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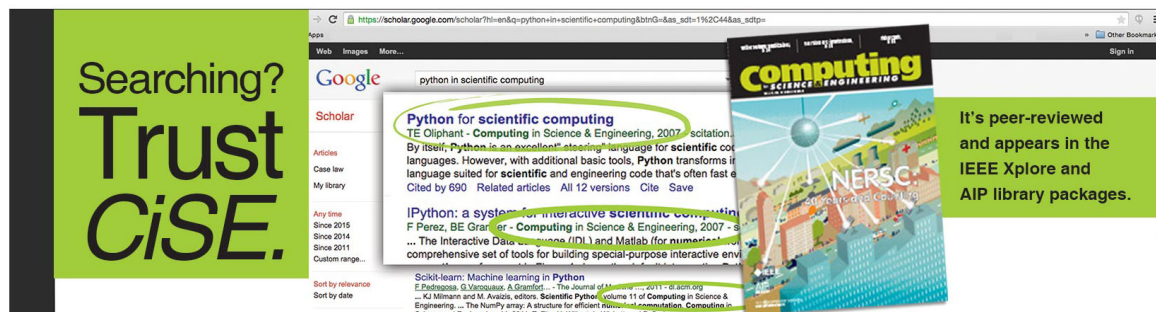
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# Becchi-Rouet-Stora-Tyutin structure for the mixed Weyl-diffeomorphism residual symmetry

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In this paper, we show the compatibility of the so-called “dressing field method,” which allows a systematic reduction of gauge symmetries, with the inclusion of diffeomorphisms in the Becchi-Rouet-Stora-Tyutin (BRST) algebra of a gauge theory. The robustness of the scheme is illustrated on two examples where Cartan connections play a significant role. The former is General Relativity, while the latter concerns the second-order conformal structure where one ends up with a BRST algebra handling both the Weyl residual symmetry and diffeomorphisms of spacetime. We thereby provide a geometric counterpart to the BRST cohomological treatment used in Boulanger [J. Math. Phys. **46**, 053508 (2005)] in the construction of a Weyl covariant tensor calculus. © 2016 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4943595>]

## I. INTRODUCTION

Modern field theory framework (classical and quantum), to this day so successful in describing nature from particles to cosmology, rests on few keystones, one of which being the notion of local symmetry. Elementary fields are subject to local transformations which are required to leave invariant the physical theory (the Lagrangian). These transformations thus form a symmetry of the theory. Requiring local symmetries is such a stringent restriction on the admissible theories and their content so as to justify Yang’s well known aphorism: “symmetry dictates interaction.”<sup>2</sup>

Confirmed fundamental theories distinguish two types of symmetries: “external” symmetries stemming from transformations of spacetime  $\mathcal{M}$ , that is, diffeomorphisms  $\text{Diff}(\mathcal{M})$ , and “internal” symmetries stemming from the action of a gauge group  $\mathcal{H}$ .

From a geometric standpoint,  $\mathcal{H}$  is (isomorphic to) the group of vertical automorphisms  $\text{Aut}_V(\mathcal{P})$  of a principal fiber bundle  $\mathcal{P}(\mathcal{M}, H)$  over spacetime  $\mathcal{M}$  with structure group  $H$ , itself a subgroup of the group of bundle automorphisms  $\text{Aut}(\mathcal{P})$ . While  $\text{Aut}_V(\mathcal{P})$  projects onto identity map of  $\mathcal{M}$ ,  $\text{Aut}(\mathcal{P})$  projects onto  $\text{Diff}(\mathcal{M})$ . The group  $\text{Aut}(\mathcal{P})$  offers a geometrical way to gather both internal and spacetime symmetries through the short exact sequence,

$$\{I\} \rightarrow \mathcal{H} \rightarrow \text{Aut}(\mathcal{P}) \rightarrow \text{Diff}(\mathcal{M}) \rightarrow \{\text{Id}_{\mathcal{M}}\}.$$

It is often easier to work with the infinitesimal version of the transformations, that is, with the Lie algebras of the symmetry groups. As a matter of fact, the infinitesimal gauge transformations are encoded in the so-called Becchi-Rouet-Stora-Tyutin (BRST) differential algebra of a gauge theory in which the infinitesimal local gauge parameter is turned into the Faddeev-Popov ghost field. This is algebraic in nature.<sup>3</sup> The Lie algebra of  $\text{Diff}(\mathcal{M})$  is the space of smooth vector fields  $\Gamma(T\mathcal{M})$  on  $\mathcal{M}$  with the Lie bracket of vector fields. This is geometric in nature. The corresponding infinitesimal symmetries are summed up in the following short exact sequence of Lie algebroids:

$$0 \rightarrow \text{Lie } \mathcal{H} \rightarrow \Gamma_H(\mathcal{P}) \rightarrow \Gamma(T\mathcal{M}) \rightarrow 0,$$

where  $\Gamma_H(\mathcal{P}) := \text{Lie } \text{Aut}(\mathcal{P})$  are the  $H$ -right-invariant vector fields on  $\mathcal{P}$ . Since both infinitesimal symmetries (internal/external or algebraic/geometric) ought to be unified in the central piece of

the above sequence, one expects to find a BRST treatment that encompasses both (pure) gauge transformations and diffeomorphisms.

This problem has been already addressed by several authors. Pioneering work is Ref. 4, and improved in Ref. 5. A refined work addressing the case of pure gravity is Ref. 6. From these papers, a general heuristic construction emerges that allows to alter a pure gauge BRST algebra so as to obtain a *shifted*<sup>7</sup> BRST algebra that describes both together gauge and diffeomorphism symmetries. Roughly, this shifting operation amounts to introducing the diffeomorphism ghost (vector field)  $\xi$  and modifying accordingly the “Russian formula” (or “horizontality condition”) and the BRST operator  $s$  itself.

In a previous work,<sup>8</sup> we proposed a systematic approach to reduce gauge symmetries by the *dressing field method*. Its relevance to recent controversies on the proton spin decomposition was advocated in Ref. 9, and its generalization to higher-order  $G$ -structures was suggested in Ref. 10 by application to the second-order conformal structure (see Ref. 11 for the general case). In the latter, it was also shown that the dressing field method adapts to the BRST framework: from an initial pure gauge BRST algebra one obtains, by dressing, a *reduced* BRST algebra describing residual gauge transformations and whose central object is the *composite ghost* which encapsulates the residual gauge symmetry (if any).

The aim of the present paper is to combine together the shifting operation and the dressing field method providing a *residual shifted* BRST algebra that describes both residual gauge transformations and diffeomorphisms, for which the central object is the *dressed shifted ghost*. In doing so, we address the issue of their compatibility and we provide the necessary condition for the two operations of shifting and dressing to commute between themselves. A pragmatic criterion for the failure of that condition is discussed. We then illustrate the construction on two examples: we shall first treat General Relativity (GR). Then, notably enough we shall deal with the second-order conformal structure where our scheme easily provides the BRST structure of the residual mixed Weyl + diffeomorphism symmetry out of the whole conformal + diffeomorphism symmetry.

The paper, which can be considered as a sequel of Ref. 10, is organized as follows. In Section II, we first recall the minimal definition of the standard BRST approach, and then give the heuristic construction allowing to include diffeomorphisms of spacetime  $\mathcal{M}$ . In Section III, we provide the basics of the dressing field method, exhibit the reduced BRST algebra and the associated composite ghost, and finally show the compatibility with the inclusion of diffeomorphisms. We also exhibit the necessary and sufficient condition securing the commutation of the shifting and dressing operations. Section IV deals with the simple application to GR. Section V details the rich example of the second-order conformal structure. Finally, we discuss our results and conclude in Section VI.

## II. MIXED BRST SYMMETRY GAUGE + DIFF: A GENERAL SCHEME

As just mentioned in the Introduction, the search for a single BRST algebra for the description of both gauge symmetries and diffeomorphisms has been quite early addressed. To the best of our knowledge, a pioneering work is Ref. 4. There, a first step was the recognition of the necessity to modify the so-called *horizontality condition*,<sup>12,13</sup> also named “Russian formula,”<sup>3</sup> encapsulating the standard BRST algebra.

Then Ref. 5 significantly improved the previous work by generalizing it (to a wide class of supersymmetric Einstein-Yang-Mills theories), but first and foremost—besides modifying the horizontality condition—the ghost field was modified as well. This change of generators has also been performed in Refs. 3, 6, and 14 in the pure gravitational case, where a BRST algebra for a Lorentz + diff symmetry (or *mixed* symmetry) in presence of a background field was given.<sup>15</sup> For further references on the subject, see, e.g., Refs. 16–18.

In this section however, we aim at giving the simplest heuristic construction allowing to modify the BRST algebra of a gauge (Yang-Mills) theory so as to include diffeomorphisms. Let us start by recalling the definition of a standard BRST gauge algebra.

## A. The BRST gauge algebra

The geometrical framework of Yang-Mills theories is that of a principal bundle  $\mathcal{P} = \mathcal{P}(\mathcal{M}, H)$  over an  $m$ -dimensional spacetime  $\mathcal{M}$ , with structure group  $H$  whose Lie algebra is  $\mathfrak{h}$ . Let  $\omega \in \Omega^1(\mathcal{P}, \mathfrak{h})$  be a (principal) connection 1-form on  $\mathcal{P}$  and let  $d\omega + \frac{1}{2}[\omega, \omega] =: \Omega \in \Omega^2(\mathcal{P}, \mathfrak{h})$  be its curvature; let  $\Psi$  denote a section of an associated bundle constructed out of a representation  $(V, \rho)$  of  $H$ .

In order to stick to the usual local description on an open set  $\mathcal{U} \subset \mathcal{M}$  (through a local trivializing section of the principal bundle  $\mathcal{P}$ ), the local connection 1-form gives the usual Yang-Mills gauge potential  $A$  with field strength  $F = dA + \frac{1}{2}[A, A]$ , and matter field  $\psi : \mathcal{U} \rightarrow V$ .

