Noncommutative equivalent to principal fiber bundles

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Outline

Derivation-based differential calculus

$\mathcal{A}$ associative algebra with unit.

- $\mathcal{Z}(\mathcal{A})$ center of $\mathcal{A}$: commutative subalgebra.
- $\mathrm{Der}(\mathcal{A})$ vector space of derivations of $\mathcal{A}$, Lie algebra, $\mathcal{Z}(\mathcal{A})$-module.
  $X : \mathcal{A} \to \mathcal{A}$ s.t. $X(ab) = X(a)b + aX(b)$.
- $\mathrm{Int}(\mathcal{A})$ vector space of inner derivations, Lie ideal and $\mathcal{Z}(\mathcal{A})$-submodule.
  $a \mapsto X_b(a) = [b, a] = ad_b(a)$.
- $\mathrm{Out}(\mathcal{A}) = \mathrm{Der}(\mathcal{A})/\mathrm{Int}(\mathcal{A})$ Lie algebra, $\mathcal{Z}(\mathcal{A})$-module.
- $\Omega_{\mathrm{Der}}(\mathcal{A})$ the set of $\mathcal{Z}(\mathcal{A})$-multilinear antisymmetric maps from $\mathrm{Der}(\mathcal{A})$ to $\mathcal{A}$.
  $\mathbb{N}$-graded algebra.
- d differential (of degree 1) defined by (Koszul formula)

$$\begin{aligned} d\omega(X_1, \dots, X_{n+1}) &= \sum_{i=1}^{n+1} (-1)^{i+1} X_i \omega(X_1, \dots \overset{i}{\vee} \dots, X_{n+1}) \\ &+ \sum_{1 \leq i < j \leq n+1} (-1)^{i+j} \omega([X_i, X_j], \dots \overset{i}{\vee} \dots \overset{j}{\vee} \dots, X_{n+1}) \end{aligned}$$

- See Complements on Cartan operations.

Noncommutative connections

Special case: the right module is $\mathcal{A}$

- N.C. connection: $\widehat{\nabla}_X : \mathcal{A} \to \mathcal{A}$

$$\widehat{\nabla}_X(ma) = m(X \cdot a) + \widehat{\nabla}_X(m)a \qquad \widehat{\nabla}_{fX} m = f\widehat{\nabla}_X m$$

  Curvature: $\hat{R}(X, Y)m = [\widehat{\nabla}_X, \widehat{\nabla}_Y]m - \widehat{\nabla}_{[X,Y]}m$
  Right $\mathcal{A}$-module homomorphism.

- $\widehat{\nabla}$ completely given by $\widehat{\nabla}_X \mathbb{1} = \omega(X)$, where $\omega \in \Omega^1_{\text{Der}}(\mathcal{A})$:

$$\widehat{\nabla}_X m = Xm + \omega(X)m$$

  Its curvature is the left multiplication by the noncommutative 2-form

$$d\omega(X, Y) + [\omega(X), \omega(Y)]$$

- Case $\mathcal{A}$ with involution: $U(\mathcal{A})$ the group of unitary elements of $\mathcal{A}$.
  Any $U \in U(\mathcal{A})$ defines on $\mathcal{A}$ a right module endomorphism $m \mapsto Um$, and induces a gauge transformation on n.c. connections:

