*} on E^* and then ∇ on $\text{End}(E)$ by the relations

$$X \cdot \langle \alpha, s \rangle = \langle \nabla_X^{E^*} \alpha, s \rangle + \langle \alpha, \nabla_X^E s \rangle \qquad \nabla_X(\alpha \otimes s) = (\nabla_X^{E^*} \alpha) \otimes s + \alpha \otimes (\nabla_X^E s)$$

Use the notation $X = \rho(\mathfrak{X}) \in \Gamma(M)$ for any $\mathfrak{X} \in \text{Der}(\mathbf{A})$.

Proposition 36

For any $X \in \Gamma(M)$, ∇_X is a derivation of \mathbf{A} . For any $\mathfrak{X} \in \text{Der}(\mathbf{A})$, $\mathfrak{X} - \nabla_X \in \text{Int}(\mathbf{A})$. $\mathfrak{X} \mapsto \alpha(\mathfrak{X}) = -i\theta(\mathfrak{X} - \nabla_X)$ defines a n.c. 1-form $\alpha \in \Omega_{\text{Der}}^1(\mathbf{A})$. One has $\mathfrak{X} = \nabla_X - \text{ad}_{\alpha(\mathfrak{X})}$.

$\forall \gamma \in \mathbf{A}_0$, $\alpha(\text{ad}_\gamma) = -\gamma$, $\forall \mathfrak{X} \in \text{Der}(\mathbf{A})$, $\text{Tr} \alpha(\mathfrak{X}) = 0$ and $\forall \mathfrak{X} \in \text{Der}_{\mathbb{R}}(\mathbf{A})$, $\alpha(\mathfrak{X})^* + \alpha(\mathfrak{X}) = 0$.

- $X \mapsto \nabla_X$ is a splitting as $C^\infty(M)$ -modules of the s.e.s.

$$0 \longrightarrow \mathbf{A}_0 \longrightarrow \text{Der}(\mathbf{A}) \xrightarrow{\nabla} \Gamma(M) \longrightarrow 0$$

The obstruction to be a splitting of Lie algebras is the curvature of ∇ : $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$.

- The relation $\alpha(\text{ad}_\gamma) = -\gamma$ shows that α extends $-i\theta : \text{Int}(\mathbf{A}) \rightarrow \mathbf{A}_0$.

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Connections (continued)

Proposition 37 (Ordinary connections and n.c. forms)

The map $\nabla^E \mapsto \alpha$ is an isomorphism between the affine spaces

$$\begin{aligned} &SU(n)\text{-connections on } E \\ &\text{Traceless antihermitean n.c. 1-forms on } \mathbf{A} \text{ s.t. } \alpha(\text{ad}_\gamma) = -\gamma \end{aligned}$$

The n.c. 2-form $(\mathfrak{X}, \mathfrak{Y}) \mapsto \Omega(\mathfrak{X}, \mathfrak{Y}) = \widehat{d}\alpha(\mathfrak{X}, \mathfrak{Y}) + [\alpha(\mathfrak{X}), \alpha(\mathfrak{Y})]$ depends only on the projections X and Y of \mathfrak{X} and \mathfrak{Y} . This means that it is a horizontal 2-form for the operation of $\text{Int}(\mathbf{A})$ on $\Omega_{\text{Der}}^\bullet(\mathbf{A})$.

$\Gamma(\wedge^\bullet T^*M \otimes \text{End}(E))$ is the graded algebra of horizontal forms in $\Omega_{\text{Der}}^\bullet(\mathbf{A})$ for this operation.

The curvature R^E of ∇^E , considered in the intermediate construction as a section of $\wedge^2 T^*M \otimes \text{Ad}P \subset \wedge^2 T^*M \otimes \text{End}(E)$, is exactly Ω .

Remark 38 (Intermediate construction)

This proposition shows that the intermediate construction can be completed using noncommutative geometry in order to take into account the connection 1-form, which is now a n.c. 1-form. The vertical and equivariant conditions at the level of P are replaced by a unique condition on inner derivations at the level of \mathbf{A} .

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Connections (continued)

Let $u \in \mathcal{G} = SU(\mathbf{A})$ and $\xi \in \text{Lie}\mathcal{G} = \mathfrak{su}(\mathbf{A})$.

The n.c. 1-form α^u corresponding to the gauge transformed connection ∇^{Eu} is

$$\alpha^u = u^* \alpha u + u^* \widehat{d}u$$

Proposition 39 (Infinitesimal gauge transformations)

The infinitesimal gauge transformation induced by ξ is

$$\alpha \mapsto -\widehat{d}\xi - [\alpha, \xi] = L_{\text{ad}_\xi} \alpha$$

This means that

Infinitesimal gauge transformations on connections on $E = \text{Lie derivative of real inner derivations on } \mathbf{A}$.

3.3 Derivations of \mathbf{A} in local trivialisations

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Derivations of \mathbf{A} in local trivialisations

Let $U_i \subset M$ be a local trivialisaton of E , and so of $\text{End}(E)$. We denote by $a_i^{\text{loc}} : U_i \rightarrow M_n$ the restriction of the global section $a \in \mathbf{A}$.

Over $U_i \cap U_j \neq \emptyset$, one has $a_j^{\text{loc}} = \text{Ad}_{g_{ij}^{-1}} a_i^{\text{loc}} = g_{ij}^{-1} a_i^{\text{loc}} g_{ij}$.

Locally a derivation $\mathfrak{X} \in \text{Der}(\mathbf{A})$ can be written $\mathfrak{X}_i^{\text{loc}} = X_i + \text{ad}_{\gamma_i}$ with $\gamma_i : U_i \rightarrow M_n$ (traceless) and X_i a vector fields on U . Using the map ρ , one gets that X_i is the restriction of $X = \rho(\mathfrak{X})$ to $U_i \rightarrow$ notation $X = X_i$.

Using compatibility with the gluing relations for sections, one finds

$$\gamma_j = g_{ij}^{-1} \gamma_i g_{ij} + g_{ij}^{-1} X \cdot g_{ij}$$

The n.c. 1-form α is locally given by the expressions

$$\alpha_i^{\text{loc}}(X + \text{ad}_{\gamma_i}) = A_i(X) - \gamma_i$$

where the A_i 's form the family of local trivialisaton of the connection 1-form.

$$\begin{aligned} \alpha_j^{\text{loc}}(X + \text{ad}_{\gamma_j}) &= A_j(X) - \gamma_j \\ &= (g_{ij}^{-1} A_i(X) g_{ij} + g_{ij}^{-1} X \cdot g_{ij}) - (g_{ij}^{-1} \gamma_i g_{ij} + g_{ij}^{-1} X \cdot g_{ij}) \\ &= g_{ij}^{-1} (A_i(X) - \gamma_i) g_{ij} = g_{ij}^{-1} \alpha_i^{\text{loc}}(X + \text{ad}_{\gamma_i}) g_{ij} \end{aligned}$$

So that these expressions indeed define a global section in \mathbf{A} .

4 Noncommutative connections on \mathbf{A}

4.1 Main properties

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Some general facts

The right \mathbf{A} -module is taken to be $M = \mathbf{A}$ with Hermitean structure $(a, b) \mapsto a^*b$.

A n.c. connection $\widehat{\nabla}$ is then given by a n.c. 1-form $\omega \in \Omega_{\text{Der}}^1(\mathbf{A})$ by the relation $\widehat{\nabla}_{\mathfrak{X}}a = \mathfrak{X}a + \omega(\mathfrak{X})a \rightarrow$ studying $\widehat{\nabla}$ is studying ω .

Proposition 40 (The n.c. connection associated to α)

Let ∇^E be a connection on E , α its associated n.c. 1-form.