To the infinitesimal generators of gauge transformations is associated a Faddeev-Popov ghost field  $v : \mathcal{U} \rightarrow \mathfrak{h}^* \otimes \mathfrak{h}$ , where  $\mathfrak{h}^*$  is the dual Lie algebra  $\mathfrak{h}$  of  $H$ .

The BRST algebra of a non-abelian gauge field theory is well-known<sup>19</sup> to be defined as

$$sA = -Dv := -dv - [A, v], \quad sF = [F, v], \quad s\psi = -\rho_*(v)\psi, \quad sv = -\frac{1}{2}[v, v]. \quad (2.1)$$

Let us remind that the BRST operator  $s$  is an antiderivation which anticommutes with the exterior differential  $d$  and with odd differential forms, and  $[\ , \ ]$  is a graded bracket with respect to the form+ghost degrees. It is easily verified that  $s^2 = 0$ . We shall denote by *BRST* the above differential algebra.

This differential algebra can be incorporated into a larger differential algebra bigraded by the form and ghost degrees, whose nilpotent operator is  $\tilde{d} := d + s$  such that  $\tilde{d}^2 = 0$ . Accordingly, one may define the ‘‘algebraic connection’’<sup>20</sup>  $\tilde{A} := A + v$  of bidegree 1. Then, due to the definition of  $s$  for the pure gauge sector of differential algebra (2.1), one has the ‘‘Russian formula’’

$$\tilde{d}\tilde{A} + \frac{1}{2}[\tilde{A}, \tilde{A}] = F. \quad (2.2)$$

Indeed, by expanding (2.2) with respect to the ghost degree, one recovers the pure gauge sector of (2.1).

In the same way,  $\psi$  being a 0-form stands alone in the bigraded algebra  $\tilde{\psi} = \psi$ , and if one requires the following horizontality condition,<sup>5</sup>

$$\tilde{D}\tilde{\psi} := \tilde{d}\tilde{\psi} + \rho_*(\tilde{A})\tilde{\psi} = D\psi, \quad (2.3)$$

one recovers the BRST variation of the matter sector in (2.1).

These horizontality conditions (2.2) and (2.3) provide a very convenient starting point that allows a systematic and straightforward inclusion of diffeomorphisms in the BRST framework.

## B. Adding diffeomorphisms

The infinitesimal generators of diffeomorphisms are vector fields. According to the usual BRST setting, let us associate a ghost vector field  $\xi$  to the infinitesimal diffeomorphism symmetry. Denote by  $i_\xi$  its usual inner product on differential forms. The inner product is of degree  $-1$  but  $\xi$  has ghost number 1, then  $i_\xi$  is of total degree 0 and is thus a derivation. The Lie derivative acting on differential forms is accordingly an antiderivation of degree  $+1$  (graded Cartan formula) and yields the Cartan operation

$$L_\xi := i_\xi d - d i_\xi, \quad \text{with} \quad [L_\xi, i_\xi] = i_{[\xi, \xi]} \quad \text{and} \quad d L_\xi + L_\xi d = 0. \quad (2.4)$$

These identities will be extensively used in the sequel.

BRST gauge algebra (2.1) is equivalently recast into horizontality conditions (2.2) and (2.3). To obtain a new BRST algebra that also takes diffeomorphisms into account, one may accordingly modify these horizontality conditions. A systematic way to do so rests on the following ansatz:<sup>5,21</sup> the new BRST operator  $\sigma$  is defined through the intertwining

$$d + \sigma := e^{i_\xi} \tilde{d} e^{-i_\xi}, \quad (2.5)$$

where  $e^{i\xi} = 1 + i\xi + \frac{1}{2}i\xi i\xi + \dots$  is the formal power series of the exponential of  $i\xi$  and is shown<sup>5</sup> to be a morphism of the exterior algebra of differential forms and also a Lie algebra homomorphism, namely,  $e^{i\xi}[\alpha, \beta] = [e^{i\xi}\alpha, e^{i\xi}\beta]$ .

With ansatz (2.5), Russian formula (2.2) becomes

$$(d + \sigma)(e^{i\xi}\tilde{A}) + \frac{1}{2} [e^{i\xi}\tilde{A}, e^{i\xi}\tilde{A}] = e^{i\xi}F. \tag{2.6}$$

One thus readily computes for the algebraic connection

$$e^{i\xi}\tilde{A} = (1 + i\xi)(A + v) =: A + v + i\xi A,$$

where, according to the ghost degree, we are led to define the *shifted ghost*,

$$v' := v + i\xi A. \tag{2.7}$$

In more detail, Russian formula (2.6) becomes

$$(d + \sigma)(A + v') + \frac{1}{2}[A + v', A + v'] = F + i\xi F + \frac{1}{2}i\xi i\xi F. \tag{2.8}$$

By sorting out the terms according to the bigrading one gets in a row:

Degree (2,0) corresponds to the usual Cartan structure equation,  $dA + \frac{1}{2}[A, A] = F$ .

Degree (1,1) gives rise to

$$\sigma A = -Dv' + i\xi F := -dv' - [A, v'] + i\xi F. \tag{2.9}$$

From these two one easily finds  $\sigma F = [F, v'] - D(i\xi F)$ . Degree (0,2) yields

$$\sigma v' = -\frac{1}{2}[v', v'] + \frac{1}{2}i\xi i\xi F. \tag{2.10}$$

In the same way, the horizontality condition for the matter fields reads

$$(d + \sigma)e^{i\xi}\psi + e^{i\xi}\rho_*(A + v)\psi = e^{i\xi}D\psi. \tag{2.11}$$

In ghost degree 1 one finds

$$\sigma\psi = -\rho_*(v') + i\xi D\psi. \tag{2.12}$$

Moreover, requiring the nilpotency  $\sigma^2 = 0$  on the generators  $(A, \psi, v')$  leads to

$$\sigma\xi = \frac{1}{2}[\xi, \xi], \tag{2.13}$$

where  $[\xi, \xi]$  is the Lie bracket of vector fields.<sup>22,43</sup> Accordingly, the new shifted BRST algebra with generators  $(A, \psi, v', \xi)$  and  $\sigma$ -operation describing both infinitesimal transformations gauge + Diff( $\mathcal{M}$ ) is defined by

$$\begin{aligned} \sigma A &= -Dv' + i\xi F, & \sigma F &= [F, v'] - D(i\xi F), & \sigma\psi &= -\rho_*(v') + i\xi D\psi, \\ \sigma v' &= -\frac{1}{2}[v', v'] + \frac{1}{2}i\xi i\xi F, & \sigma\xi &= \frac{1}{2}[\xi, \xi]. \end{aligned} \tag{2.14}$$

It is somewhat hard to disentangle the two symmetries with the above presentation of the shifted algebra. But one can give an alternative presentation which relies on the fact that the shifted ghost  $v'$  assumes form (2.7). Having taken this into account, one can give the action of  $\sigma$  on the generators  $(A, \psi, v, \xi)$ , and (2.14) becomes

$$\begin{aligned} \sigma A &= -Dv + L_\xi A, & \sigma F &= [F, v] + L_\xi F, & \sigma\psi &= -\rho_*(v)\psi + L_\xi\psi, \\ \sigma v &= -\frac{1}{2}[v, v] + L_\xi v, & \sigma\xi &= \frac{1}{2}[\xi, \xi]. \end{aligned} \tag{2.15}$$

This presentation shows that  $\sigma = s + L_\xi$ , so that the actions of gauge and diffeomorphisms symmetries turn out to be decoupled on this set of generators. For convenience and subsequent purpose, let us denote  $BRST^\xi$  either of the two presentations (2.14) or (2.15) for the resulting shifted BRST algebra.

### III. THE DRESSING FIELD METHOD AND DIFFEOMORPHISM SYMMETRY

A systematic approach to reduce gauge symmetries has been proposed and applied to various examples in Refs. 8–11 (see also Ref. 23). It is already compatible with the BRST framework, as

it will be briefly outlined in the following. It remains to study the compatibility with the shifting procedure described above. This will be the main issue of this section.

### A. The dressing field method: A primer

The gauge group of a Yang-Mills theory is defined as  $\mathcal{H} := \{\gamma : \mathcal{U} \rightarrow H\}$  and it carries the canonical action on itself  $\gamma_1^{\gamma_2} = \gamma_2^{-1}\gamma_1\gamma_2$  for any  $\gamma_1, \gamma_2 \in \mathcal{H}$ . It, respectively, acts on gauge potential, field strength, and matter fields according to

$$A^\gamma = \gamma^{-1}A\gamma + \gamma d\gamma, \quad F^\gamma = \gamma^{-1}F\gamma, \quad \text{and} \quad \psi^\gamma = \rho(\gamma^{-1})\psi. \quad (3.1)$$

Suppose the theory also contains a (Lie) group-valued field  $u : \mathcal{U} \rightarrow G'$  defined by its transformation under  $\mathcal{H}' = \{\gamma' : \mathcal{U} \rightarrow H'\}$ , where  $H' \subseteq H$  is a subgroup, according to

$$u^{\gamma'} := \gamma'^{-1}u, \quad \gamma' \in \mathcal{H}'.$$

One can define the following *composite fields*:<sup>24</sup>

$$\widehat{A} := u^{-1}Au + u^{-1}du, \quad \widehat{F} := u^{-1}Fu, \quad \text{and} \quad \widehat{\psi} := \rho(u^{-1})\psi. \quad (3.2)$$

The Cartan structure equation still holds,  $\widehat{F} = d\widehat{A} + \frac{1}{2}[\widehat{A}, \widehat{A}]$ .

Despite the formal similarity with (3.1), the composite fields given in (3.2) are not mere gauge transformations since  $u \notin \mathcal{H}$ , as is testified by its transformation property under  $\mathcal{H}'$  and the fact that, in general,  $G'$  may be different from  $H$ . This fact clearly implies also that the composite field  $\widehat{A}$  does no longer belong to the space of local connections.