$$\widehat{\nabla}^U_X m = U^* \widehat{\nabla}_X(Um)$$

Three important examples

- $\mathcal{A} = C^\infty(M)$, algebra of smooth complex-valued functions on a finite dimensional regular manifold M.
  - $\text{Der}(C^\infty(M)) = \Gamma(TM)$ is the ordinary Lie algebra of vector fields on M.
  - $(\Omega_{\text{Der}}(\mathcal{A}), d)$ is the de Rham complex $(\Omega(M), d)$.
- $\mathcal{A} = M_n := M_n(\mathbb{C})$ algebra of $n \times n$ complex matrices.
  - $\mathcal{Z}(\mathcal{A}) = \mathbb{C}$.
  - $\text{Der}(M_n) = \text{Int}(M_n) \simeq \mathfrak{sl}_n := \mathfrak{sl}(n, \mathbb{C})$.
  - $\Omega_{\text{Der}}(M_n) \simeq M_n \otimes \bigwedge \mathfrak{sl}_n^*$, d' its differential.
  - $i\theta \in \Omega^1_{\text{Der}}(M_n)$ canonical 1-form: $i\theta(ad_\gamma) = \gamma - \frac{1}{n}\text{Tr}(\gamma)\mathbb{1}$.
- $\mathcal{A} = C^\infty(M) \otimes M_n$ algebra of matrix valued functions.
  - $\mathcal{Z}(\mathcal{A}) = C^\infty(M)$
  - $\text{Der}(\mathcal{A}) = [\text{Der}(C^\infty(M)) \otimes \mathbb{1}] \oplus [C^\infty(M) \otimes \text{Der}(M_n)]$ as $C^\infty(M)$-modules and Lie algebras.
  - $\Omega_{\text{Der}}(\mathcal{A}) = \Omega(M) \otimes \Omega_{\text{Der}}(M_n)$. Differential $\hat{d} = d + d'$.
  - $i\theta \in \Omega^1_{\text{Der}}(\mathcal{A})$, extended to $\text{Der}(\mathcal{A})$ by zero on $\text{Der}(C^\infty(M)) \otimes \mathbb{1}$.

The algebra we will consider is a generalization of this last algebra.

The algebra and its derivations

$\mathcal{E}$ a $SU(n)$-vector bundle over M. $\mathrm{End}(\mathcal{E})$ fiber bundle of endomorphisms of $\mathcal{E}$.
$\mathcal{A}$ the algebra of sections of $\mathrm{End}(\mathcal{E})$. $\mathcal{Z}(\mathcal{A}) = C^\infty(M)$.
Trivial case: $\mathcal{E}$ trivial bundle ➜ $\mathcal{A} = C^\infty(M) \otimes M_n$.
Involution, trace map and determinant ($\mathrm{Tr}, \det : \mathcal{A} \to C^\infty(M)$) well defined.
$\rho : \mathrm{Der}(\mathcal{A}) \to \mathrm{Der}(\mathcal{A})/\mathrm{Int}(\mathcal{A}) = \mathrm{Out}(\mathcal{A}) \simeq \mathrm{Der}(C^\infty(M)) = \Gamma(TM)$

$$0 \to \mathrm{Int}(\mathcal{A}) \to \mathrm{Der}(\mathcal{A}) \xrightarrow{\rho} \Gamma(TM) \to 0$$

$$\mathcal{X} \longmapsto X$$

Generalizes decomposition in the trivial case.
No canonical splitting in the non trivial case.
$i\theta$ defined here on $\mathrm{Int}(\mathcal{A})$ only: $i\theta(ad_\gamma) = \gamma - \frac{1}{n}\mathrm{Tr}(\gamma)\mathbb{1}$

$\nabla^{\mathcal{E}}$ any connection on $\mathcal{E}$ ➜ ∇ on $\mathrm{End}(\mathcal{E})$ ➜ $\nabla_X \in \mathrm{Der}(\mathcal{A})$
➜ $\mathcal{X} - \nabla_X \in \mathrm{Int}(\mathcal{A})$ ➜ $\alpha(\mathcal{X}) = -i\theta(\mathcal{X} - \nabla_X)$
➜ $\alpha \in \Omega^1_{\mathrm{Der}}(\mathcal{A})$ ➜ $\mathcal{X} = \nabla_X - ad_{\alpha(\mathcal{X})}$
➜ $X \mapsto \nabla_X$ splitting as $C^\infty(M)$-modules

Notice that: $\alpha(ad_\gamma) = -\gamma$ ➜ α generalizes $i\theta$

Ordinary connections

As far as ordinary connections are concerned, $\mathcal{A}$ plays a similar role to a principal bundle.

- $\nabla^{\mathcal{E}} \mapsto \alpha$, isomorphism between the affines spaces:

  $SU(n)$-connections on $\mathcal{E}$

  Traceless antihermitien n.c. 1-forms on $\mathcal{A}$ s.t. $\alpha(ad_\gamma) = -\gamma$

- $R^{\mathcal{E}}$ curvature of $\nabla^{\mathcal{E}}$, $R^{\mathcal{E}}(X, Y) = \hat{d}\alpha(\mathcal{X}, \mathcal{Y}) + [\alpha(\mathcal{X}), \alpha(\mathcal{Y})]$
- $ad_\gamma \in \text{Int}(\mathcal{A})$, $\mathcal{L}_{ad_\gamma}\alpha = -\hat{d}\gamma - [\alpha, \gamma]$

  Infinitesimal gauge transformations on connections

  =

  Lie derivative of real inner derivations on $\mathcal{A}$.

