Noncommutative equivalent to principal fiber bundles

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Outline

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- Cartan operations well defined.

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Any $U \in U(\mathcal{A})$ defines on $\mathcal{A}$ a right module endomorphism $m \mapsto Um$, and induces a gauge transformation on n.c. connections:
$$\widehat{\nabla}^U_X m = U^* \widehat{\nabla}_X(Um)$$

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The algebra we will consider is a generalization of this last algebra.

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Generalizes decomposition in the trivial case.

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$$\mathcal{X} \longmapsto X$$

No canonical splitting in the non trivial case.

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  The Lie algebra of the gauge group of $\mathcal{E}$ is exactly the traceless antihermitean elements in $\mathcal{A}$.

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- Relations between the Lie algebras of derivations. ▸ Diagram for derivations

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for any $\xi \in \mathfrak{su}(n)$
Ordinary connections on E: ϕ replaced by $i\theta$
➔ vertical condition on a: $i_{\xi^E} a = \xi$

Ordinary *vs.* noncommutative connections

$\Omega_{\mathrm{Der}}(\mathcal{A})$ identified with the corresponding basic subalgebra of $\Omega_{\mathrm{Der}}(\mathcal{B})$.
$\omega \in \Omega^1_{\mathrm{Der}}(\mathcal{A})$ decomposed as

$$\omega = a - \phi \in [\Omega^1(E) \otimes M_n] \oplus [C^\infty(E) \otimes M_n \otimes \mathfrak{sl}_n^*]$$

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- ➔ a is the connection 1-form on E.

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- Possible similar constructions with other bundles of algebras.

The End

(realized with the LaTeX package Beamer)

Complement

- Cartan operations
- Local vision of derivations on $\mathcal{A}$
- N.C. connections and hermitian structure
- Derivations on $\mathcal{A}$ and $\mathcal{B}$
- Splittings coming from connections

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    $$\gamma'_{\mathrm{loc}} = Ad_{g^{-1}} \gamma_{\mathrm{loc}} + g^{-1} X_{|U} \cdot g$$

N.C. connections and hermitian structure

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This is equivalent to

$$\omega(\mathcal{X})^* + \omega(\mathcal{X}) = 0$$

for the n.c. 1-form ω giving rise to $\widehat{\nabla}$.

Derivations on $\mathcal{A}$ and $\mathcal{B}$

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & \mathcal{Z}_{\text{Der}}(\mathcal{A}) \arrow[r] \arrow[d] & \Gamma(TVE) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \text{Int}(\mathcal{A}) \arrow[r] \arrow[d] & \mathcal{N}_{\text{Der}}(\mathcal{A}) \arrow[r, "\rho_{\boldsymbol{E}}"] \arrow[d, "\tau"] & \Gamma_M(E) \arrow[r] \arrow[d, "\pi_*"] & 0 \\
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\end{tikzcd}$$

This exact commutative diagram shows some relations between derivations on $\mathcal{A}$, derivations on $\mathcal{B}$ and vector fields on E.

◂ Main text

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Short exact sequence which relates vector fields on M, derivations on $\mathcal{A}$ and inner derivations on $\mathcal{A}$.

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$\mathcal{N}_{\mathrm{Der}}(\mathcal{A}) \subset \mathrm{Der}(\mathcal{B})$ subset of derivations on $\mathcal{B}$ which preserve $\mathcal{A} \subset \mathcal{B}$.
$\mathcal{Z}_{\mathrm{Der}}(\mathcal{A}) \subset \mathrm{Der}(\mathcal{B})$ subset of derivations on $\mathcal{B}$ which vanish on $\mathcal{A}$.
The Lie algebra $\mathcal{Z}_{\mathrm{Der}}(\mathcal{A})$ is generated as a $C^\infty(E)$-module by the particular elements $\xi^E + ad_\xi$ for any $\xi \in \mathfrak{su}(n)$.

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Geometrical objects. $\Gamma(TVE)$ vertical vector fields on E.
$\Gamma_M(E) = \{\hat{X} \in \Gamma(E) / \pi_* \hat{X}(p) = \pi_* \hat{X}(p') \ \forall p, p' \in E \text{ s.t. } \pi(p) = \pi(p')\}$
Lie algebra of vector fields on E which can be mapped to vector fields on M using the tangent maps $\pi_* : T_pE \to T_{\pi(p)}M$.