Finally, as it can be easily checked, composite fields (3.2) are  $\mathcal{H}'$ -invariant and are only subject to residual gauge transformation laws in  $\mathcal{H} \setminus \mathcal{H}'$ .<sup>25</sup> In the case where  $H' = H$ , these composite fields are  $\mathcal{H}$ -gauge invariants and may become good candidates to be observables.

It is easy to show that the BRST algebra pertaining to a pure gauge theory is modified by the dressing as

$$s\widehat{A} = -\widehat{D}\widehat{v} = -d\widehat{v} - [\widehat{A}, \widehat{v}], \quad s\widehat{F} = [\widehat{F}, \widehat{v}], \quad s\widehat{\psi} = -\rho_*(\widehat{v})\widehat{\psi}, \quad s\widehat{v} = -\frac{1}{2}[\widehat{v}, \widehat{v}], \quad (3.3)$$

upon defining the *composite ghost*

$$\widehat{v} := u^{-1}vu + u^{-1}su. \quad (3.4)$$

To the best of our knowledge, first occurrences of such a change of generator in a BRST setting for specific cases can be found in Refs. 26 and 27. Let us denote by  $\widehat{\text{BRST}}$  the above dressed BRST algebra. Results (3.3) and (3.4) are actually strictly formal. They do not depend on the fact that  $u$  is a dressing field, namely, on an explicit expression of the variation  $su$ . See Ref. 10 for a detailed discussion on this point. When  $u$  is indeed a dressing field, (3.4) may thus encode the infinitesimal residual gauge symmetry, if any. If  $\widehat{v} = 0$ , obviously the differential algebra  $\widehat{\text{BRST}}$  becomes trivial, thus expressing the whole gauge invariance of the composite fields.

As in the usual case, upon defining the *composite algebraic connection*

$$\widehat{A} + \widehat{v} = u^{-1}\widetilde{A}u + u^{-1}\widetilde{d}u,$$

dressed algebra (3.3) can be compactly encapsulated into the following two horizontality conditions:

$$\begin{aligned} (d+s)(\widehat{A} + \widehat{v}) + \frac{1}{2}[\widehat{A} + \widehat{v}, \widehat{A} + \widehat{v}] &= \widehat{F}, \\ (d+s)\widehat{\psi} + \rho_*(\widehat{A} + \widehat{v})\widehat{\psi} &= \widehat{D}\widehat{\psi}. \end{aligned} \quad (3.5)$$

### B. Shifting and dressing

We now investigate the compatibility of the two operations of shifting (adding diffeomorphisms) and dressing. Two approaches are available to us.



First, one can proceed as for the dressing of the initial BRST algebra in order to obtain the algebra  $\widehat{\text{BRST}}$ . This amounts to expressing the initial gauge variables  $(A, F, \psi)$  as functions of the dressed variables  $(\widehat{A}, \widehat{F}, \widehat{\psi})$  and the dressing field  $u$ , and replacing them into the *first* presentation (2.14) of  $\text{BRST}^\xi$ . One then obtains

$$\begin{aligned} \sigma \widehat{A} &= -\widehat{D}\widehat{v}' + i_\xi \widehat{F}, & \sigma \widehat{F} &= [\widehat{F}, \widehat{v}'] - \widehat{D}(i_\xi \widehat{F}), & \sigma \widehat{\psi} &= -\rho_*(\widehat{v}')\widehat{\psi} + i_\xi \widehat{D}\widehat{\psi}, \\ \sigma \widehat{v}' &= -\frac{1}{2}[\widehat{v}', \widehat{v}'] + \frac{1}{2}i_\xi i_\xi \widehat{F}, & \sigma \xi &= \frac{1}{2}[\xi, \xi], \end{aligned} \tag{3.6}$$

with the composite shifted ghost defined by

$$\widehat{v}' := u^{-1}v'u + u^{-1}\sigma u. \tag{3.7}$$

Let us denote by  $\widehat{\text{BRST}}^\xi$  this algebra. Since it assumes the same formal presentation as (2.14), one verifies that  $\sigma^2 = 0$  on  $(\widehat{A}, \widehat{\psi}, \widehat{v}')$  implies  $\sigma \xi = \frac{1}{2}[\xi, \xi]$  in the same way.<sup>28</sup>

The second possible route amounts to modifying dressed horizontality conditions (3.5) by using ansatz (2.5),

$$\begin{aligned} (d + \sigma)e^{i\xi}(\widehat{A} + \widehat{v}) + \frac{1}{2}[e^{i\xi}(\widehat{A} + \widehat{v}), e^{i\xi}(\widehat{A} + \widehat{v})] &= e^{i\xi}\widehat{F}, \\ (d + \sigma)e^{i\xi}\widehat{\psi} + e^{i\xi}\rho_*(\widehat{A} + \widehat{v})\psi &= e^{i\xi}\widehat{D}\widehat{\psi}. \end{aligned} \tag{3.8}$$

Expansion according to the ghost degree provides, besides the Cartan structure equation for  $\widehat{F}$ ,

$$\begin{aligned} \sigma \widehat{A} &= -\widehat{D}\widehat{v}' + i_\xi \widehat{F}, & \sigma \widehat{F} &= [\widehat{F}, \widehat{v}'] - \widehat{D}(i_\xi \widehat{F}), & \sigma \widehat{\psi} &= -\rho_*(\widehat{v}')\widehat{\psi} + i_\xi \widehat{D}\widehat{\psi}, \\ \sigma \widehat{v}' &= -\frac{1}{2}[\widehat{v}', \widehat{v}'] + \frac{1}{2}i_\xi i_\xi \widehat{F}, & \sigma \xi &= \frac{1}{2}[\xi, \xi], \end{aligned} \tag{3.9}$$

with shifted composite ghost defined by

$$\widehat{v}' := \widehat{v} + i_\xi \widehat{A}. \tag{3.10}$$

Let us denote  $\widehat{\text{BRST}}^\xi$  this algebra. Due to (3.10)  $\widehat{\text{BRST}}^\xi$  also assumes the second presentation, see (2.15),

$$\begin{aligned} \sigma \widehat{A} &= -\widehat{D}\widehat{v} + L_\xi \widehat{A}, & \sigma \widehat{F} &= [\widehat{F}, \widehat{v}] + L_\xi \widehat{F}, & \sigma \psi &= -\rho_*(\widehat{v})\psi + L_\xi \psi, \\ \sigma \widehat{v} &= -\frac{1}{2}[\widehat{v}, \widehat{v}] + L_\xi \widehat{v}, & \sigma \xi &= \frac{1}{2}[\xi, \xi]. \end{aligned} \tag{3.11}$$

The above form clearly shows the decoupling between the residual gauge symmetry  $(\widehat{v})$  and the diffeomorphisms  $(\xi)$ .

The question now is to see whether the operations of shifting and dressing do commute, that is, whether  $\widehat{\text{BRST}}^\xi$  (3.6) is the same as  $\widehat{\text{BRST}}^\xi$  (3.9). This is clearly the case if  $\widehat{v}' = \widehat{v}$  and the latter is true if and only if the condition

$$\sigma u = (s + L_\xi)u \tag{3.12}$$

is satisfied. Indeed,

$$\begin{aligned} \widehat{v}' &= u^{-1}v'u + u^{-1}\sigma u \stackrel{(3.12)}{=} u^{-1}(v + i_\xi A)u + u^{-1}(s + L_\xi)u \\ &= \underbrace{u^{-1}vu + u^{-1}su}_{=: \widehat{v}} + \underbrace{u^{-1}i_\xi Au + u^{-1}i_\xi du}_{=: i_\xi(u^{-1}Au + u^{-1}du) = i_\xi \widehat{A}} \end{aligned}$$

since  $i_\xi u = 0$ . Hence, one has proven that  $\widehat{v}' = \widehat{v} + i_\xi \widehat{A} =: \widehat{v}'$ . Conversely,  $\widehat{v}' = \widehat{v}$  infers (3.12) as it can be easily shown. Notice that the second presentation of  $\widehat{\text{BRST}}^\xi$  (3.6) is exactly (3.11) when equation (3.12) holds: the ghost mentioned in Note<sup>28</sup> turns out to be  $(u^{-1}vu + u^{-1}\sigma u) - u^{-1}L_\xi u = u^{-1}vu + u^{-1}su =: \widehat{v}$ . Symbolically, one may write [shifting, dressing] = 0. On the other hand, if

$$\sigma u \neq (s + L_\xi)u, \tag{3.13}$$

then [shifting, dressing]  $\neq$  0. This can be summarized in the diagram reported in Fig. 1.

A criterion to decide in advance whether (3.12) or (3.13) holds is the following. By definition, the gauge transformation of a dressing field  $u$  is known and given by the  $s$ -operation. If  $u$  has no (free)

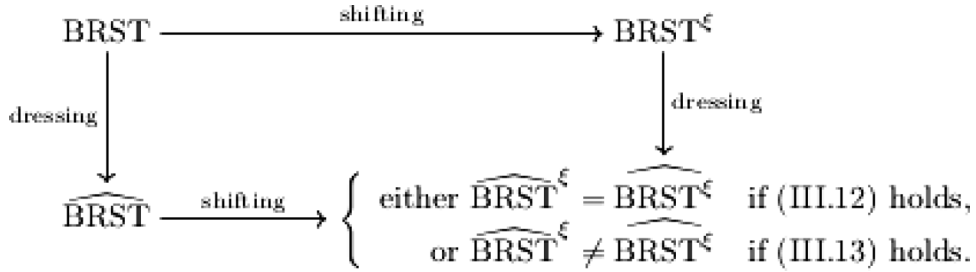


FIG. 1. Commutativity or not between the dressing and the shifting operations.

spacetime index it is a 0-form, so that its transformation under  $\text{Diff}(\mathcal{M})$  is actually given by the Lie derivative on differential forms,  $L_\xi = i_\xi d - di_\xi$ . Therefore, (3.12) holds in that case indeed. On the contrary, if  $u$  carries (free) spacetime indices, its transformation under  $\text{Diff}(\mathcal{M})$  is given by the Lie derivative  $\mathcal{L}_\xi$  of tensors<sup>29</sup> or pseudo-tensors (as the case may be), and accordingly, (3.13) holds.