  The Lie algebra of the gauge group of $\mathcal{E}$ is exactly the traceless antihermitean elements in $\mathcal{A}$.

Noncommutative connections on $\mathcal{A}$

- Noncommutative connection defined by $\omega \in \Omega^1_{\mathrm{Der}}(\mathcal{A})$.
- The (n.c.) gauge group $SU(\mathcal{A})$ is exactly the (ordinary) gauge group on $\mathcal{E}$.
- Ordinary connection on $\mathcal{E}$ ➜ n.c. 1-form α ➜ n.c. connection $\widehat{\nabla}^\alpha$:

$$\widehat{\nabla}^\alpha_{\mathcal{X}} m := \nabla_X m + m\alpha(\mathcal{X}) = \mathcal{X} m + \alpha(\mathcal{X}) m$$

The curvatures coincide, gauge transformations are compatible...

- N.C. connections on $\mathcal{A}$ are generalizations of ordinary connections on $\mathcal{E}$.
- Purely n.c. part of the connection can be interpreted as Higgs fields.
  Recall: $\alpha(ad_\gamma) = -\gamma$.

The need for a bigger algebra

E the principal $SU(n)$-bundle over M for which $\mathcal{E}$ is associated.
Look at geometrical relations between E and $\mathcal{A}$.

Introduce the algebra $\mathcal{B} = C^\infty(E) \otimes M_n$

- $\text{Der}(\mathcal{B}) = [\Gamma(E) \otimes \mathbb{1}] \oplus [C^\infty(E) \otimes \text{Der}(M_n)]$
  $(\Omega_{\text{Der}}(\mathcal{B}), \hat{d}) = (\Omega(E) \otimes \Omega_{\text{Der}}(M_n), d + d')$.
- $\mathfrak{su}(n) \ni \xi \mapsto \xi^E$ associated vertical vector field on E ➔ derivation on $\mathcal{B}$.
- $\mathfrak{su}(n) \ni \xi \mapsto ad_\xi$ associated inner derivation on $\mathcal{B}$.
- $\mathfrak{g}_1 = \{ad_\xi \ / \ \xi \in \mathfrak{su}(n)\}$ Lie subalgebra of $\text{Der}(\mathcal{B})$.
  $C^\infty(E)$ is the invariant subalgebra for the Lie derivative in the $\mathfrak{g}_1$ directions.
- $\mathfrak{g}_2 = \{\xi^E + ad_\xi \ / \ \xi \in \mathfrak{su}(n)\}$ Lie subalgebra of $\text{Der}(\mathcal{B})$.
  $\mathcal{A}$ is the invariant subalgebra for the Lie derivative in the $\mathfrak{g}_2$ directions.

Global relations

These relations generalize to the differential calculi:

$$
\begin{array}{ccc}
\Omega(E) \otimes \Omega_{\mathrm{Der}}(M_n) & \xleftarrow[\mathfrak{su}(n) \ni \xi \mapsto ad_\xi]{\text{basic elements}} & \Omega(E) \\
{\scriptstyle \text{basic elements}}\ \uparrow\ {\scriptstyle \mathfrak{su}(n) \ni \xi \mapsto \xi^{\boldsymbol{E}} + ad_\xi} & & {\scriptstyle \text{basic elements}}\ \uparrow\ {\scriptstyle \mathfrak{su}(n) \ni \xi \mapsto \xi^{\boldsymbol{E}}} \\
\Omega_{\mathrm{Der}}(\mathcal{A}) & \xleftarrow[\mathrm{Int}(\mathcal{A})]{\text{basic elements}} & \Omega(M)
\end{array}
$$

Other relations:

- N.C. integration "along the n.c. fibers".

$$
\begin{aligned}
\Omega(E) \otimes \Omega_{\mathrm{Der}}(M_n) &\to \Omega(E) \\
\Omega_{\mathrm{Der}}(\mathcal{A}) &\to \Omega(M)
\end{aligned}
$$

- Relations between the Lie algebras of derivations.
  See Complements.