- The n.c. connection $\widehat{\nabla}^\alpha$ defined by the n.c. 1-form α is given by

$$\widehat{\nabla}_{\mathfrak{X}}^\alpha a = \nabla_X a + a\alpha(\mathfrak{X})$$

- $\widehat{\nabla}^\alpha$ is compatible with the Hermitean structure.
- The curvature of $\widehat{\nabla}^\alpha$ is $\widehat{R}^\alpha(\mathfrak{X}, \mathfrak{Y}) = R^E(X, Y)$
- A gauge transformation induced by $u \in \mathcal{G} = SU(\mathbf{A})$ on the connection ∇^E induces a (n.c.) gauge transformation on $\widehat{\nabla}^\alpha$.
- Recall that $\mathfrak{X} = \nabla_X - \text{ad}_{\alpha(\mathfrak{X})}$ and $\widehat{\nabla}_{\mathfrak{X}}^\alpha a = \mathfrak{X}a + \alpha(\mathfrak{X})a$.
- Curvature of $\widehat{\nabla}^\alpha = \widehat{d}\alpha(\mathfrak{X}, \mathfrak{Y}) + [\alpha(\mathfrak{X}), \alpha(\mathfrak{Y})] =$ curvature of ∇ .
- In a gauge transformation, one has $\alpha^u = u^*\alpha u + u^*\widehat{d}u$, which is also the n.c. gauge transformation applied to $\widehat{\nabla}^\alpha$.

.....

The space of n.c. connections

We arrive at the main point of this lecture:

Theorem 41 (Ordinary connections as n.c. connections)

The space of n.c. connections on the right \mathbf{A} -module \mathbf{A} compatible with the Hermitean structure $(a, b) \mapsto a^*b$ contains the space of ordinary connections on E .

This inclusion is compatible with the corresponding definitions of curvature and gauge transformations.

- From now on one can consider that an ordinary connection is a n.c. connection.
- This point of view generalizes the notion of connection through the intermediate construction.
- A natural question is: what are n.c. connections from a physical point of view?

4.2 Decomposition of noncommutative connections

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Decomposition of n.c. connections

Let us fix a connection ∇^E on E , and denote by α its associated n.c. 1-form.

Any n.c. connection $\widehat{\nabla}$ can be decomposed as

$$\widehat{\nabla}_{\mathfrak{X}}a = \widehat{\nabla}_{\mathfrak{X}}^\alpha a + \mathcal{A}(\mathfrak{X})a$$

with $\mathcal{A} \in \Omega_{\text{Der}}^1(\mathbf{A})$, so that $\omega = \alpha + \mathcal{A}$ is the n.c. 1-form for $\widehat{\nabla}$.

Using the relation $\mathfrak{X} = \nabla_X - \text{ad}_{\alpha(\mathfrak{X})}$, one decomposes \mathcal{A} as $\mathcal{A}(\mathfrak{X}) = \mathfrak{a}(X) - \mathfrak{b}(\alpha(\mathfrak{X}))$, where $\mathfrak{b} : \mathbf{A}_0 \rightarrow \mathbf{A}$ is defined by $\mathfrak{b}(\gamma) = \mathcal{A}(\text{ad}_\gamma)$.

The curvature of $\widehat{\nabla}$ can be written

$$\begin{aligned} \widehat{R}(\mathfrak{X}, \mathfrak{Y}) &= R^E(X, Y) + \nabla_X \mathcal{A}(\mathfrak{Y}) - \nabla_Y \mathcal{A}(\mathfrak{X}) - \mathcal{A}([\mathfrak{X}, \mathfrak{Y}]) + [\mathcal{A}(\mathfrak{X}), \mathcal{A}(\mathfrak{Y})] \\ &= R^{E, \alpha}(X, Y) - \nabla_{\mathfrak{X}}^\alpha \mathfrak{b}(\alpha(\mathfrak{Y})) + \nabla_{\mathfrak{Y}}^\alpha \mathfrak{b}(\alpha(\mathfrak{X})) \\ &\quad + [\mathfrak{b}(\alpha(\mathfrak{X})), \mathfrak{b}(\alpha(\mathfrak{Y}))] + \mathfrak{b}(\alpha([\mathfrak{X}, \mathfrak{Y}])) \end{aligned}$$

where $R^{E,\mathbf{a}}$ is the curvature of the connection $\nabla_X^{E,\mathbf{a}}s = \nabla_X^E s + \mathbf{a}(X)s$ on E and $\nabla^{\mathbf{a}}$ is its associated connection on $\text{End}(E)$.

Performing a gauge transformation with $u \in \mathcal{G} = SU(\mathbf{A})$, one has

$$\mathcal{A}^u = u^* \mathcal{A}u + u^*(\nabla u) \qquad \mathbf{a}^u = u^* \mathbf{a}u + u^*(\nabla u) \qquad \mathbf{b}^u = u^* \mathbf{b}u$$

Notice the replacement of the differential by ∇ in many of these expressions.

•••••
The trivial case

Consider the situation $E = M \times \mathbb{C}^n$ and $\mathbf{A} = C^\infty(M) \otimes M_n$.

As a reference (ordinary) connection, take $\nabla_X^E s = X \cdot s$, so that, using the local expression of α , one has

$$\alpha(\mathfrak{X}) = \alpha(X + \text{ad}_\gamma) = -\gamma = -i\theta(\mathfrak{X})$$

with $\text{Tr } \gamma = 0$. Then $\widehat{\nabla}^\alpha = \widehat{\nabla}^{-i\theta}$. Moreover, $\mathbf{b}(\alpha(\mathfrak{X})) = \mathbf{b}(-\gamma) = -\mathbf{b}(\gamma)$, so that

$$\widehat{\nabla}_{\mathfrak{X}} a = X \cdot a + \mathbf{a}(X)a + \mathbf{b}(\gamma)a = \overline{\nabla}_X^{\mathbf{a}} a + \mathbf{b}(\gamma)a$$

where $\overline{\nabla}^{\mathbf{a}}$ is an ordinary connection on $\text{End}(E)$, but is not $\nabla^{\mathbf{a}}$.

$X \mapsto \mathbf{a}(X)$ behave like a gauge potential with respect to gauge transformations (here $\nabla = d$). The difference between ordinary connections and n.c. connections is the presence of \mathbf{b} , which represents some additional fields in physics. It has homogeneous gauge transformations.

The curvature can be written, for $\mathfrak{X} = X + \text{ad}_\gamma$ and $\mathfrak{Y} = Y + \text{ad}_\eta$,

$$\widehat{R}(\mathfrak{X}, \mathfrak{Y}) = R^{E,\mathbf{a}}(X, Y) + (\widetilde{\nabla}_X^{\mathbf{a}} \mathbf{b})(\eta) - (\widetilde{\nabla}_Y^{\mathbf{a}} \mathbf{b})(\gamma) + [\mathbf{b}(\gamma), \mathbf{b}(\eta)] - \mathbf{b}([\gamma, \eta])$$

where $\widetilde{\nabla}^{\mathbf{a}}$ is the connection $(\widetilde{\nabla}_X^{\mathbf{a}} \mathbf{b})(\eta) = X \cdot \mathbf{b}(\eta) - \mathbf{b}(X \cdot \eta) + [\mathbf{a}(X), \mathbf{b}(\eta)]$ on the space of $C^\infty(M)$ -linear maps $\mathbf{A}_0 \rightarrow \mathbf{A}$.

4.3 Yang-Mills-Higgs Lagrangian

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Yang-Mills-Higgs Lagrangian

Consider the trivial case $\mathbf{A} = C^\infty(M) \otimes M_n$ and the right \mathbf{A} -module \mathbf{A} .

Let $\mathbf{a} = \mathbf{a}_\mu dx^\mu$ and $\mathbf{b} = \mathbf{b}_k \theta^k$, with $\mathbf{a}_\mu, \mathbf{b}_k \in C^\infty(M) \otimes M_n$.