The two examples treated respectively in Secs. IV and V show that the differential algebra which implements the correct infinitesimal gauge + diffeomorphism symmetries will be  $\widehat{\text{BRST}}^\xi$  (3.6), that is, the one obtained by shifting first and then dressing. Notice that it provides the most general form of ghost (3.7) which takes into account the possible tensorial character of the dressing field  $u$  through the inhomogeneous term  $u^{-1}\sigma u$ .

The two following examples concern the gauge formulation of pure gravitational theories (GR and conformal gravity), see, e.g., Ref. 30. The natural language will be that of Cartan geometry,<sup>31,32</sup> and the gravitational gauge potential is given by a (local) Cartan connection.

**IV. EXAMPLE: THE GEOMETRY OF GENERAL RELATIVITY**

**A. Mixed Lorentz +  $\text{Diff}(\mathcal{M})$  symmetry**

The geometry of GR, seen as a gauge theory, is a Cartan geometry  $(\mathcal{P}, \varpi)$  where  $\mathcal{P}(\mathcal{M}, H)$  is a principal bundle with  $H = SO(1, m - 1)$  the Lorentz group, and  $\varpi \in \Omega^1(\mathcal{U}, \mathfrak{g})$  is a (local) Cartan connection on  $\mathcal{U} \subset \mathcal{M}$  with values in  $\mathfrak{g}$  the Lie algebra of the Poincaré group  $G = SO(1, m - 1) \times \mathbb{R}^{(1, m-1)}$ . One has the matrix representation of a gravitational field on spacetime  $\mathcal{M}$ ,

$$\varpi = \begin{pmatrix} A & \theta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A^a_{b,\mu} & e^a_\mu \\ 0 & 0 \end{pmatrix} dx^\mu,$$

with  $A \in \Omega^1(\mathcal{U}, \mathfrak{h})$  the Lorentz connection (or spin connection) and  $\theta \in \Omega^1(\mathcal{U}, \mathbb{R}^{(1, m-1)})$  the soldering form (or vielbein 1-form). The Greek indices are spacetime indices, while Latin indices are “internal” (gauge)-Minkowski indices. The curvature is

$$\Omega = d\varpi + \frac{1}{2}[\varpi, \varpi] = d\varpi + \varpi \wedge \varpi \rightarrow \begin{pmatrix} F & \Theta \\ 0 & 0 \end{pmatrix} := \begin{pmatrix} dA + A \wedge A & d\theta + A \wedge \theta \\ 0 & 0 \end{pmatrix},$$

with  $F$  the curvature 2-form of  $A$  and  $\Theta$  the torsion 2-form. The Lorentz ghost is

$$v = \begin{pmatrix} v_L & 0 \\ 0 & 0 \end{pmatrix} : \mathcal{U} \rightarrow \mathfrak{h}^* \otimes \mathfrak{h},$$

and the associated BRST algebra reads

$$\begin{aligned} s\varpi &= -dv - [\varpi, v] \rightarrow \begin{pmatrix} sA & s\theta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -Dv_L & -v_L\theta \\ 0 & 0 \end{pmatrix} := \begin{pmatrix} -dv_L - [\varpi, v_L] & -v_L\theta \\ 0 & 0 \end{pmatrix}, \\ s\Omega &= [\Omega, v] \rightarrow \begin{pmatrix} sF & s\Theta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} [F, v_L] & -v_L\Theta \\ 0 & 0 \end{pmatrix}, \quad sv = -v^2 \rightarrow \begin{pmatrix} sv_L & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -v_L^2 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$



This algebra handles the infinitesimal SO-gauge transformations of the variables of the theory. Defining the algebraic Cartan connection  $\tilde{\omega} := \omega + v$ , one recovers the BRST algebra for GR from the horizontality condition  $\tilde{d}\tilde{\omega} + \frac{1}{2}[\tilde{\omega}, \tilde{\omega}] = \Omega$ .

Let us use the results of Section II B and write the shifted algebra  $BRST^\xi$  for GR. The Lorentz ghost is shifted by the Cartan connection according to

$$v' = v + i_\xi \omega = \begin{pmatrix} v_L + i_\xi A & i_\xi \theta \\ 0 & 0 \end{pmatrix} =: \begin{pmatrix} v'_L & i_\xi \theta \\ 0 & 0 \end{pmatrix}$$

and thus acquiring an effective ghost term  $i_\xi \theta$  in the translation entry. Hence, according to the general scheme given in Section II A, we readily get

$$\sigma \omega = -Dv' + i_\xi \Omega = -Dv + L_\xi \omega,$$

where  $D = d + [A, \cdot]$ . In matrix notation, it reads

$$\begin{aligned} \begin{pmatrix} \sigma A & \sigma \theta \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} -dv'_L & -d(i_\xi \theta) \\ 0 & 0 \end{pmatrix} - \left[ \begin{pmatrix} A & \theta \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} v'_L & i_\xi \theta \\ 0 & 0 \end{pmatrix} \right] + \begin{pmatrix} i_\xi F & i_\xi \Theta \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -Dv'_L + i_\xi F & -D(i_\xi \theta) - v'_L \theta + i_\xi \Theta \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -Dv_L + L_\xi \omega & -v_L \theta + L_\xi \theta \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{4.1}$$

Similarly, we have for the curvature

$$\sigma \Omega = [\Omega, v'] - D(i_\xi \Omega) = [\Omega, v] + L_\xi \Omega, \text{ i.e. } \begin{pmatrix} \sigma F & \sigma \Theta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} [F, v_L] + L_\xi F & -v_L \Theta + L_\xi \Theta \\ 0 & 0 \end{pmatrix}$$

and for the ghost field

$$\sigma v' = -v'^2 + \frac{1}{2} i_\xi i_\xi \Omega, \text{ i.e. } \begin{pmatrix} \sigma v'_L & \sigma i_\xi \theta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -v'^2_L + \frac{1}{2} i_\xi i_\xi F & -v'_L i_\xi \theta + \frac{1}{2} i_\xi i_\xi \Theta \\ 0 & 0 \end{pmatrix}. \tag{4.2}$$

One can also check that one recovers the presentation

$$\sigma v = -v^2 + L_\xi v, \text{ i.e. } \begin{pmatrix} \sigma v_L & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -v^2_L + L_\xi v_L & 0 \\ 0 & 0 \end{pmatrix}.$$

Formulas (4.1) and (4.2), respectively, reproduce equations (8a-b) and (8-c) of “parallel transport” given in Ref. 6 (once the background connection has been reabsorbed in the redefinition of the generators). See also Ref. 14.

The algebra  $BRST^\xi$  handles the full mixed symmetry Lorentz + Diff( $\mathcal{M}$ ) of GR. Now, thanks to the dressing field method, it is possible to reduce it so as to obtain a strict Diff( $\mathcal{M}$ ) algebra, in other words, to get the diffeomorphism symmetry only.

**B. Residual Diff( $\mathcal{M}$ ) symmetry**

As is detailed in Refs. 8, 10, and 11, the dressing field in GR is nothing but the vielbein,  $u := \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix}$ ,  $e = e^a{}_\mu$ . The composite fields are, for the connection

$$\hat{\omega} = u^{-1} \omega u + u^{-1} du = \begin{pmatrix} e^{-1} A e + e^{-1} de & e^{-1} \theta \\ 0 & 0 \end{pmatrix} =: \begin{pmatrix} \Gamma & dx \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \Gamma^\rho{}_{\nu, \mu} & \delta^\rho_\mu \\ 0 & 0 \end{pmatrix} dx^\mu,$$

where the  $\Gamma$ 's are the Christoffel symbols of a metric connection for the metric  $g = e^T \eta e$ , while for the curvature

$$\hat{\Omega} = u^{-1} \Omega u = \begin{pmatrix} e^{-1} F e & e^{-1} \Theta \\ 0 & 0 \end{pmatrix} =: \begin{pmatrix} R & T \\ 0 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} R^\rho{}_{\nu, \mu \sigma} & T^\rho{}_{\nu, \mu \sigma} \\ 0 & 0 \end{pmatrix} dx^\mu \wedge dx^\sigma,$$

where  $R$  and  $T$  are the corresponding Riemann and torsion tensors, respectively.