Ordinary *vs.* noncommutative connections

$\Omega_{\mathrm{Der}}(\mathcal{A})$ identified with the corresponding basic subalgebra of $\Omega_{\mathrm{Der}}(\mathcal{B})$.
$\omega \in \Omega^1_{\mathrm{Der}}(\mathcal{A})$ decomposed as

$$\omega = a - \phi \in [\Omega^1(E) \otimes M_n] \oplus [C^\infty(E) \otimes M_n \otimes \mathfrak{sl}_n^*]$$

Basic conditions

$$(\mathcal{L}_{\xi^E} + \mathcal{L}_{ad_\xi})a = 0 \qquad (\mathcal{L}_{\xi^E} + \mathcal{L}_{ad_\xi})\phi = 0$$
$$i_{\xi^E} a - i_{ad_\xi}\phi = 0$$

for any $\xi \in \mathfrak{su}(n)$
Ordinary connections on E: ϕ replaced by $i\theta$
- ➔ vertical condition on a: $i_{\xi^E} a = \xi$
- ➔ a is the connection 1-form on E.

Conclusion

- $\mathcal{A}$ extends the trivial situation of matrix valued functions
  → canonical *versus* connection dependant structures.
- $\mathcal{A}$ is a good candidate to be some generalizations of principal bundles.
  - Strongly related to the geometry of the principal bundle.
  - Contains ordinary connections and related objects.
  - Introduces the Higgs fields.
  - Some constructions are more natural in this framework.
    See talk by E. Serié.
- Possible similar constructions with other bundles of algebras.

Cartan operations

$\mathcal{A}$ any n.c. associative algebra. $\mathfrak{g}$ a Lie subalgebra of $\text{Der}(\mathcal{A})$.
$\mathfrak{g}$ defines a natural operation in the sense of H. Cartan on $(\Omega_{\text{Der}}(\mathcal{A}), d)$.

- Interior product. Graded derivation of degree -1 on $\Omega_{\text{Der}}(\mathcal{A})$.

$$i_X : \Omega^n_{\text{Der}}(\mathcal{A}) \to \Omega^{n-1}_{\text{Der}}(\mathcal{A}) \qquad (i_X\omega)(X_1, \dots, X_{n-1}) = \omega(X, X_1, \dots, X_{n-1})$$

  $\forall X \in \mathfrak{g}$, $\forall \omega \in \Omega^n_{\text{Der}}(\mathcal{A})$ and $\forall X_i \in \text{Der}(\mathcal{A})$.
  i_X is 0 on $\Omega^0_{\text{Der}}(\mathcal{A}) = \mathcal{A}$.

- Lie derivative. Graded derivation of degree 0 on $\Omega_{\text{Der}}(\mathcal{A})$.

$$L_X = i_X d + d i_X : \Omega^n_{\text{Der}}(\mathcal{A}) \to \Omega^n_{\text{Der}}(\mathcal{A})$$

- $$i_X i_Y + i_Y i_X = 0 \qquad L_X i_Y - i_Y L_X = i_{[X,Y]}$$
  $$L_X L_Y - L_Y L_X = L_{[X,Y]} \qquad L_X d - d L_X = 0$$

- Subspaces of $\Omega_{\text{Der}}(\mathcal{A})$ associated to this operation:
  - Horizontal subspace: kernel of all the i_X for $X \in \mathfrak{g}$. Gr. Alg.
  - Invariant subspace: kernel of all the L_X for $X \in \mathfrak{g}$. Gr. Diff. Alg.
  - Basic subspace: kernel of all the i_X and L_X for $X \in \mathfrak{g}$. Gr. Diff. Alg.

Local vision of derivations on $\mathcal{A}$

U open subset over which $\mathrm{End}(\mathcal{E})$ is trivialized.