The curvature is then the n.c. 2-form

$$\widehat{R} = \frac{1}{2}(\partial_\mu \mathbf{a}_\nu - \partial_\nu \mathbf{a}_\mu + [\mathbf{a}_\mu, \mathbf{a}_\nu])dx^\mu dx^\nu + (\partial_\mu \mathbf{b}_k + [\mathbf{a}_\mu, \mathbf{b}_k])dx^\mu \theta^k + \frac{1}{2}([\mathbf{b}_k, \mathbf{b}_\ell] - C_{k\ell}^m \mathbf{b}_m)\theta^k \theta^\ell$$

Using a metric (here euclidean) on $\text{Der}(\mathbf{A})$ and an associated Hodge star operation, one can define a Lagrangian. Using ordinary and n.c. integration, one then defines the action:

$$S(\widehat{R}) = \int dx \text{Tr} \left\{ \sum_{\mu,\nu} \frac{1}{4}(\partial_\mu \mathbf{a}_\nu - \partial_\nu \mathbf{a}_\mu + [\mathbf{a}_\mu, \mathbf{a}_\nu])^2 + m^2 \sum_{\mu,k} (\partial_\mu \mathbf{b}_k + [\mathbf{a}_\mu, \mathbf{b}_k])^2 + m^4 \sum_{k,\ell} \frac{1}{4}([\mathbf{b}_k, \mathbf{b}_\ell] - C_{k\ell}^m \mathbf{b}_m)^2 \right\}$$

The integrand is zero when

$$\mathbf{a} \text{ gauge equivalent to } 0 \qquad d\mathbf{b} = 0 \qquad [\mathbf{b}_k, \mathbf{b}_\ell] = C_{k\ell}^m \mathbf{b}_m$$

so that the \mathbf{b}_k 's are constant and induce a representation of \mathfrak{sl}_n in M_n .

For the right \mathbf{A} -module $\mathbf{M} = C^\infty(M) \otimes M_{r,n}$, one would get similar results: flat connections are classified by inequivalent representations of \mathfrak{sl}_n in M_r .

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Yang-Mills-Higgs Lagrangian

From a fields theory point of view:

- The \mathbf{a}_μ fields behave like ordinary Yang-Mills fields, for a $SU(n)$ gauge theory.
- The \mathbf{b}_k fields behave as Higgs fields: the vacuum states can be non trivial and the Higgs mechanism of mass generation is possible.

- The coupling between these fields is a covariant derivative in the adjoint representation.

For a more general situation where \mathbf{A} is not the trivial case, one can proceed in the same line:

- One has to use a reference connection on E to help to decompose n.c. connections.
- The curvature looks similar except for the presence of the reference connection.
- The Hodge star operator is defined.
- The action splits into three terms, and the vacuum states are related to the global structure of the vector fiber bundle E .

5 Relations with the principal fiber bundle

5.1 A bigger algebra

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The algebra $\mathbf{B} = C^\infty(P) \otimes M_n$

Let P the $SU(n)$ -principal fiber bundle for which E is associated.

Consider the associative algebra $\mathbf{B} = C^\infty(P) \otimes M_n$. Then

- $\mathcal{Z}(\mathbf{B}) = C^\infty(P)$, $\text{Der}(\mathbf{B}) = \Gamma(P) \oplus [C^\infty(P) \otimes \mathfrak{su}(n)]$ and $\Omega_{\text{Der}}^\bullet(\mathbf{B}) = \Omega^\bullet(P) \otimes \Omega_{\text{Der}}^\bullet(M_n)$ with the differential $\widehat{d} = d + d'$.
- $\mathfrak{su}(n)$ is a real Lie subalgebra of $\text{Der}(\mathbf{B})$ for two inclusions:
 $\xi \mapsto \xi^v$ vertical vector field on P $\xi \mapsto \text{ad}_\xi$ inner derivation
- $\mathfrak{g}_{\text{ad}} = \{\text{ad}_\xi / \xi \in \mathfrak{su}(n)\}$ and $\mathfrak{g}_{\text{equ}} = \{\xi^v + \text{ad}_\xi / \xi \in \mathfrak{su}(n)\}$ are Lie subalgebras of $\text{Der}(\mathbf{B})$.

Proposition 42

The algebra $C^\infty(P)$ (resp. \mathbf{A}) is the invariants of \mathfrak{g}_{ad} (resp. $\mathfrak{g}_{\text{equ}}$) in \mathbf{B} .

Proof. $C^\infty(P)$ is the invariants of \mathfrak{g}_{ad} because $\text{ad}_\xi b = 0$ for any $\xi \in \mathfrak{su}(n)$ implies $b \in \mathcal{Z}(\mathbf{B})$. \mathbf{A} is the invariants of $\mathfrak{g}_{\text{equ}}$ because \mathbf{A} is the set of sections of $\text{End}(E)$, which is $\mathcal{F}_{SU(n)}(P, M_n)$. The relation $\xi^v \cdot b + \text{ad}_\xi b = 0$ for any $\xi \in \mathfrak{su}(n)$ is the infinitesimal version of the equivariance. □

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Cartan operations

The Lie subalgebras \mathfrak{g}_{ad} and $\mathfrak{g}_{\text{equ}}$ define Cartan operations on $(\Omega_{\text{Der}}^\bullet(\mathbf{B}), \widehat{d})$.

The previous proposition tells us that the algebras \mathbf{B} , $C^\infty(P)$ and \mathbf{A} are related by these operations.

Moreover, $C^\infty(M)$ is itself the invariant elements for $\xi \mapsto \xi^v$ in $C^\infty(P)$ and the invariants in \mathbf{A} for the operation of $\text{Int}(\mathbf{A})$.

Proposition 43 (Relations between the differential calculi)

At the level of differential calculi, all these relations generalize in the following structure:

$$\begin{array}{ccc}
 \Omega^\bullet(P) \otimes \Omega_{\text{Der}}^\bullet(M_n) & \xleftarrow[\mathfrak{su}(n) \ni \xi \mapsto \text{ad}_\xi]{\text{basic elements}} & \Omega^\bullet(P) \\
 \uparrow \scriptstyle \text{basic elements} & & \uparrow \scriptstyle \text{basic elements} \\
 \Omega_{\text{Der}}^\bullet(\mathbf{A}) & \xleftarrow[\text{Int}(\mathbf{A})]{\text{basic elements}} & \Omega^\bullet(M)
 \end{array}$$

$\mathfrak{su}(n) \ni \xi \mapsto \xi^v + \text{ad}_\xi$ $\mathfrak{su}(n) \ni \xi \mapsto \xi^v$

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Derivations of A and B

There exist some strong relations between the derivations of **A**, some derivations of **B**, and some vector fields on P and M .

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 & & 0 & \longrightarrow & \mathcal{Z}_{\text{Der}}(\mathbf{A}) & \longrightarrow & \Gamma(VP) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \text{Int}(\mathbf{A}) & \longrightarrow & \mathcal{N}_{\text{Der}}(\mathbf{A}) & \xrightarrow{\rho_P} & \Gamma_M(P) \longrightarrow 0 \\
 & & \downarrow & & \downarrow \tau & & \downarrow \pi_* \\
 0 & \longrightarrow & \text{Int}(\mathbf{A}) & \longrightarrow & \text{Der}(\mathbf{A}) & \xrightarrow{\rho} & \Gamma(M) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

This exact commutative diagram illustrated these relations.

- $0 \longrightarrow \text{Int}(\mathbf{A}) \longrightarrow \text{Der}(\mathbf{A}) \xrightarrow{\rho} \Gamma(M) \longrightarrow 0$

Short exact sequence which relates vector fields on M , derivations on **A** and inner derivations on **A**.

- $0 \longrightarrow \mathcal{Z}_{\text{Der}}(\mathbf{A}) \longrightarrow \mathcal{N}_{\text{Der}}(\mathbf{A}) \xrightarrow{\tau} \text{Der}(\mathbf{A}) \longrightarrow 0$

$\mathcal{N}_{\text{Der}}(\mathbf{A}) \subset \text{Der}(\mathbf{B})$ subset of derivations on **B** which preserve $\mathbf{A} \subset \mathbf{B}$. $\mathcal{Z}_{\text{Der}}(\mathbf{A}) \subset \text{Der}(\mathbf{B})$ subset of derivations on **B** which vanish on **A**. The Lie algebra $\mathcal{Z}_{\text{Der}}(\mathbf{A})$ is generated as a $C^\infty(P)$ -module by the elements $\xi^v + \text{ad}_\xi$ for any $\xi \in \mathfrak{su}(n)$.