Let us use the results of Section III B and write the algebra  $\widehat{\text{BRST}}^\xi$  for GR. The composite shifted ghost is  $\widehat{v}' = u^{-1}v'u + u^{-1}\sigma u$ , which requires to know  $\sigma u$  explicitly, that is,  $\sigma e$ . The latter can be obtained from  $\sigma\theta$ , using the natural assumption that  $\sigma x = 0$ , where  $x$  is considered as a background system of local coordinates pertaining to the differentiable structure of the spacetime  $\mathcal{M}$ . One has

$$\begin{aligned} \sigma\theta &= s\theta + L_\xi\theta, \\ \sigma(e \cdot dx) &= s(e \cdot dx) + (i_\xi d - di_\xi)(e \cdot dx) = (se) \cdot dx + i_\xi(de \wedge dx) - d(e \cdot \xi), \\ \sigma e \cdot dx &= (se + i_\xi de + e \cdot \partial\xi) \cdot dx. \end{aligned} \tag{4.3}$$

Here “ $\cdot$ ” is a shorthand for Greek index summation, e.g.,  $\theta = e \cdot dx := e^\alpha_\mu dx^\mu$ . One has then

$$\sigma u = \begin{pmatrix} \sigma e & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} se + i_\xi de + e \cdot \partial\xi & 0 \\ 0 & 0 \end{pmatrix}.$$

According to Ref. 17, let us define  $v_\xi = \begin{pmatrix} \partial\xi & 0 \\ 0 & 0 \end{pmatrix}$ , and since  $i_\xi u = 0$ , we finally obtain

$$\sigma u = su + L_\xi u + uv_\xi. \tag{4.4}$$

We are thus in a case where (3.13) holds, so  $\widehat{\text{BRST}}^\xi \neq \widehat{\text{BRST}}^\xi$  (the diagram does not commute). Remark that  $L_\xi u + uv_\xi := \mathcal{L}_\xi u$  is the Lie derivative of the tensor  $u \sim e^\alpha_\mu$  (that is obvious since  $L_\xi\theta = (\mathcal{L}_\xi e^\alpha_\mu)dx^\mu$ ). This is a situation discussed at the very end of Section III B: the dressing  $u$  has a free spacetime index and is a tensor, so its variation under  $\text{Diff}(\mathcal{M})$  is indeed given by  $\mathcal{L}_\xi$ . The composite shifted Lorentz ghost is thus

$$\begin{aligned} \widehat{v}' &= u^{-1}v'u + u^{-1}\sigma u = u^{-1}(v + i_\xi\varpi)u + u^{-1}(su + L_\xi u + uv_\xi) \\ &= (u^{-1}vu + u^{-1}su) + (u^{-1}i_\xi\varpi u + u^{-1}i_\xi du) + v_\xi, \\ \widehat{v}' &= \widehat{v} + i_\xi\widehat{\varpi} + v_\xi. \end{aligned}$$

Of course, as expected  $\widehat{v}' \neq \widehat{v} + i_\xi\widehat{\varpi} =: \widehat{v}'$ . Moreover, in the situation at hand, we have from the initial Lorentz BRST algebra,  $se = -v_L e$ . Therefore, the composite Lorentz ghost vanishes,

$$\widehat{v} = u^{-1}vu + u^{-1}su = \begin{pmatrix} e^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_L & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} e^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} se & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

This means that the algebra  $\widehat{\text{BRST}}$  is trivial in this case,

$$s\widehat{\varpi} = \begin{pmatrix} s\Gamma & sdx \\ 0 & 0 \end{pmatrix} = 0 \quad \text{and} \quad s\widehat{\Omega} = \begin{pmatrix} sR & sT \\ 0 & 0 \end{pmatrix} = 0 \quad (\text{and obviously } s\widehat{v} = 0).$$

This expresses the Lorentz invariance of the composite fields. This is an instance of complete gauge neutralization as described in Ref. 10.

At last, the composite shifted Lorentz ghost is thus simply

$$\widehat{v}' = i_\xi\widehat{\varpi} + v_\xi = \begin{pmatrix} i_\xi\Gamma & i_\xi dx \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \partial\xi & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \Gamma^\rho_{\nu,\mu}\xi^\mu + \partial_\nu\xi^\rho & \xi^\rho \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \nabla_\nu\xi^\rho & \xi^\rho \\ 0 & 0 \end{pmatrix}. \tag{4.5}$$

It is worth noticing that it depends *covariantly* on the diffeomorphism ghost only. Straightforward matrix calculations now easily provide the algebra  $\widehat{\text{BRST}}^\xi$ . First,

$$\begin{aligned} \sigma\widehat{\varpi} &= -D\widehat{v}' + i_\xi\widehat{\Omega} \\ &= -d(i_\xi\widehat{\varpi} + v_\xi) - [\widehat{\varpi}, i_\xi\widehat{\varpi}] - [\widehat{\varpi}, v_\xi] + i_\xi d\widehat{\varpi} + i_\xi \frac{1}{2}[\widehat{\varpi}, \widehat{\varpi}] = (i_\xi d - di_\xi)\widehat{\varpi} - [\widehat{\varpi}, v_\xi] - dv_\xi \\ &= L_\xi\widehat{\varpi} - [\widehat{\varpi}, v_\xi] - dv_\xi =: \mathcal{L}_\xi\widehat{\varpi} - dv_\xi. \end{aligned} \tag{4.6}$$

In matrix form the last line reads

$$\begin{aligned} \sigma \widehat{\omega} &= \begin{pmatrix} \sigma \Gamma & \sigma dx \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} L_\xi \Gamma - [\Gamma, \partial \xi] - d\partial \xi & L_\xi dx - \partial \xi \cdot dx \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \xi^\alpha \partial_\alpha \Gamma^\rho_{\mu\nu} + \Gamma^\rho_{\alpha\nu} \partial_\mu \xi^\alpha + \Gamma^\rho_{\mu\alpha} \partial_\nu \xi^\alpha - \partial_\alpha \xi^\rho \Gamma^\alpha_{\mu\nu} + \partial_\mu (\partial_\nu \xi^\rho) & 0 \\ 0 & 0 \end{pmatrix} dx^\mu \\ &=: \begin{pmatrix} \mathcal{L}_\xi \Gamma^\rho_{\mu\nu} + \partial_\mu (\partial_\nu \xi^\rho) & 0 \\ 0 & 0 \end{pmatrix} dx^\mu \end{aligned}$$

which gives the Lie derivative of the Christoffel symbols. Also

$$\begin{aligned} \sigma \widehat{\Omega} &= [\widehat{\Omega}, \widehat{v}'] - \widehat{D}(i_\xi \widehat{\Omega}) \\ &= [\widehat{\Omega}, i_\xi \widehat{\omega}] + [\widehat{\Omega}, v_\xi] - di_\xi \widehat{\Omega} - [\widehat{\omega}, i_\xi \widehat{\Omega}] = i_\xi d\widehat{\Omega} - di_\xi \widehat{\Omega} + [\widehat{\Omega}, v_\xi] \\ &= L_\xi \widehat{\Omega} + [\widehat{\Omega}, v_\xi] =: \mathcal{L}_\xi \widehat{\Omega}, \end{aligned} \tag{4.7}$$

where in the course of the computation the Bianchi identity  $i_\xi d\widehat{\Omega} = -i_\xi [\widehat{\omega}, \widehat{\Omega}]$  has been used in the third equality. In matrix notation, one thus gets

$$\sigma \widehat{\Omega} = \begin{pmatrix} \sigma R & \sigma T \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} L_\xi R + [R, \partial \xi] & L_\xi T - \partial \xi \cdot T \\ 0 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mathcal{L}_\xi R^\rho_{\nu, \mu\sigma} & \mathcal{L}_\xi T^\rho_{\mu\sigma} \\ 0 & 0 \end{pmatrix} dx^\mu \wedge dx^\sigma$$

from which one reads the well-known Lie derivatives of the Riemann and torsion tensors, respectively,

$$\begin{aligned} \mathcal{L}_\xi R^\rho_{\nu, \mu\sigma} &= \xi^\alpha \partial_\alpha R^\rho_{\nu, \mu\sigma} + R^\rho_{\nu, \alpha\sigma} \partial_\mu \xi^\alpha + R^\rho_{\nu, \mu\alpha} \partial_\sigma \xi^\alpha + R^\rho_{\alpha, \mu\sigma} \partial_\nu \xi^\alpha - \partial_\alpha \xi^\rho R^\alpha_{\nu, \mu\sigma}, \\ \mathcal{L}_\xi T^\rho_{\mu\sigma} &= \xi^\alpha \partial_\alpha T^\rho_{\mu\sigma} + T^\rho_{\alpha\sigma} \partial_\mu \xi^\alpha + T^\rho_{\mu\alpha} \partial_\sigma \xi^\alpha - \partial_\alpha \xi^\rho T^\alpha_{\mu\sigma}. \end{aligned} \tag{4.8}$$

At this stage, since we know that  $\sigma^2 = 0$  on  $\widehat{\omega}$  (and  $\widehat{\Omega}$ ) requires (2.13),<sup>33</sup> the action of  $\sigma$  on the relevant variables  $\widehat{\omega}$ ,  $\widehat{\Omega}$ , and  $\xi$  ( $\widehat{v}$  being vanishing) is known. But, for the sake of completeness, we nevertheless write the last relation

$$\begin{aligned} \sigma \widehat{v}' &= -\frac{1}{2} [\widehat{v}', \widehat{v}'] + \frac{1}{2} i_\xi i_\xi \widehat{\Omega} \\ &= L_\xi i_\xi \widehat{\omega} - [i_\xi \widehat{\omega}, v_\xi] + L_\xi v_\xi - \frac{1}{2} [v_\xi, v_\xi] - i_\xi dv_\xi - i_{\frac{1}{2}[\xi, \xi]} \widehat{\omega} \\ &= \mathcal{L}_\xi \widehat{v}' - i_\xi dv_\xi - i_{\frac{1}{2}[\xi, \xi]} \widehat{\omega}. \end{aligned}$$

This is redundant with (4.6) and provides only an indirect checking on the variation  $\sigma \xi$  to secure that  $\sigma^2 \widehat{v}' = 0$ .

The algebra  $\widehat{\text{BRST}}^\xi$  thus gives the correct transformations of the Christoffel symbols, the Riemann, and the torsion tensors under infinitesimal diffeomorphisms. Since in this case the Lorentz-gauge symmetry is neutralized, the shifted algebra  $\widehat{\text{BRST}}^\xi$  handles the spacetime symmetry only as a residual symmetry.

### V. EXAMPLE: THE SECOND-ORDER CONFORMAL STRUCTURE

The joint treatment of Weyl symmetry and diffeomorphisms within the BRST framework has been addressed (algebraically) by several authors, see, e.g., Refs. 34–37 or later in Refs. 1, 38, and 39. We shall use here the geometrical view that the Weyl symmetry is involved in the second order conformal structure which is well described in the framework of a Cartan geometry. We shall apply the scheme depicted in Section III to the corresponding BRST algebra.

We refer the reader to Ref. 31 and to Refs. 32 and 40 for mathematical details. Here, we just sketch the necessary material to follow our scheme, but we also heavily rely on results detailed in Ref. 10.