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Elements in $\text{Int}(\mathcal{A})$ considered as ad_γ for $\gamma \in \mathcal{A} \subset \mathcal{B}$.
ρ_E restriction to $\mathcal{N}_{\text{Der}}(\mathcal{A})$ of the projection on the first term in $\text{Der}(\mathcal{B}) = [\Gamma(E) \otimes \mathbb{1}] \oplus [C^\infty(E) \otimes \text{Der}(M_n)]$

◂ Main text

Splittings coming from connections

An ordinary connection $\omega_E \in \Omega^1(E) \otimes \mathfrak{su}(n)$ splits some short exact sequences in the diagram:

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 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & \mathcal{Z}_{\mathrm{Der}}(\mathcal{A}) \arrow[r] \arrow[d] & \Gamma(TVE) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Int}(\mathcal{A}) \arrow[r] \arrow[d] & \mathcal{N}_{\mathrm{Der}}(\mathcal{A}) \arrow[r, "\rho_{\boldsymbol{E}}"] \arrow[d, "\tau"'] & \Gamma_M(E) \arrow[r] \arrow[d, "\pi_*"] & 0 \\
0 \arrow[r] & \mathrm{Int}(\mathcal{A}) \arrow[r] \arrow[d] & \mathrm{Der}(\mathcal{A}) \arrow[r, "\rho"'] \arrow[d] & \Gamma(TM) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Splittings coming from connections

Look more closely at the central square:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & \mathcal{Z}_{\mathrm{Der}}(\mathcal{A}) \arrow[r] \arrow[d] & \Gamma(TVE) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Int}(\mathcal{A}) \arrow[r] \arrow[d] & \mathcal{N}_{\mathrm{Der}}(\mathcal{A}) \arrow[r, "\rho_{\boldsymbol{E}}"] \arrow[d, "\tau"'] & \Gamma_M(E) \arrow[r] \arrow[d, "\pi_*"] & 0 \\
0 \arrow[r] & \mathrm{Int}(\mathcal{A}) \arrow[r] \arrow[d] & \mathrm{Der}(\mathcal{A}) \arrow[r, "\rho"'] \arrow[d] & \Gamma(TM) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Splittings coming from connections

$$\begin{tikzcd}
\mathcal{N}_{\mathrm{Der}}(\mathcal{A}) \arrow[r, two heads, "\rho_{\boldsymbol{E}}"] \arrow[d, two heads, "\tau"'] & \Gamma_M(E) \arrow[d, two heads, "\pi_*"] \\
\mathrm{Der}(\mathcal{A}) \arrow[r, two heads, "\rho"'] & \Gamma(TM)
\end{tikzcd}$$

Splittings coming from connections

$$\begin{tikzcd}
\mathcal{N}_{\mathrm{Der}}(\mathcal{A}) \arrow[rr, two heads, "\rho_{\boldsymbol{E}}"] \arrow[ddd, two heads, "\tau"'] & & \Gamma_M(E) \arrow[ddd, two heads, "\pi_*"] \\
& \color{red}{X^h} & \\
& \color{red}{X} \arrow[u, mapsto, red] & \\
\mathrm{Der}(\mathcal{A}) \arrow[rr, two heads, "\rho"'] & & \Gamma(TM)
\end{tikzcd}$$

$$0 \longrightarrow \Gamma(TVE) \longrightarrow \Gamma_M(E) \xrightarrow{\pi_*} \Gamma(TM) \longrightarrow 0$$

Lift vector fields on M into horizontal vector fields X^h on E.

Splittings coming from connections

$$\begin{tikzcd}
\mathcal{N}_{\mathrm{Der}}(\mathcal{A}) \arrow[rrrr, two heads, "\rho_{\boldsymbol{E}}"] \arrow[dddd, "\tau"'] & & & & \Gamma_M(E) \arrow[dddd, "\pi_*"] \\
& & & X^h & \\
& & & X \arrow[u, mapsto] & \\
& \color{red}{\nabla_X} & \color{red}{X} \arrow[l, mapsto, red] & & \\
\mathrm{Der}(\mathcal{A}) \arrow[rrrr, two heads, "\rho"'] & & & & \Gamma(TM)
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Int}(\mathcal{A}) \arrow[r] & \mathrm{Der}(\mathcal{A}) \arrow[r, "\rho"] & \Gamma(TM) \arrow[r] & 0
\end{tikzcd}$$

Lift vector fields on M into derivations on $\mathcal{A}$.

Splittings coming from connections

$$\begin{tikzcd}
\mathcal{N}_{\text{Der}}(\mathcal{A}) \arrow[rrr, two heads, "\rho_{\boldsymbol{E}}"] \arrow[dddd, "\tau"'] & & & \Gamma_M(E) \arrow[dddd, "\pi_*"] \\
& \color{red}{\rho(\mathcal{X})^h - ad_{\alpha(\mathcal{X})^{\boldsymbol{E}}}} & X^h & \\
& \color{red}{\mathcal{X}} \arrow[u, mapsto, red] & X \arrow[u, mapsto] & \\
& \nabla_X & X \arrow[l, mapsto] & \\
\text{Der}(\mathcal{A}) \arrow[rrr, two heads, "\rho"'] & & & \Gamma(TM)
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{Z}_{\text{Der}}(\mathcal{A}) \arrow[r] & \mathcal{N}_{\text{Der}}(\mathcal{A}) \arrow[r, "\tau"] & \text{Der}(\mathcal{A}) \arrow[r] & 0
\end{tikzcd}$$

Lift derivations on $\mathcal{A}$ into derivations on $\mathcal{B}$.
$\alpha(\mathcal{X})^E$ is the basic element in $\mathcal{B}$ associated to $\alpha(\mathcal{X}) \in \mathcal{A}$.