- Restricted to U, $\mathcal{A}$ is isomorphic to $\mathcal{A}_{\mathrm{loc}} := C^\infty(U) \otimes M_n$.
- $a \in \mathcal{A} \rightarrow a_{\mathrm{loc}} \in \mathcal{A}_{\mathrm{loc}}$.
- $\mathcal{X} \in \mathrm{Der}(\mathcal{A}) \rightarrow \mathcal{X}_{\mathrm{loc}} \in \mathrm{Der}(C^\infty(U) \otimes M_n)$.
  Decomposed into two parts: $\mathcal{X}_{\mathrm{loc}} = X_{|U} + ad_{\gamma_{\mathrm{loc}}}$.
- $U \cap U' \neq \varnothing$.
  - Transition function $g : U \cap U' \to SU(n)$.
  - a_{loc} and a'_{loc} are related by
  $$a'_{\mathrm{loc}} = Ad_{g^{-1}} a_{\mathrm{loc}}$$
  - $\mathcal{X}_{\mathrm{loc}}$ and $\mathcal{X}'_{\mathrm{loc}}$ are related by
  $$\begin{aligned} X'_{|U} &= X_{|U} \\ \gamma'_{\mathrm{loc}} &= Ad_{g^{-1}} \gamma_{\mathrm{loc}} + g^{-1} X_{|U} \cdot g \end{aligned}$$

N.C. connections and hermitian structure

Natural hermitian structure on the right module $\mathcal{A}$:

$$\langle m, m' \rangle = m^* m' \in \mathcal{A} \text{ as an algebra}$$

satisfies

$$\langle ma, m'a' \rangle = a^* \langle am, m' \rangle a'$$

A n.c. connection is compatible with the hermitian structure if

$$\mathcal{X}\langle m, m' \rangle = \langle \widehat{\nabla}_{\mathcal{X}} m, m' \rangle + \langle m, \widehat{\nabla}_{\mathcal{X}} m' \rangle$$

This is equivalent to

$$\omega(\mathcal{X})^* + \omega(\mathcal{X}) = 0$$

for the n.c. 1-form ω giving rise to $\widehat{\nabla}$.

Derivations on $\mathcal{A}$ and $\mathcal{B}$

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \mathcal{Z}_{\mathrm{Der}}(\mathcal{A}) & \longrightarrow & \Gamma(TVE) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Int}(\mathcal{A}) & \longrightarrow & \mathcal{N}_{\mathrm{Der}}(\mathcal{A}) & \xrightarrow{\rho_{\boldsymbol{E}}} & \Gamma_M(E) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \pi_*} & & \\
0 & \longrightarrow & \mathrm{Int}(\mathcal{A}) & \longrightarrow & \mathrm{Der}(\mathcal{A}) & \xrightarrow{\rho} & \Gamma(TM) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

This exact commutative diagram shows some relations between derivations on $\mathcal{A}$, derivations on $\mathcal{B}$ and vector fields on E.

Derivations on $\mathcal{A}$ and $\mathcal{B}$

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \mathcal{Z}_{\mathsf{Der}}(\mathcal{A}) & \longrightarrow & \Gamma(TVE) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathsf{Int}(\mathcal{A}) & \longrightarrow & \mathcal{N}_{\mathsf{Der}}(\mathcal{A}) & \xrightarrow{\rho_{\boldsymbol{E}}} & \Gamma_M(E) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \pi_*} & & \\
0 & \longrightarrow & \mathsf{Int}(\mathcal{A}) & \longrightarrow & \mathsf{Der}(\mathcal{A}) & \xrightarrow{\rho} & \Gamma(TM) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Short exact sequence which relates vector fields on M, derivations on $\mathcal{A}$ and inner derivations on $\mathcal{A}$.

Derivations on $\mathcal{A}$ and $\mathcal{B}$

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \mathcal{Z}_{\mathrm{Der}}(\mathcal{A}) & \longrightarrow & \Gamma(TVE) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Int}(\mathcal{A}) & \longrightarrow & \mathcal{N}_{\mathrm{Der}}(\mathcal{A}) & \xrightarrow{\rho_E} & \Gamma_M(E) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \tau & & \downarrow \pi_* & & \\
0 & \longrightarrow & \mathrm{Int}(\mathcal{A}) & \longrightarrow & \mathrm{Der}(\mathcal{A}) & \xrightarrow{\rho} & \Gamma(TM) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

$\mathcal{N}_{\mathrm{Der}}(\mathcal{A}) \subset \mathrm{Der}(\mathcal{B})$ subset of derivations on $\mathcal{B}$ which preserve $\mathcal{A} \subset \mathcal{B}$.
$\mathcal{Z}_{\mathrm{Der}}(\mathcal{A}) \subset \mathrm{Der}(\mathcal{B})$ subset of derivations on $\mathcal{B}$ which vanish on $\mathcal{A}$.
The Lie algebra $\mathcal{Z}_{\mathrm{Der}}(\mathcal{A})$ is generated as a $C^\infty(E)$-module by the particular elements $\xi^E + ad_\xi$ for any $\xi \in \mathfrak{su}(n)$.