The whole structure is modeled on the Klein pair of Lie groups  $(G, H)$  where  $G$  is the Möbius group and  $H$  is the subgroup such that  $G/H \simeq (S^{m-1} \times S^1)/\mathbb{Z}^2$  (the compactified Minkowski space

considered as homogeneous space, see, e.g., Ref. 41) and has the following factorized matrix presentation:

$$H = K_0 K_1 = \left\{ \begin{pmatrix} z & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & z^{-1} \end{pmatrix} \begin{pmatrix} 1 & r & \frac{1}{2} r r^t \\ 0 & \mathbb{1} & r^t \\ 0 & 0 & 1 \end{pmatrix} \middle| z \in \mathbb{W} = \mathbb{R}_+^*, S \in SO(1, m - 1), r \in \mathbb{R}^{m*} \right\}. \tag{5.1}$$

Here  ${}^t$  stands for the  $\eta$ -transposition, namely, for the row vector  $r$  one has  $r^t = (r\eta^{-1})^T$  (the operation  $T$  being the usual matrix transposition), and  $\mathbb{R}^{m*}$  is the dual of  $\mathbb{R}^m$ . We refer to  $\mathbb{W}$  as the Weyl group of rescaling. Obviously  $K_0 \simeq CO(1, m - 1)$ , and  $K_1$  is the abelian group of inversions (or special conformal transformations).

Infinitesimally, we have the Klein pair  $(\mathfrak{g}, \mathfrak{h})$  of graded Lie algebras.<sup>32</sup> They decompose, respectively, as  $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \simeq \mathbb{R}^m \oplus \mathfrak{co}(1, m - 1) \oplus \mathbb{R}^{m*}$ , a splitting which gives the different sectors of the conformal rigid symmetry: translations + (Weyl  $\times$  Lorentz) + inversions, and  $\mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \simeq \mathfrak{co}(1, m - 1) \oplus \mathbb{R}^{m*}$ . In matrix notation, we have

$$\mathfrak{g} = \left\{ \begin{pmatrix} \epsilon & \iota & 0 \\ \tau & v & \iota^t \\ 0 & \tau^t & -\epsilon \end{pmatrix} \middle| (v - \epsilon \mathbb{1}) \in \mathfrak{co}, \tau \in \mathbb{R}^m, \iota \in \mathbb{R}^{m*} \right\} \supset \mathfrak{h} = \left\{ \begin{pmatrix} \epsilon & \iota & 0 \\ 0 & v & \iota^t \\ 0 & 0 & -\epsilon \end{pmatrix} \right\},$$

with the  $\eta$ -transposition  $\tau^t = (\eta\tau)^T$  of the column vector  $\tau$ . The graded structure of the Lie algebras,  $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$ ,  $i, j = 0, \pm 1$  with the abelian Lie subalgebras  $[\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] = 0 = [\mathfrak{g}_1, \mathfrak{g}_1]$ , is automatically handled by the matrix commutator.

One can thus “localized” the conformal group by considering the second-order conformal structure. The latter is a Cartan geometry  $(\mathcal{P}, \varpi)$  where  $\mathcal{P} = \mathcal{P}(\mathcal{M}, H)$  is a principal bundle over  $\mathcal{M}$  with structure group  $H = K_0 K_1$ , and  $\varpi \in \Omega^1(\mathcal{U}, \mathfrak{g})$  is a (local) Cartan connection. The curvature is given by  $\Omega = d\varpi + \frac{1}{2}[\varpi, \varpi] = d\varpi + \varpi^2$ . Both have a matrix representation

$$\varpi = \begin{pmatrix} a & \alpha & 0 \\ \theta & A & \alpha^t \\ 0 & \theta^t & -a \end{pmatrix} \quad \text{and} \quad \Omega = \begin{pmatrix} f & \Pi & 0 \\ \Theta & F & \Pi^t \\ 0 & \Theta^t & -f \end{pmatrix}.$$

One can single out the so-called normal conformal Cartan connection (which is unique) by imposing the constrains  $\Theta = 0$  (torsion free) and  $F^a{}_{bad} = 0$ . Together with the  $\mathfrak{g}_{-1}$ -sector of the Bianchi identity,  $d\Omega + [\varpi, \Omega] = 0$ , these imply  $f = 0$  (trace free), so that the curvature of the normal Cartan connection reduces to

$$\Omega = \begin{pmatrix} 0 & \Pi & 0 \\ 0 & F & \Pi^t \\ 0 & 0 & 0 \end{pmatrix} \quad (\text{normal case}).$$

In the normal geometry,  $\alpha$  is the Schouten 1-form,  $\Pi$  and  $F$  are the Cotton and Weyl 2-forms, respectively.

The ghost field,  $v : \mathcal{U} \rightarrow \mathfrak{h}^* \otimes \mathfrak{h}$ , associated with the  $\mathcal{H}$ -gauge symmetry is given by

$$v = v_W + v_L + v_i = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\epsilon \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & v_L & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \iota & 0 \\ 0 & 0 & \iota^t \\ 0 & 0 & 0 \end{pmatrix}.$$

With this matrix representation, the associated BRST algebra reads as usual

$$s\varpi = -Dv := -dv - [\varpi, v], \quad s\Omega = [\Omega, v], \quad \text{and} \quad sv = -\frac{1}{2}[v, v] = -v^2. \tag{5.2}$$

Defining the corresponding algebraic conformal Cartan connection  $\tilde{\varpi} := \varpi + v$ , algebra (5.2) is recovered from the Russian formula,  $\tilde{d}\tilde{\varpi} + \frac{1}{2}[\tilde{\varpi}, \tilde{\varpi}] = \Omega$ .

The principal bundle  $\mathcal{P}(\mathcal{M}, H)$  is a second order  $G$ -structure, a reduction of the second order frame bundle  $L^2\mathcal{M}$ ; it is thus a “2-stage bundle.” The bundle  $\mathcal{P}(\mathcal{M}, H)$  over  $\mathcal{M}$  can also be seen as a principal bundle  $\mathcal{P}_1 := \mathcal{P}(\mathcal{P}_0, K_1)$  with structure group  $K_1$  over the principal bundle  $\mathcal{P}_0 := \mathcal{P}(\mathcal{M}, K_0)$ .

Accordingly, in Ref. 10, we showed how (locally) the structure group  $H$  could be reduced in two steps: first from  $H$  to  $K_0$  by neutralizing  $K_1$  with a first dressing field  $u_1$ , then from  $K_0$  to  $\mathbf{W}$  thanks to a second dressing field  $u_0$ . We displayed the corresponding sequence of reduced BRST algebras<sup>42</sup>

$$\text{BRST} : (\varpi, \Omega, s, v) \xrightarrow{u_1} \text{BRST}_1 : (\varpi_1, \Omega_1, s, v_1) \xrightarrow{u_0} \text{BRST}_0 : (\varpi_0, \Omega_0, s, v_0).$$

We also showed that it is possible to define  $u := u_1 u_0$  and to proceed in a single step,

$$\text{BRST} : (\varpi, \Omega, s, v) \xrightarrow{u} \text{BRST}_0 : (\varpi_0, \Omega_0, s, v_0).$$

### A. Shifting and first dressing field

The shift of differential algebra (5.2) in order to include infinitesimal diffeomorphisms is performed as in the general case,

$$\text{BRST} : (\varpi, \Omega, s, v) \xrightarrow{\xi} \text{BRST}^\xi : (\varpi, \Omega, \sigma, v' = v + i_\xi \varpi),$$

and due to the decomposition of the ghost,  $v' = v + i_\xi \varpi$ , we know that  $\sigma = s + L_\xi$  on  $(\varpi, \Omega, v)$ .

It is interesting to see what happens under the first dressing operation by  $u_1$ . The latter gives the sequence,

$$\text{BRST}^\xi : (\varpi, \Omega, \sigma, v') \xrightarrow{u_1} (\text{BRST}^\xi)_1 : (\varpi_1, \Omega_1, \sigma, (v')_1 := u_1^{-1} v' u_1 + u_1^{-1} \sigma u_1).$$

The crux is of course to determine  $\sigma u_1$ . In Ref. 10, we defined the dressing field  $u_1 : \mathcal{U} \rightarrow K_1$  by

$$u_1 := \begin{pmatrix} 1 & q & \frac{1}{2} q q^t \\ 0 & \mathbb{1} & q^t \\ 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad q := a \cdot e^{-1} \in \mathbb{R}^{(1, m-1)*},$$

where  $a = a \cdot dx$  and  $\theta = e \cdot dx$  (with indices,  $q_a := a_\mu (e^{-1})^\mu_a$ ). Hence,

$$\sigma u_1 \sim \sigma q = \sigma(a \cdot e^{-1}) = (\sigma a) \cdot e^{-1} - a \cdot e^{-1} (\sigma e) e^{-1},$$

and we need to know  $\sigma a$  and  $\sigma e$ , that is,  $\sigma \varpi$ . Since  $\sigma \varpi = s \varpi + L_\xi \varpi$ , one has

$$\sigma a = (s + L_\xi) a \quad \text{and} \quad \sigma \theta = (s + L_\xi) \theta.$$

The first equation reads

$$\begin{aligned} \sigma(a \cdot dx) &= s(a \cdot dx) + (i_\xi d - di_\xi)(a \cdot dx) = (sa) \cdot dx + i_\xi(da \wedge dx) - d(a \cdot \xi) \\ &= (sa) \cdot dx + i_\xi da \cdot dx + \cancel{da \cdot \xi} - \cancel{da \cdot \xi} - a \cdot d\xi. \end{aligned}$$

This gives  $\sigma a = sa + i_\xi da + a \cdot \partial \xi$ . The second equation is already known from (4.3). Finally,

$$\begin{aligned} \sigma q &= (\sigma a) \cdot e^{-1} - a \cdot e^{-1} (\sigma e) e^{-1} \\ &= (sa + i_\xi da + a \cdot \partial \xi) \cdot e^{-1} - a \cdot e^{-1} (se + i_\xi de + e \cdot \partial \xi) e^{-1} \\ &= (sa) \cdot e^{-1} + i_\xi da \cdot e^{-1} + \cancel{(a \cdot \partial \xi) \cdot e^{-1}} + a \cdot se^{-1} + a \cdot i_\xi de^{-1} - \cancel{(a \cdot \partial \xi) \cdot e^{-1}} \\ &= s(a \cdot e^{-1}) + i_\xi d(a \cdot e^{-1}) \\ &= sq + i_\xi dq. \end{aligned}$$

Noticing that  $i_\xi q = 0$ , we end up with the result

$$\sigma q = (s + L_\xi) q \quad \text{so that} \quad \sigma u_1 = (s + L_\xi) u_1. \tag{5.3}$$

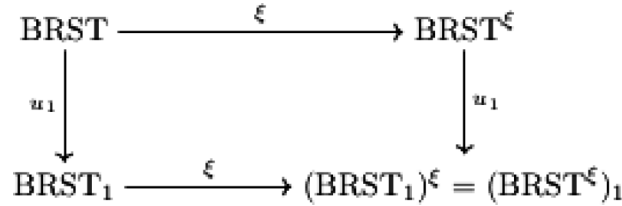


FIG. 2. Commutativity between a dressing and shifting.