Splittings coming from connections

$$\begin{tikzcd}
\mathcal{N}_{\mathrm{Der}}(\mathcal{A}) \arrow[rrr, two heads, "\rho_{\boldsymbol{E}}"] \arrow[ddddd, "\tau"'] & & & \Gamma_M(E) \arrow[ddddd, "\pi_*"] \\
& (\pi_* \hat{X})^h + \omega_E(\hat{X})^E + ad_{\omega_{\boldsymbol{E}}(\hat{X})} & \hat{X} \arrow[l, maps to] & \\
& \rho(\mathcal{X})^h - ad_{\alpha(\mathcal{X})^{\boldsymbol{E}}} & X^h & \\
& \mathcal{X} \arrow[u, maps to] & X \arrow[u, maps to] & \\
& \nabla_X & X \arrow[l, maps to] & \\
\mathrm{Der}(\mathcal{A}) \arrow[rrr, two heads, "\rho"'] & & & \Gamma(TM)
\end{tikzcd}$$

$$0 \longrightarrow \mathrm{Int}(\mathcal{A}) \longrightarrow \mathcal{N}_{\mathrm{Der}}(\mathcal{A}) \xrightarrow{\rho_{\boldsymbol{E}}} \Gamma_M(E) \longrightarrow 0$$

Lift π_*-projectable vector fields on E into derivations on $\mathcal{B}$.

Splittings coming from connections

$$\begin{tikzcd}
\mathcal{N}_{\text{Der}}(\mathcal{A}) \arrow[rrr, two heads, "\rho_{\boldsymbol{E}}"] \arrow[ddddd, "\tau"'] & & & \Gamma_M(E) \arrow[ddddd, "\pi_*"] \\
& (\pi_*\hat{X})^h + \omega_E(\hat{X})^E + ad_{\omega_{\boldsymbol{E}}(\hat{X})} & \hat{X} \arrow[l, mapsto] & \\
& \rho(\mathcal{X})^h - ad_{\alpha(\mathcal{X})^{\boldsymbol{E}}} & X^h & \\
& \mathcal{X} \arrow[u, mapsto] & X \arrow[u, mapsto] & \\
& \nabla_X & X \arrow[l, mapsto] & \\
\text{Der}(\mathcal{A}) \arrow[rrr, two heads, "\rho"'] & & & \Gamma(TM)
\end{tikzcd}$$

Notice that

$$(\pi_*\hat{X})^h + \omega_E(\hat{X})^E + ad_{\omega_{\boldsymbol{E}}(\hat{X})} \neq \rho(\mathcal{X})^h - ad_{\alpha(\mathcal{X})^{\boldsymbol{E}}}$$

Splittings coming from connections

$$\begin{tikzcd}
\mathcal{N}_{\mathrm{Der}}(\mathcal{A}) \arrow[rrrr, two heads, "\rho_{\boldsymbol{E}}"] \arrow[ddddd, "\tau"'] & & & & \Gamma_M(E) \arrow[ddddd, "\pi_*"] \\
& (\pi_*\hat{X})^h + \omega_E(\hat{X})^E + ad_{\omega_{\boldsymbol{E}}(\hat{X})} & \hat{X} \arrow[l, mapsto] & & \\
& \rho(\mathcal{X})^h - ad_{\alpha(\mathcal{X})^{\boldsymbol{E}}} & & X^h & \\
& \mathcal{X} \arrow[u, mapsto] & & X \arrow[u, mapsto] & \\
& \nabla_X & X \arrow[l, mapsto] & & \\
\mathrm{Der}(\mathcal{A}) \arrow[rrrr, two heads, "\rho"'] & & & & \Gamma(TM)
\end{tikzcd}$$

Indeed one has:

$$\mathrm{Der}(\mathcal{B}) \ni \mathfrak{X} = X^h + \underbrace{ad_Z}_{\in \mathrm{Int}(\mathcal{A})} + \underbrace{\omega_E(\hat{X})^E + ad_{\omega_{\boldsymbol{E}}(\hat{X})}}_{\in \mathcal{Z}_{\mathbf{Der}}(\mathcal{A})}$$