Derivations on $\mathcal{A}$ and $\mathcal{B}$

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \mathcal{Z}_{\mathrm{Der}}(\mathcal{A}) & \longrightarrow & \Gamma(TVE) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Int}(\mathcal{A}) & \longrightarrow & \mathcal{N}_{\mathrm{Der}}(\mathcal{A}) & \xrightarrow{\rho_E} & \Gamma_M(E) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \tau & & \downarrow \pi_* & & \\
0 & \longrightarrow & \mathrm{Int}(\mathcal{A}) & \longrightarrow & \mathrm{Der}(\mathcal{A}) & \xrightarrow{\rho} & \Gamma(TM) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Geometrical objects. $\Gamma(TVE)$ vertical vector fields on E.
$\Gamma_M(E) = \{\hat{X} \in \Gamma(E) / \pi_* \hat{X}(p) = \pi_* \hat{X}(p') \ \forall p, p' \in E \text{ s.t. } \pi(p) = \pi(p')\}$
Lie algebra of vector fields on E which can be mapped to vector fields on M using the tangent maps $\pi_* : T_pE \to T_{\pi(p)}M$.

Derivations on $\mathcal{A}$ and $\mathcal{B}$

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \mathcal{Z}_{\mathrm{Der}}(\mathcal{A}) & \longrightarrow & \Gamma(TVE) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Int}(\mathcal{A}) & \longrightarrow & \mathcal{N}_{\mathrm{Der}}(\mathcal{A}) & \xrightarrow{\rho_{\boldsymbol{E}}} & \Gamma_M(E) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \tau & & \downarrow \pi_* & & \\
0 & \longrightarrow & \mathrm{Int}(\mathcal{A}) & \longrightarrow & \mathrm{Der}(\mathcal{A}) & \xrightarrow{\rho} & \Gamma(TM) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Elements in $\mathrm{Int}(\mathcal{A})$ considered as ad_γ for $\gamma \in \mathcal{A} \subset \mathcal{B}$.
ρ_E restriction to $\mathcal{N}_{\mathrm{Der}}(\mathcal{A})$ of the projection on the first term in
$\mathrm{Der}(\mathcal{B}) = [\Gamma(E) \otimes \mathbb{1}] \oplus [C^\infty(E) \otimes \mathrm{Der}(M_n)]$

Splittings coming from connections

$$
\begin{array}{ccc}
\mathcal{N}_{\mathrm{Der}}(\mathcal{A}) & \xrightarrow{\quad\rho_{\boldsymbol{E}}\quad} & \Gamma_M(E) \\
\Big\downarrow \tau & (\pi_*\hat{X})^h + \omega_E(\hat{X})^E + ad_{\omega_{\boldsymbol{E}}(\hat{X})} \longleftarrow\!\shortmid \hat{X} & \Big\downarrow \pi_* \\
 & \rho(\mathcal{X})^h - ad_{\alpha(\mathcal{X})^{\boldsymbol{E}}} \;\longleftarrow\!\shortmid\; \mathcal{X} \qquad X^h \longleftarrow\!\shortmid X & \\
 & \nabla_X \longleftarrow\!\shortmid X & \\
\mathrm{Der}(\mathcal{A}) & \xrightarrow{\quad\rho\quad} & \Gamma(TM)
\end{array}
$$

Notice that

$$(\pi_*\hat{X})^h + \omega_E(\hat{X})^E + ad_{\omega_{\boldsymbol{E}}(\hat{X})} \neq \rho(\mathcal{X})^h - ad_{\alpha(\mathcal{X})^{\boldsymbol{E}}}$$

Indeed one has:

$$\mathrm{Der}(\mathcal{B}) \ni \mathfrak{X} = X^h + \underbrace{ad_Z}_{\in \mathrm{Int}(\mathcal{A})} + \underbrace{\omega_E(\hat{X})^E + ad_{\omega_{\boldsymbol{E}}(\hat{X})}}_{\in \mathcal{Z}_{\mathbf{Der}}(\mathcal{A})}$$