This shows that the first dressing of the conformal structure satisfies (3.12). Therefore, from our general discussion of Section III B, we can conclude that

$$(v')_1 = u_1^{-1}v'u_1 + u_1^{-1}\sigma u_1 = v_1 + i_\xi \varpi_1 =: (v_1)'. \tag{5.4}$$

This means that  $(\text{BRST}^\xi)_1 = (\text{BRST}_1)^\xi$ , and we have the commutative diagram as shown in Fig. 2. The ghost  $(v')_1 = (v_1)'$  encodes the residual symmetry  $CO(1, m - 1) + \text{Diff}(\mathcal{M})$  and the reduced shifted algebra  $(\text{BRST}^\xi)_1 = (\text{BRST}_1)^\xi$  handles the transformation of the composites fields  $\varpi_1$  and  $\Omega_1$  under these symmetries. In its second, decoupled, presentation it reads

$$\sigma \varpi_1 = -Dv_1 + L_\xi \varpi_1, \quad \sigma \Omega_1 = [\Omega_1, v_1] + L_\xi \Omega_1, \quad \text{and} \quad \sigma v_1 = -\frac{1}{2}[v_1, v_1] + L_\xi v_1.$$

We refer to Ref. 10 for the detailed results concerning the algebra  $\text{BRST}_1$  associated to the first composite ghost  $v_1$ .

As already mentioned, it is possible to further reduce the gauge symmetry with a second dressing field  $u_0$ .

### B. Shifting and the second dressing field

We here face a situation which is analogous to the GR case, the second dressing field  $u_0$  we now use is the vielbein, extracted from the Cartan connection  $\varpi$ . Again, from general results of Section III B, we know that upon dressing  $(\text{BRST}^\xi)_1$  with  $u_0$  we have the change of differential algebras

$$(\text{BRST}^\xi)_1 : (\varpi_1, \Omega_1, \sigma, (v')_1) \xrightarrow{u_0} (\text{BRST}^\xi)_{1,0} : (\varpi_0, \Omega_0, \sigma, (v')_{1,0} := u_0^{-1}(v')_1 u_0 + u_0^{-1} \sigma u_0)$$

whose outcoming ghost reads

$$\begin{aligned}
 (v')_{1,0} &= u_0^{-1}(v')_1 u_0 + u_0^{-1} \sigma u_0 = u_0^{-1}(u_1^{-1}v'u_1 + u_1^{-1}\sigma u_1)u_0 + u_0^{-1} \sigma u_0 \\
 &= (u_1 u_0)^{-1} v' (u_1 u_0) + (u_1 u_0)^{-1} \sigma (u_1 u_0), \\
 (v')_{1,0} &= u^{-1} v' u + u^{-1} \sigma u, \quad \text{by defining } u := u_1 u_0.
 \end{aligned} \tag{5.5}$$

Equation (5.5) shows that one can start from  $\text{BRST}^\xi$  and use the dressing field  $u = u_1 u_0$  to obtain the algebra  $(\text{BRST}^\xi)_{1,0}$  in a single step. This is possible because the two dressing fields satisfy the following compatibility conditions:

$$u_1^S = S^{-1} u_1 S \quad \text{and} \quad u_0^{\gamma_1} = u_0$$

regarding their transformations under the gauge subgroups  $\mathcal{SO}$  and  $\mathcal{K}_1$ . These, together with the defining transformations properties  $u_1^{\gamma_1} = \gamma_1^{-1} u_1$  and  $u_0^S = S^{-1} u_0$ , entail that  $u$  is indeed a dressing under the product  $\mathcal{SO} \mathcal{K}_1 \subset \mathcal{H}$ .

In an alternative but equivalent way, upon dressing  $(\text{BRST}_1)^\xi$  we have the change of differential algebras

$$(\text{BRST}_1)^\xi : (\varpi_1, \Omega_1, \sigma, (v_1)') \xrightarrow{u_0} [(\text{BRST}_1)^\xi]_0 : (\varpi_0, \Omega_0, \sigma, [(v_1)']_0 := u_0^{-1}(v_1)'u_0 + u_0^{-1} \sigma u_0)$$



whose outgoing ghost reads

$$\begin{aligned} [(v_1)']_0 &= u_0^{-1}(v_1 + i_\xi \varpi_1)u_0 + u_0^{-1}\sigma u_0 \\ &= u_0^{-1}(u_1^{-1}vu_1 + u_1^{-1}su_1 + u_1^{-1}i_\xi \varpi u_1 + u_1^{-1}i_\xi du_1)u_0 + u_0^{-1}\sigma u_0, \\ [(v_1)']_0 &= u^{-1}(v + i_\xi \varpi)u + u^{-1}(su_1 + i_\xi du_1)u_0 + u_0^{-1}\sigma u_0. \end{aligned} \tag{5.6}$$

Of course, both (5.5) and (5.6) are the same, and in any case the key question is the value of  $\sigma u_0$ . But the answer is already at hand since we know that  $\sigma e = se + i_\xi de + e \cdot \partial \xi$ . With the definitions

$$u_0 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ given in Ref. 10 and } v_\xi := \begin{pmatrix} 0 & 0 & 0 \\ 0 & \partial \xi & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

we have then

$$\sigma u_0 = su_0 + L_\xi u_0 + u_0 v_\xi \tag{5.7}$$

which is an instance of (3.13). This means that the final ghost has the decomposition

$$(v')_{1,0} = [(v_1)']_0 = u^{-1}vu + u^{-1}su + i_\xi(u^{-1}\varpi u + u^{-1}du) + v_\xi.$$

One can easily recognize the final Weyl ghost  $v_0 := u^{-1}vu + u^{-1}su$ , and the final composite field  $\varpi_0 := u^{-1}\varpi u + u^{-1}du$ , obtained in a single step upon dressing  $v$  and  $\varpi$  with  $u := u_1 u_0$ . By simply denoting  $v'_0 := (v')_{1,0} = [(v_1)']_0$ , we have

$$v'_0 = v_0 + i_\xi \varpi_0 + v_\xi. \tag{5.8}$$

As expected  $v'_0 \neq v_0 + i_\xi \varpi_0 = (v_0)'$ , and we can sum up the steps in a diagram as presented in Fig. 3. On the right hand side, the curved arrow illustrates the implication of (5.5). The bottom dashed arrow indicates that the diagram does not close there, as is clear from (5.8), consequence of (5.7).

As an illustration of the criterion discussed at the end of Section III B, let us remark that (3.12) holds for  $u_1 \sim q_a = a_\mu (e^{-1})^\mu_a$  since it has only a free internal index and no free spacetime index. It is a 0-form and is thus scalar under coordinate changes. Whereas (3.13) holds for  $u_0 \sim e^a_\mu$  since it carries both an internal index and a free spacetime index. Thus it is tensorial (covector) under coordinate changes.

Using the matrix form of the Weyl ghost  $v_0$  and of the composite field  $\varpi_0$  found in Ref. 10, final ghost (5.8) explicitly reads

$$v'_0 = \begin{pmatrix} \epsilon & \partial \epsilon + P \cdot \xi & 0 \\ \xi & \epsilon \delta + \nabla \xi & g^{-1} \cdot (\partial \epsilon + \xi P^T) \\ 0 & \xi \cdot g & -\epsilon \end{pmatrix} = \begin{pmatrix} \epsilon & \partial_\nu \epsilon + P_{\nu\lambda} \xi^\lambda & 0 \\ \xi^\rho & \epsilon \delta^\rho_\nu + \partial_\nu \xi^\rho + \Gamma^\rho_{\nu\lambda} \xi^\lambda & g^{\rho\alpha} (\partial_\alpha \epsilon + \xi^\lambda P_{\lambda\alpha}) \\ 0 & \xi^\lambda g_{\lambda\nu} & -\epsilon \end{pmatrix},$$

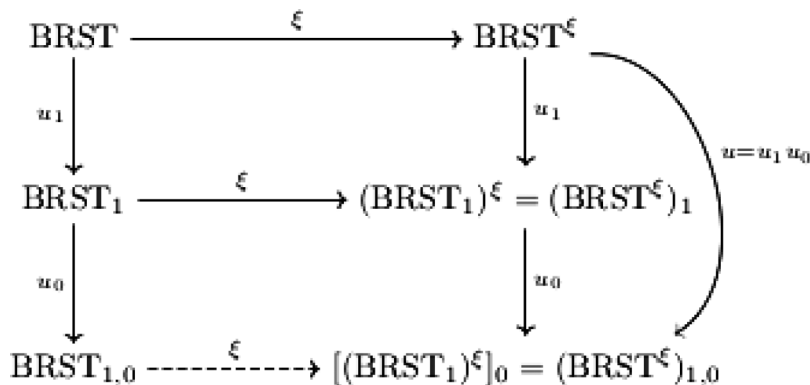


FIG. 3. Final dressing and shifting in the conformal case.

where  $g = e^T \eta e$  is the metric,  $\Gamma$  is a linear connection, and  $P$  is a generalization of the Schouten tensor. The final ghost encodes in a covariant way the residual mixed symmetry Weyl+Diff( $\mathcal{M}$ ) with generators  $(\epsilon, \xi)$ .