- $0 \longrightarrow \Gamma(VP) \longrightarrow \Gamma_M(P) \xrightarrow{\pi_*} \Gamma(M) \longrightarrow 0$

Geometrical objects.

$\Gamma(VP)$ vertical vector fields on P . $\Gamma_M(P) = \{\mathcal{X} \in \Gamma(P) / \pi_* \mathcal{X}(p) = \pi_* \mathcal{X}(p') \ \forall p, p' \in P \text{ s.t. } \pi(p) = \pi(p')\}$ Lie algebra of vector fields on P which can be mapped to vector fields on M using the tangent maps $\pi_* : T_p P \rightarrow T_{\pi(p)} M$.

- $0 \longrightarrow \text{Int}(\mathbf{A}) \longrightarrow \mathcal{N}_{\text{Der}}(\mathbf{A}) \xrightarrow{\rho_P} \Gamma_M(P) \longrightarrow 0$

Elements in $\text{Int}(\mathbf{A})$ considered as ad_γ for $\gamma \in \mathbf{A}_0 \subset \mathbf{B}$. ρ_P restriction to $\mathcal{N}_{\text{Der}}(\mathbf{A})$ of the projection on the first term in $\text{Der}(\mathbf{B}) = \Gamma(P) \oplus [C^\infty(P) \otimes \text{Der}(M_n)]$.

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Integration

The n.c. integration induces a map

$$\int_{\text{n.c.}} : \Omega_{\text{Der}}^r(\mathbf{B}) \rightarrow \Omega^{r-(n^2-1)}(P)$$

Proposition 44 (Integration)

If $\omega \in \Omega_{\text{Der}}^r(\mathbf{B})$ is a horizontal (resp. basic) n.c. form for one of the operation of \mathfrak{g}_{ad} or $\mathfrak{g}_{\text{equ}}$, then $\int_{\text{n.c.}} \omega \in \Omega^{r-(n^2-1)}(P)$ is horizontal (resp. basic) for the corresponding operation restricted to $\Omega^\bullet(P) \subset \Omega_{\text{Der}}^\bullet(\mathbf{B})$.

This n.c. integration then restricts to a “n.c. integration along the n.c. fiber” $\Omega_{\text{Der}}^\bullet(\mathbf{A}) \rightarrow \Omega^{\bullet-(n^2-1)}(M)$.

This n.c. integration is compatible with the differentials, and it induces maps in cohomologies

$$\begin{aligned}
 \int_{\text{n.c.}} & : H^\bullet(\Omega_{\text{Der}}^\bullet(\mathbf{B}), \widehat{d}) \rightarrow H_{\text{dR}}^{\bullet-(n^2-1)}(P) \\
 \int_{\text{n.c.}} & : H^\bullet(\Omega_{\text{Der}}^\bullet(\mathbf{A}), \widehat{d}) \rightarrow H_{\text{dR}}^{\bullet-(n^2-1)}(M)
 \end{aligned}$$

5.2 Ordinary vs. noncommutative connections

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Ordinary and n.c. connections

Let $\widehat{\nabla}$ be a n.c. connection on the right \mathbf{A} -module \mathbf{A} , and denote by $\alpha \in \Omega_{\text{Der}}^1(\mathbf{A})$ its associated n.c. 1-form. As a basic n.c. 1-form in $\Omega_{\text{Der}}^1(\mathbf{B})$ for the operation of $\mathfrak{g}_{\text{equ}}$, one can write

$$\alpha = \omega - \phi \in [\Omega^1(P) \otimes M_n] \oplus [C^\infty(P) \otimes M_n \otimes \mathfrak{sl}_n^*]$$

The basic condition implies the relations

$$(L_{\xi^v} + L_{\text{ad}_\xi})\omega = 0 \qquad (L_{\xi^v} + L_{\text{ad}_\xi})\phi = 0 \qquad i_{\xi^v}\omega - i_{\text{ad}_\xi}\phi = 0$$

for any $\xi \in \mathfrak{su}(n)$.

Proposition 45 (Ordinary connection)

Let ∇^E be an ordinary connection on E and $\alpha \in \Omega_{\text{Der}}^1(\mathbf{A})$ its associated n.c. 1-form. Then, as a basic element in $\Omega_{\text{Der}}^1(\mathbf{B})$, one has

$$\alpha = \omega - i\theta$$

where $\omega \in \Omega^1(P) \otimes \mathfrak{su}(n) \subset \Omega^1(P) \otimes M_n$ is the connection 1-form on P and $i\theta$ the canonical n.c. 1-form defined in $\Omega_{\text{Der}}^1(\mathbf{B})$.

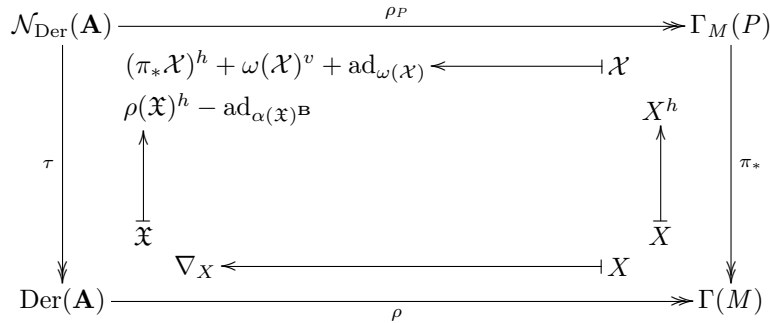
In order to prove this formula, one has to use the equivariance and the vertical condition for ω , and some of the properties listed before on $i\theta$.

5.3 Splittings coming from connections

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Connection as splittings

An ordinary connection $\omega \in \Omega^1(P) \otimes \mathfrak{su}(n)$ splits some short exact sequence in the diagram relating derivations of \mathbf{A} , derivations of \mathbf{B} and some vector fields on P and M .

Look more closely at the central square of this diagram.



- $0 \longrightarrow \Gamma(VP) \longrightarrow \Gamma_M(P) \xrightarrow{\pi_*} \Gamma(M) \longrightarrow 0$
Lift vector fields on M into horizontal vector fields X^h on P .
- $0 \longrightarrow \text{Int}(\mathbf{A}) \longrightarrow \text{Der}(\mathbf{A}) \xrightarrow{\rho} \Gamma(M) \longrightarrow 0$
Lift vector fields on M into derivations on \mathbf{A} .
- $0 \longrightarrow \mathcal{Z}_{\text{Der}}(\mathbf{A}) \longrightarrow \mathcal{N}_{\text{Der}}(\mathbf{A}) \xrightarrow{\tau} \text{Der}(\mathbf{A}) \longrightarrow 0$
Lift derivations on \mathbf{A} into derivations on \mathbf{B} . $\alpha(\mathfrak{X})^{\mathbf{B}}$ is the basic element in \mathbf{B} associated to $\alpha(\mathfrak{X}) \in \mathbf{A}$.
- $0 \longrightarrow \text{Int}(\mathbf{A}) \longrightarrow \mathcal{N}_{\text{Der}}(\mathbf{A}) \xrightarrow{\rho_P} \Gamma_M(P) \longrightarrow 0$
Lift π_* -projectable vector fields on P into derivations on \mathbf{B} .
- Notice that $(\pi_*\mathcal{X})^h + \omega(\mathcal{X})^v + \text{ad}_{\omega(\mathcal{X})} \neq \rho(\mathfrak{X})^h - \text{ad}_{\alpha(\mathfrak{X})^{\mathbf{B}}}$.
Indeed one has $\text{Der}(\mathbf{B}) \ni \mathfrak{X} = X^h + \underbrace{\text{ad}_Z}_{\in \text{Int}(\mathbf{A})} + \underbrace{\omega(\mathcal{X})^v + \text{ad}_{\omega(\mathcal{X})}}_{\in \mathcal{Z}_{\text{Der}}(\mathbf{A})}$.

6 Cohomology and characteristic classes

6.1 The cohomology of $\Omega_{\text{Der}}^\bullet(\mathbf{A})$

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About the cohomology of a fiber bundle

Let us recall the Leray theorem in ordinary differential geometry.

Theorem 46 (Leray)

For any fiber bundle $F \rightarrow E \xrightarrow{\pi} M$, there exists a spectral sequence $\{E_r\}$ converging to the cohomology of the total space $H_{\text{dR}}^\bullet(E)$ with

$$E_2^{p,q} = H^p(\mathfrak{U}; \mathcal{H}^q)$$

where $\mathcal{H}^q(U) = H_{\text{dR}}^q(\pi^{-1}U)$ is a locally constant presheaf on the good covering \mathfrak{U} of M .

If M is simply connected and $H_{\text{dR}}^q(F)$ is finite dimensional, then

$$E_2^{p,q} = H_{\text{dR}}^p(M) \otimes H_{\text{dR}}^q(F)$$

One of the proofs of this theorem relies on the construction of a Čech-de Rham bicomplex:

$$\mathbf{K}^{p,q} = \prod_{\alpha_0 < \dots < \alpha_p} \Omega^q(E_{U_{\alpha_0 \dots \alpha_p}}) = \prod_{\alpha_0 < \dots < \alpha_p} \Omega^q(\pi^{-1}U_{\alpha_0 \dots \alpha_p})$$

where $U_{\alpha_0 \dots \alpha_p} = U_{\alpha_0} \cap \dots \cap U_{\alpha_p}$ for $U_{\alpha_i} \in \mathfrak{U}$, $d : \mathbf{K}^{p,q} \rightarrow \mathbf{K}^{p,q+1}$ is the ordinary de Rham differential on the spaces $\Omega^\bullet(E_{U_{\alpha_0 \dots \alpha_p}})$, and $\delta : \mathbf{K}^{p,q} \rightarrow \mathbf{K}^{p+1,q}$ is the Čech differential

$$(\delta\omega_p)_{\alpha_0 \dots \alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \omega_{\alpha_0 \dots \widehat{\alpha_i} \dots \alpha_{p+1}}|_{U_{\alpha_0 \dots \alpha_{p+1}}}$$

.....

The noncommutative Čech-de Rham bicomplex

One can introduce a noncommutative Čech-de Rham bicomplex for \mathbf{A} .

Denote by $\mathbf{A}(U) \simeq C^\infty(U) \otimes M_n$ the sections of $\text{End}(E)$ restricted over a local trivialisation $U \subset M$. Denote by $g_{UV} : U \cap V \rightarrow SU(n)$ the transition functions for E . Let \mathfrak{U} be a good cover of M .

For any n.c. p -form $\omega = a_0 \widehat{d}a_1 \dots \widehat{d}a_p \in \Omega_{\text{Der}}^p(\mathbf{A}(U))$ and any differential function $g : U \rightarrow SU(n)$, define the action of g by $\omega^g = (g^{-1}a_0 g) \widehat{d}(g^{-1}a_1 g) \dots \widehat{d}(g^{-1}a_p g)$.

Lemma 47 (The presheaf $\Omega_{\text{Der}}^\bullet(\mathbf{A}(U))$)

For any $V \subset U$, the maps $i_V^U : \Omega_{\text{Der}}^\bullet(\mathbf{A}(U)) \rightarrow \Omega_{\text{Der}}^\bullet(\mathbf{A}(V))$ given by $\omega \mapsto (\omega|_V)^{g_{UV}}$ (restriction to V and action of g_{UV}) give to $U \mapsto \Omega_{\text{Der}}^\bullet(\mathbf{A}(U))$ a structure of presheaf on M , which we denote by \mathcal{F} .

Let $\mathbf{C}^{p,q}(\mathfrak{U}; \mathcal{F}) = \prod_{\alpha_0 < \dots < \alpha_p} \Omega_{\text{Der}}^q(\mathbf{A}(U_{\alpha_0 \dots \alpha_p}))$, where by convention the trivialisation over $U_{\alpha_0 \dots \alpha_p}$ is the one over U_{α_p} . Let $\widehat{d} : \mathbf{C}^{p,q} \rightarrow \mathbf{C}^{p,q+1}$ be the n.c. differential, and define $\delta : \mathbf{C}^{p,q}(\mathfrak{U}; \mathcal{F}) \rightarrow \mathbf{C}^{p+1,q}(\mathfrak{U}; \mathcal{F})$ by $(g_{\alpha\beta} = g_{U_\alpha U_\beta})$

$$(\delta\omega)_{\alpha_0 \dots \alpha_{p+1}} = \sum_{i=0}^p (-1)^i (\omega_{\alpha_0 \dots \widehat{\alpha_i} \dots \alpha_{p+1}})|_{U_{\alpha_0 \dots \alpha_{p+1}}} + (-1)^{p+1} (\omega_{\alpha_0 \dots \alpha_p})|_{U_{\alpha_0 \dots \alpha_{p+1}}}^{g_{\alpha_p \alpha_{p+1}}}$$

.....

The cohomology of $\Omega_{\text{Der}}^\bullet(\mathbf{A})$

Denote by $\mathbf{C}^{-1,q}(\mathfrak{U}; \mathcal{F}) = \Omega_{\text{Der}}^q(\mathbf{A})$ and define $\delta : \mathbf{C}^{-1,q}(\mathfrak{U}; \mathcal{F}) \rightarrow \mathbf{C}^{0,q}(\mathfrak{U}; \mathcal{F})$ as the restrictions to the trivialisations.

One has the following results about the cohomology of $\Omega_{\text{Der}}^\bullet(\mathbf{A})$:

Proposition 48 (Noncommutative Leray theorem)

The cohomology of the total complex of the bicomplex $(\mathbf{C}^{\bullet,\bullet}(\mathfrak{U}; \mathcal{F}), \widehat{d}, \delta)$ is the cohomology of $\Omega_{\text{Der}}^\bullet(\mathbf{A})$.

The spectral sequence $\{E_r\}$ associated to the filtration

$$F^p \mathbf{C}(\mathfrak{U}; \mathcal{F}) = \bigoplus_{s \geq p} \bigoplus_{q \geq 0} \mathbf{C}^{s,q}(\mathfrak{U}; \mathcal{F})$$

converges to the cohomology of $\Omega_{\text{Der}}^\bullet(\mathbf{A})$ and satisfies

$$E_2 = H_{dR}^\bullet(M) \otimes \mathcal{I}(\wedge^\bullet \mathfrak{sl}_n^*)$$

Recall that the structure of $\mathcal{I}(\wedge^\bullet \mathfrak{sl}_n^*)$ is known.

6.2 Characteristic classes and short exact sequences of Lie algebras

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Splitting of short exact sequences of Lie algebras

Let $0 \rightarrow \mathfrak{i} \rightarrow \mathfrak{g} \xrightarrow{\pi} \mathfrak{h} \rightarrow 0$ be a s.e.s. of Lie algebras. Let $\varphi : \mathfrak{h} \rightarrow \mathfrak{g}$ be a morphism which splits it as vector spaces. Define $R_\varphi = d_{\mathfrak{h}}\varphi + \frac{1}{2}[\varphi, \varphi] : \wedge^2 \mathfrak{h}^* \otimes \mathfrak{g}$ with $d_{\mathfrak{h}}$ the differential on $\wedge^\bullet \mathfrak{h}^* \otimes \mathfrak{g}$ for the trivial representation of \mathfrak{h} on \mathfrak{g} . For any $x, y \in \mathfrak{h}$, one has $R_\varphi(x, y) = -\varphi([x, y]) + [\varphi(x), \varphi(y)]$ (obstruction to be a Lie algebra morphism).

\rightarrow R_φ looks like a curvature: R_φ belongs to $\wedge^2 \mathfrak{h}^* \otimes \mathfrak{i}$ and satisfies the Bianchi identity $d_{\mathfrak{h}}R_\varphi + [R_\varphi, R_\varphi] = 0$. Let \mathbf{V} be a vector space and ρ a representation of \mathfrak{h} on \mathbf{V} . Denote by $S_\rho^q(\mathfrak{i}, \mathbf{V})$ the space of linear symmetric maps $\otimes^q \mathfrak{i} \rightarrow \mathbf{V}$ which intertwine the adjoint representation $\text{ad}^{\otimes q}$ of \mathfrak{g} on $\otimes^q \mathfrak{i}$ and the representation $\rho \circ \pi$ of \mathfrak{g} on \mathbf{V} . Let ϵ be the antisymmetrisation map $\otimes^\bullet \mathfrak{h}^* \rightarrow \wedge^\bullet \mathfrak{h}^*$.

Proposition 49 (Lecomte)

For any $\alpha \in S_\rho^q(\mathfrak{i}, \mathbf{V})$, let $\alpha_\varphi = \epsilon \circ \alpha(R_\varphi \otimes \dots \otimes R_\varphi) \in \wedge^{2q} \mathfrak{h}^* \otimes \mathbf{V}$. Then one has $d\alpha_\varphi = 0$ where d is the differential of the complex $\wedge^\bullet \mathfrak{h}^* \otimes \mathbf{V}$.

The cohomology class of α_φ in $H^{2q}(\mathfrak{h}; \mathbf{V})$ does not depend on the choice of φ .

If the s.e.s. is split exact as a Lie algebra s.e.s. then this cohomology class is zero.

.....

Characteristic classes constructed from \mathbf{A}

Consider the s.e.s. $0 \rightarrow \text{Int}(\mathbf{A}) \rightarrow \text{Der}(\mathbf{A}) \xrightarrow{\rho} \Gamma(M) \rightarrow 0$.

It is possible to generalise the previous construction in order to take into account the structures of $\mathcal{Z}(\mathbf{A})$ -modules.

We identify $\text{Int}(\mathbf{A})$ with \mathbf{A}_0 . The adjoint representation of $\text{Der}(\mathbf{A})$ on $\text{Int}(\mathbf{A})$ is given by $\text{ad}_{\mathfrak{X}}(\text{ad}_a) = [\mathfrak{X}, \text{ad}_a] = \text{ad}_{\mathfrak{X}(a)}$ so that it is $(\mathfrak{X}, a) \mapsto \mathfrak{X}(a)$ on \mathbf{A}_0 .

The vector space (and $\mathcal{Z}(\mathbf{A})$ -module) we consider is $\mathcal{Z}(\mathbf{A})$ itself, on which the representation ρ is $(\mathfrak{X}, f) \mapsto \rho(\mathfrak{X}) \cdot f$.

Let $S_{\mathcal{Z}(\mathbf{A})}^q(\mathbf{A}_0, \mathcal{Z}(\mathbf{A}))$ be the space of $\mathcal{Z}(\mathbf{A})$ -linear symmetric maps $\otimes_{\mathcal{Z}(\mathbf{A})}^q \mathbf{A}_0 \rightarrow \mathcal{Z}(\mathbf{A})$ which intertwine the adjoint representation $\text{ad}^{\otimes q}$ of $\text{Der}(\mathbf{A})$ on $\otimes_{\mathcal{Z}(\mathbf{A})}^q \text{Int}(\mathbf{A}) = \otimes_{\mathcal{Z}(\mathbf{A})}^q \mathbf{A}_0$ and the representation ρ of $\text{Der}(\mathbf{A})$ on $\mathcal{Z}(\mathbf{A})$.

Thanks to the $\mathcal{Z}(\mathbf{A})$ -linearity, maps in $S_{\mathcal{Z}(\mathbf{A})}^q(\mathbf{A}_0, \mathcal{Z}(\mathbf{A}))$ are local on M

\rightarrow look at them in local trivialisation of E .

The intertwining relations can be written, with $\mathfrak{X}^{\text{loc}} = X + \text{ad}_\gamma$:

$$\begin{aligned} \sum_{i=1}^q \phi(a_1 \otimes \dots \otimes X \cdot a_i \otimes \dots \otimes a_q) &= X \cdot \phi(a_1 \otimes \dots \otimes a_q) \\ \sum_{i=1}^q \phi(a_1 \otimes \dots \otimes [\gamma, a_i] \otimes \dots \otimes a_q) &= 0 \end{aligned}$$

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Characteristic classes constructed from \mathbf{A}

Proposition 50 (Characteristic classes)

The space $S_{\mathcal{Z}(\mathbf{A})}^q(\mathbf{A}_0, \mathcal{Z}(\mathbf{A}))$ is well defined ($\mathcal{Z}(\mathbf{A})$ -linearity and intertwiners are compatible) and

$$S_{\mathcal{Z}(\mathbf{A})}^q(\mathbf{A}_0, \mathcal{Z}(\mathbf{A})) = \mathcal{P}_I^q(\mathfrak{sl}_n)$$

the space of invariant polynomials on the Lie algebra \mathfrak{sl}_n .

The differential complex in which the characteristic classes for the splitting are defined is

$$\text{Hom}_{\mathcal{Z}(\mathbf{A})}(\wedge_{\mathcal{Z}(\mathbf{A})}^\bullet \Gamma(M), \mathcal{Z}(\mathbf{A}))$$

which is the de Rham complex of differential forms on M .

The characteristic classes one computes in this way are precisely the ordinary characteristic classes of the principal fiber bundle P or of the vector bundle E .

The last statement relies on the fact that any ordinary connection ∇^E on E gives rise to a splitting of the s.e.s. whose curvature is exactly the obstruction to be a morphism of Lie algebras. The construction based on $S_{\mathcal{Z}(\mathbf{A})}^q(\mathbf{A}_0, \mathcal{Z}(\mathbf{A})) = \mathcal{P}_1^q(\mathfrak{sl}_n)$ is then the Chern-Weil morphism.

One then concludes that the s.e.s. $0 \rightarrow \text{Int}(\mathbf{A}) \rightarrow \text{Der}(\mathbf{A}) \xrightarrow{\rho} \Gamma(M) \rightarrow 0$ contains the characteristic classes of E .

7 Invariant noncommutative connections

7.1 Action of a Lie group on a principal fiber bundle

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Action of a Lie group

$G \rightarrow P \xrightarrow{\pi} M$ principal fiber bundle. H Lie group acting on the left on P , such that the action commutes with the right action of G .

- H induces a left action on M . Assumption: this action is simple, which means M admits the fiber bundle structure $H/H_0 \rightarrow M \rightarrow M/H$ where H_0 is an isotropy subgroup: $H_0 = H_{x_0} = \{h \in H / h \cdot x_0 = x_0\}$.
- Fix H_0 an isotropy subgroup.
 - $N = \{x \in M / H_x = H_0\}$.
 - $\mathcal{N}_H(H_0) = \{h \in H / hH_0 = H_0h\}$ normalizer of H_0 in H .
 - H_0 normal subgroup of $\mathcal{N}_H(H_0)$, one has the principal fiber bundle

$$\mathcal{N}_H(H_0)/H_0 \rightarrow N \rightarrow M/H$$

$H/H_0 \rightarrow M \rightarrow M/H$ is associated to this bundle for the natural action of $\mathcal{N}_H(H_0)/H_0$ on H/H_0 (left multiplication).

- $S = H \times G$ acts on the right on P : $(h, g) \cdot p = h^{-1} \cdot p \cdot g$.
 - For $p \in P$, let $\lambda_p : H_{\pi(p)} \rightarrow G$ such that $h \cdot p = p \cdot \lambda_p(h)$.
 - $S_p = \{(h, \lambda_p(h)) / h \in H_{\pi(p)}\}$ is the isotropy subgroup of $p \in P$
 ➔ the action of S on P is simple.
 - Fix S_0 an isotropy subgroup, let $Q = \{p \in P / S_p = S_0\}$, then

$$S/S_0 \rightarrow P \rightarrow M/H \text{ is associated to } \mathcal{N}_S(S_0)/S_0 \rightarrow Q \rightarrow M/H$$

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The reduced fiber bundle Q

Proposition 51 (Some properties of Q)

The map $\lambda_p : H_{\pi(p)} \rightarrow G$ such that $h \cdot p = p \cdot \lambda_p(h)$ satisfies

$$\lambda_{p \cdot g}(h) = g^{-1} \lambda_p(h) g$$

For any $q \in Q$, λ_q depends only on $\pi(q) \in M$: $\lambda_q(h) = \lambda_{q \cdot g}(h)$.

For a fixed x_0 in M , we denote it by $\lambda : H_0 \rightarrow G$ with $H_0 = H_{x_0}$.

The projection $\pi : P \rightarrow M$ induces the fiber bundle structure

$$\mathcal{Z}_G(\lambda(H_0)) \rightarrow Q \xrightarrow{\pi_Q} \pi(Q)$$

with $\mathcal{Z}_G(\lambda(H_0)) = \{g \in G / g\lambda(h_0) = \lambda(h_0)g, \forall h_0 \in H_0\}$, the centralizer of $\lambda(H_0)$ in G , and $\pi(Q) \subset N$.

In the following, we will need the diagram of fibrations:

$$\begin{array}{ccccc} \mathcal{N}_S(S_0)/S_0 & \longrightarrow & Q & \longrightarrow & M/H \\ \downarrow & & \downarrow & & \parallel \\ S/S_0 & \longrightarrow & P & \longrightarrow & M/H \end{array}$$

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Lie algebras decompositions

We are interested in the Lie algebras of different groups introduced before:

- Over the group H : \mathfrak{h} Lie algebra of H .
 - \mathfrak{h}_0 Lie algebra of H_0 , the fixed isotropy group.
 - \mathfrak{k} Lie algebra of the quotient group $\mathcal{N}_H(H_0)/H_0$.
 - $\mathfrak{n}_0 = \mathfrak{h}_0 \oplus \mathfrak{k}$ Lie algebra of $\mathcal{N}_H(H_0)$, the normalizer of H_0 in H .
 - \mathfrak{l} vector space in the orthogonal decomposition $\mathfrak{h} = \mathfrak{n}_0 \oplus \mathfrak{l}$ s. t. $[\mathfrak{n}_0, \mathfrak{l}] \subset \mathfrak{l}$.
- Over the group G : \mathfrak{g} Lie algebra of G .
 - \mathfrak{z}_0 Lie algebra of $\mathcal{Z}_G(\lambda(H_0))$, the centralizer of $\lambda(H_0)$ in G .
 - \mathfrak{m} vector space in the orthogonal decomposition $\mathfrak{g} = \mathfrak{z}_0 \oplus \mathfrak{m}$ s.t. $[\mathfrak{z}_0, \mathfrak{m}] \subset \mathfrak{m}$.
- Over the group $S = H \times G$: $\mathfrak{s} = \mathfrak{h} \oplus \mathfrak{g}$ Lie algebra of S .
 - $\mathfrak{s}_0 = \{(x_0, \lambda_* x_0) / x_0 \in \mathfrak{h}_0\}$ Lie algebra of S_0 , the fixed isotropy group.
 - $\mathfrak{s}_0 \oplus \mathfrak{k} \oplus \mathfrak{z}_0$ Lie algebra of $\mathcal{N}_S(S_0)$, the normalizer of S_0 in S .
 - $\mathfrak{k} \oplus \mathfrak{z}_0$ Lie algebra of the quotient group $\mathcal{N}_S(S_0)/S_0$.

Proposition 52 (Decomposition of TP)

For any $q \in Q$, one has $\mathfrak{k}_q^Q \oplus (\mathfrak{z}_0)_q^Q \subset T_q Q$ and $T_q P = T_q Q \oplus \mathfrak{l}_q^P \oplus \mathfrak{m}_q^P$.

\mathfrak{a}_q^R are the tangent vectors associated to $x \in \mathfrak{a}$ through the fundamental vector fields on R for the action of the corresponding group H or G .

7.2 Invariant noncommutative connections

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Action of a Lie group on \mathbf{A}

Let H be a compact connected Lie group acting on the $SL(n)$ -principal fiber bundle P (so that $G = SL(n)$).

Proposition 53 (Operations of \mathfrak{h} on $\Omega_{\text{Der}}^\bullet(\mathbf{B})$ and $\Omega_{\text{Der}}^\bullet(\mathbf{A})$)

The operation of \mathfrak{h} on $\Omega^\bullet(P)$ induced by the action of H on P extends to an operation of \mathfrak{h} on $\Omega_{\text{Der}}^\bullet(\mathbf{B}) = \Omega^\bullet(P) \otimes \Omega_{\text{Der}}^\bullet(M_n)$ by a trivial action on the second factor. This operation commutes with the operations of \mathfrak{g}_{ad} and $\mathfrak{g}_{\text{equ}}$, and so reduces to the operation of \mathfrak{h} on $\Omega^\bullet(P)$ and to an operation of \mathfrak{h} on $\Omega_{\text{Der}}^\bullet(\mathbf{A})$.

Definition 54 (Invariant n.c. connection)

A n.c. connection $\widehat{\nabla}$ on the right \mathbf{A} -module $M = \mathbf{A}$ is \mathfrak{h} -invariant if, $\forall y \in \mathfrak{h}, \forall \mathfrak{X} \in \text{Der}(\mathbf{A})$ and $\forall a \in \mathbf{A}$, $L_y(\widehat{\nabla}_{\mathfrak{X}} a) = \widehat{\nabla}_{[y, \mathfrak{X}]} a + \widehat{\nabla}_{\mathfrak{X}}(L_y a)$ for the action of \mathfrak{h} on $\Omega_{\text{Der}}^\bullet(\mathbf{A})$.

Proposition 55 (Invariance of the n.c. 1-form of a n.c. connection)

The n.c. connection $\widehat{\nabla}$ is \mathfrak{h} -invariant if and only if its n.c. 1-form α is invariant: $L_y \alpha = 0$ for all $y \in \mathfrak{h}$.

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The space of invariant n.c. connections

The problem to solve: find all the n.c. 1-forms written as $\alpha = \omega - \phi \in [\Omega^1(P) \otimes M_n] \oplus [C^\infty(P) \otimes M_n \otimes \mathfrak{sl}_n^*]$ satisfying

$$(L_{\xi^v} + L_{\text{ad}_\xi})\omega = 0 \quad (L_{\xi^v} + L_{\text{ad}_\xi})\phi = 0 \quad i_{\xi^v}\omega - i_{\text{ad}_\xi}\phi = 0 \quad L_y(\omega - \phi) = 0$$

for all $\xi \in \mathfrak{g} = \mathfrak{sl}_n$ and $y \in \mathfrak{h}$.

- $L_y \omega = 0$ and $L_y \phi = 0$ for all $y \in \mathfrak{h}$ (H -invariance): permits to study ω and ϕ over $Q \subset P$ only. For all $q \in Q$

$$\omega_q : T_q P = T_q Q \oplus \mathfrak{l}_q^P \oplus \mathfrak{m}_q^P \rightarrow M_n \quad \phi_q : \mathfrak{g} \rightarrow M_n$$

- $i_{\xi^v}\omega - i_{\text{ad}_\xi}\phi = 0$ for all $\xi \in \mathfrak{g}$ (\mathfrak{g} -horizontalness): $\phi_q(\xi)$ is determined by $\omega_q(\xi^v)$.

Let us first consider the $T_q Q$ part of $T_q P$. Denote by μ_q the restriction of ω_q to $T_q Q$. One has $(\mathfrak{z}_0)_q^Q \subset T_q Q \rightarrow \mu$ and ϕ both defined on $\mathfrak{z}_0 \subset \mathfrak{g}$, $\mu(z^Q) = \phi(z)$, $\forall z \in \mathfrak{z}_0$. Denote by η_q the restriction of ϕ_q to \mathfrak{z}_0 .

Proposition 56 (The algebra \mathbf{W})

Let $\mathbf{W} = \mathcal{Z}_{M_n}(\lambda_* \mathfrak{g}_0)$ be the centralizer of $\lambda_* \mathfrak{g}_0$ in M_n . It is an associative algebra and $\mathfrak{z}_0 \subset \text{Der}(\mathbf{W})$. Let $\Omega_{\mathfrak{z}_0}^\bullet(\mathbf{W}) = \mathbf{W} \otimes \bigwedge^* \mathfrak{z}_0^*$ be the restricted derivation-based differential calculus associated to it. There is a natural operation of \mathfrak{z}_0 on $\Omega^\bullet(Q) \otimes \Omega_{\mathfrak{z}_0}^\bullet(\mathbf{W})$, and $\mu - \eta \in (\Omega^\bullet(Q) \otimes \Omega_{\mathfrak{z}_0}^\bullet(\mathbf{W}))_{\mathfrak{z}_0\text{-basic}}^1$.

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The space of invariant n.c. connections

Proposition 57 (The differential calculus $\Omega_{\mathfrak{k} \oplus \mathfrak{z}_0}^\bullet(M/H; \mathbf{W})$)

There are natural operations of $\mathfrak{k} \subset \mathfrak{h}$ and $\mathfrak{z}_0 \subset \mathfrak{g}$ on the differential algebra $\Omega^\bullet(Q) \otimes \Omega_{\mathfrak{z}_0}^\bullet(\mathbf{W}) \otimes \bigwedge^* \mathfrak{k}^*$. Define

$$\Omega_{\mathfrak{k} \oplus \mathfrak{z}_0}^\bullet(M/H; \mathbf{W}) = (\Omega^\bullet(Q) \otimes \Omega_{\mathfrak{z}_0}^\bullet(\mathbf{W}) \otimes \bigwedge^* \mathfrak{k}^*)_{\mathfrak{k} \oplus \mathfrak{z}_0\text{-basic}}$$

The algebra $\mathbf{C} = \Omega_{\mathfrak{k} \oplus \mathfrak{z}_0}^0(M/H; \mathbf{W}) = (C^\infty(Q) \otimes \mathbf{W})_{\mathfrak{k} \oplus \mathfrak{z}_0\text{-invariants}}$ is the algebra of sections of a \mathbf{W} -fiber bundle associated to the principal fiber bundle

$$\mathcal{N}_S(S_0)/S_0 \rightarrow Q \rightarrow M/H$$

Recall that $\mathfrak{k}_q^Q \subset T_q Q$. For any $k \in \mathfrak{k}$, define $\nu(k) = \mu(k^Q)$, so that $\nu \in C^\infty(Q) \otimes \mathbf{W} \otimes \mathfrak{k}^*$.

Proposition 58 (The $T_q Q$ part of $T_q P$)

One has $\mu - \eta - \nu \in \Omega_{\mathfrak{k} \oplus \mathfrak{z}_0}^1(M/H; \mathbf{W})$.

This expression contains all the information about the restriction of ω to TQ and about ϕ .

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The space of invariant n.c. connections

Consider now the $\mathfrak{l}_q^P \oplus \mathfrak{m}_q^P$ part of $T_q P$.

Recall that $[\mathfrak{h}_0 \oplus \mathfrak{k}, \mathfrak{l}] \subset \mathfrak{l}$ and $[\mathfrak{z}_0, \mathfrak{m}] \subset \mathfrak{m}$, so that there are natural actions $[\mathfrak{h}_0, \mathfrak{l} \oplus \mathfrak{m}] \subset \mathfrak{l} \oplus \mathfrak{m}$ and $[\mathfrak{k} \oplus \mathfrak{z}_0, \mathfrak{l} \oplus \mathfrak{m}] \subset \mathfrak{l} \oplus \mathfrak{m}$.

Recall that $\mathfrak{s}_0 \oplus \mathfrak{k} \oplus \mathfrak{z}_0$ is the Lie algebra of $\mathcal{N}_S(S_0)$ and $\mathfrak{k} \oplus \mathfrak{z}_0$ is the Lie algebra of $\mathcal{N}_S(S_0)/S_0$, with $\mathfrak{s}_0 = \{(x_0, \lambda_* x_0) / x_0 \in \mathfrak{h}_0\}$.

For the restriction of ω to Q , the H -invariance and the G -invariance combine together into a $\mathcal{N}_S(S_0)$ -invariance.

One can treat this invariance in two steps: one for S_0 and one for $\mathcal{N}_S(S_0)/S_0$.

Define the vector space of S_0 -invariant linear maps $\mathfrak{l} \oplus \mathfrak{m} \rightarrow M_n$:

$$\mathbf{F} = \{f : \mathfrak{l} \oplus \mathfrak{m} \rightarrow M_n / f([x_0, v]) - [\lambda_* x_0, f(v)] = 0, \forall x_0 \in \mathfrak{h}_0, \forall v \in \mathfrak{l} \oplus \mathfrak{m}\}$$

on which $\mathfrak{k} \oplus \mathfrak{z}_0$ acts naturally: $(L_{k+z} f)(v) = -f([k, v]) + [z, f(v)]$.

Let $\mathbf{M} = (C^\infty(Q) \otimes \mathbf{F})_{\mathfrak{k} \oplus \mathfrak{z}_0\text{-invariants}}$.

Proposition 59 (The $\mathfrak{l}_q^P \oplus \mathfrak{m}_q^P$ part of $T_q P$)

\mathbf{M} is the space of sections of the \mathbf{F} -fiber bundle associated to the principal fiber bundle

$$\mathcal{N}_S(S_0)/S_0 \rightarrow Q \rightarrow M/H$$

It is a \mathbf{C} -bimodule.

The restriction of ω to the subspaces $\mathfrak{l}_q^P \oplus \mathfrak{m}_q^P$ is in \mathbf{M} .

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The space of invariant n.c. connections

One get the final identification:

Theorem 60 (The space of H -invariant n.c. connections)

The space of H -invariant n.c. connections on \mathbf{A} is $\Omega_{\mathfrak{k} \oplus \mathfrak{z}_0}^1(M/H; \mathbf{W}) \oplus \mathbf{M}$.

Remark 61 (Naturality of the spaces)

In this result, all the spaces are constructed from the principal fiber bundle $\mathcal{N}_S(S_0)/S_0 \rightarrow Q \rightarrow M/H$ with the help of geometrical or algebraic methods which are natural in this n.c. framework:

- $\mathbf{C} = (C^\infty(Q) \otimes \mathbf{W})_{\mathfrak{k} \oplus \mathfrak{so}\text{-invariants}}$ is modeled on the finite dimensional algebra $\mathbf{W} \subset M_n$. It looks like a “reduced algebra” of \mathbf{A} .
- $\Omega_{\mathfrak{k} \oplus \mathfrak{so}}^\bullet(M/H; \mathbf{W})$ is a natural differential calculus over \mathbf{C} .
- $\mathbf{M} = (C^\infty(Q) \otimes \mathbf{F})_{\mathfrak{k} \oplus \mathfrak{so}\text{-invariants}}$ is a natural \mathbf{C} -bimodule.
- The space $SU(\mathbf{C})$ acts naturally on the space $\Omega_{\mathfrak{k} \oplus \mathfrak{so}}^1(M/H; \mathbf{W}) \oplus \mathbf{M}$ as restriction of n.c. gauge transformations.
- All these spaces are sections of fiber bundles over the base space M/H .

Perspectives

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Perspectives

- Look at other ordinary constructions about P using the noncommutative geometry of \mathbf{A} .
- Extend this procedure to other kinds of fiber bundles: Clifford algebra of a metric manifold, Dixmier-Douady bundle of algebras, ...
- Look at the physical consequences of the extra fields. Could they help to understand the “geometry” behind the Higgs fields?

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