It results that the composite shifted algebraic connection,  $\varpi_0 + v'_0$ , gives a geometrical interpretation to the results obtained in Ref. 1. Indeed, in this paper which aims at constructing a Weyl-covariant tensor calculus, the entries of  $\varpi_0 + v'_0$  are found as fields (called *generalized connections*) belonging to a space of variables identified through BRST-cohomological techniques. To some extent, one ought to say that the BRST cohomologies capture pieces of the geometry. Now, it is easy to write the final algebra  $(\text{BRST}^\xi)_{1,0}$

$$\sigma \varpi_0 = -D_0 v'_0 + i_\xi \Omega_0, \quad \sigma \Omega_0 = [\Omega_0, v'_0] - D_0(i_\xi \Omega_0), \quad \text{and} \quad \sigma v'_0 = -\frac{1}{2}[v'_0, v'_0] + \frac{1}{2}i_\xi i_\xi \Omega_0.$$

On account of decomposition (5.8) of  $v'_0$  we have first

$$\begin{aligned} \sigma \varpi_0 &= -dv_0 - di_\xi \varpi_0 - dv_\xi - [\varpi_0, v_0] - [\varpi_0, i_\xi \varpi_0] - [\varpi_0, v_\xi] + i_\xi d\varpi_0 + \frac{1}{2}i_\xi [\varpi_0, \varpi_0] \\ &= -dv_0 - [\varpi_0, v_0] + \mathcal{L}_\xi \varpi_0 - [\varpi_0, v_\xi] - dv_\xi, \\ \sigma \varpi_0 &= s_W \varpi_0 + \mathcal{L}_\xi \varpi_0 - dv_\xi, \end{aligned} \tag{5.9}$$

where  $s_W$  is the Weyl BRST operator associated with the Weyl ghost  $v_0$ . Next we have

$$\begin{aligned} \sigma \Omega_0 &= [\Omega_0, v_0] + [\Omega_0, i_\xi \varpi_0] + [\Omega_0, v_\xi] - di_\xi \Omega_0 - [\varpi_0, i_\xi \Omega_0] \\ &= [\Omega_0, v_0] - i_\xi [\varpi_0, \Omega_0] + [\Omega_0, v_\xi] - di_\xi \Omega_0 = [\Omega_0, v_0] + \mathcal{L}_\xi \Omega_0 + [\Omega_0, v_\xi], \end{aligned}$$

where in the third equality, by the Bianchi identity,  $i_\xi d\Omega_0 + i_\xi [\varpi_0, \Omega_0] = 0$  has been used. One thus gets

$$\sigma \Omega_0 = s_W \Omega_0 + \mathcal{L}_\xi \Omega_0. \tag{5.10}$$

Finally, with a little bit more of effort, one finds

$$\sigma v'_0 = (s_W + \mathcal{L}_\xi)v'_0 - i_\xi dv_\xi - i_{\frac{1}{2}[\xi, \xi]} \varpi_0. \tag{5.11}$$

Most part of this equation is redundant with (5.9) and the fact that  $\sigma \xi = \frac{1}{2}[\xi, \xi]$ , implied by  $\sigma^2 = 0$ . But the Weyl subalgebra part,  $s_W v_0$ , expresses the fact that the Weyl group of scale transformations is abelian:  $s_W \epsilon = 0$ . Equations (5.9) and (5.10) express the transformation laws of the composite fields  $\varpi_0$  and  $\Omega_0$  under the mixed symmetry Weyl+Diff( $\mathcal{M}$ ).

### C. The normal case

In the instance of the normal conformal Cartan connection, the normality conditions on the curvature are preserved through the successive dressing operations<sup>10</sup> and the final composite fields are

$$\varpi_0 = \begin{pmatrix} 0 & P & 0 \\ dx & \Gamma & g^{-1} \cdot P^T \\ 0 & dx^T \cdot g & 0 \end{pmatrix} = \begin{pmatrix} 0 & P_{\mu\nu} & 0 \\ \delta_\mu^\rho & \Gamma^\rho_{\mu\nu} & g^{\rho\lambda} P_{\lambda\mu} \\ 0 & g_{\mu\nu} & 0 \end{pmatrix} dx^\mu,$$

where  $g$  is the metric tensor,  $\Gamma$  is the Levi-Civita connection,  $P$  is the Schouten tensor (expressed in terms of the Ricci tensor and Ricci scalar),

$$P_{\mu\nu} = \frac{-1}{(m-2)} \left( R_{\mu\nu} - \frac{R}{2(m-1)} g_{\mu\nu} \right),$$

and for the curvature

$$\Omega_0 = \begin{pmatrix} 0 & C & 0 \\ 0 & W & g^{-1} \cdot C^T \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & C_{\nu, \mu\sigma} & 0 \\ 0 & W^\rho_{\nu, \mu\sigma} & g^{\rho\lambda} C_{\lambda, \mu\sigma} \\ 0 & 0 & 0 \end{pmatrix} dx^\mu \wedge dx^\sigma,$$

where  $C$  and  $W$  are the Cotton and Weyl tensors, respectively. This is known as the *Riemannian parametrization of the normal conformal Cartan connection*.

The Weyl subalgebra  $s_W\varpi_0$  and  $s_W\Omega_0$  of  $(BRST^\xi)_{1,0}$  then easily gives, through a simple matrix calculation, the transformations of the above mentioned various objects under infinitesimal Weyl rescaling. This was detailed in Ref. 10. Instead, let us give the explicit matrix form of the  $\text{Diff}(\mathcal{M})$  subalgebra in (5.9) and (5.10). Again this is simple matrix computation. First, let us compute the combination

$$\begin{aligned}
 \mathcal{L}_\xi\varpi_0 - dv_\xi &= L_\xi\varpi_0 - [\varpi_0, v_\xi] - dv_\xi \\
 &= L_\xi \begin{pmatrix} 0 & P & 0 \\ dx & \Gamma & g^{-1} \cdot P^T \\ 0 & dx^T \cdot g & 0 \end{pmatrix} - \left[ \begin{pmatrix} 0 & P & 0 \\ dx & \Gamma & g^{-1} \cdot P^T \\ 0 & dx^T \cdot g & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \partial\xi & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] - \begin{pmatrix} 0 & 0 & 0 \\ 0 & d\partial\xi & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
 &= \begin{pmatrix} 0 & L_\xi P - P \cdot \partial\xi & 0 \\ 0 & L_\xi \Gamma - [\Gamma, \partial\xi] - d\partial\xi & L_\xi(g^{-1} \cdot P^T) - \partial\xi \cdot (g^{-1} \cdot P^T) \\ 0 & L_\xi(dx^T \cdot g) - (dx^T \cdot g) \cdot \partial\xi & 0 \end{pmatrix} \\
 &= \begin{pmatrix} 0 & \xi^\alpha \partial_\alpha P_{\mu\nu} + P_{\alpha\nu} \partial_\mu \xi^\alpha + P_{\mu\alpha} \partial_\nu \xi^\alpha & 0 \\ 0 & \xi^\alpha \partial_\alpha \Gamma^\rho_{\mu\nu} + \Gamma^\rho_{\alpha\nu} \partial_\mu \xi^\alpha + \Gamma^\rho_{\mu\alpha} \partial_\nu \xi^\alpha - \partial_\alpha \xi^\rho \Gamma^\alpha_{\mu\nu} + \partial_\mu(\partial_\nu \xi^\rho) & * \\ 0 & \xi^\alpha \partial_\alpha g_{\mu\nu} + g_{\alpha\nu} \partial_\mu \xi^\alpha + g_{\mu\alpha} \partial_\nu \xi^\alpha & 0 \end{pmatrix} dx^\mu \\
 &=: \begin{pmatrix} 0 & \mathcal{L}_\xi P_{\mu\nu} & 0 \\ 0 & \mathcal{L}_\xi \Gamma^\rho_{\mu\nu} + \partial_\mu(\partial_\nu \xi^\rho) & * \\ 0 & \mathcal{L}_\xi g_{\mu\nu} & 0 \end{pmatrix} dx^\mu. \tag{5.12}
 \end{aligned}$$

The entries are the correct infinitesimal transformations under active diffeomorphisms of the metric tensor, the Christoffel symbols and Schouten tensor. Entry (2,3) is of course redundant with entries (1,2) and entry (3,2). In the same way,

$$\begin{aligned}
 \mathcal{L}_\xi\Omega_0 &= L_\xi\Omega_0 + [\Omega_0, v_\xi] \\
 &= L_\xi \begin{pmatrix} 0 & C & 0 \\ 0 & W & g^{-1} \cdot C^T \\ 0 & 0 & 0 \end{pmatrix} + \left[ \begin{pmatrix} 0 & C & 0 \\ 0 & W & g^{-1} \cdot C^T \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \partial\xi & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] \\
 &= \begin{pmatrix} 0 & L_\xi C + C \cdot \partial\xi & 0 \\ 0 & L_\xi W + [W, \partial\xi] & L_\xi(g^{-1} \cdot C^T) - \partial\xi \cdot (g^{-1} \cdot C^T) \\ 0 & 0 & 0 \end{pmatrix} \\
 &= \frac{1}{2} \begin{pmatrix} 0 & \xi^\alpha \partial_\alpha C_{\nu,\mu\sigma} + C_{\nu,\alpha\sigma} \partial_\mu \xi^\alpha + C_{\nu,\mu\alpha} \partial_\sigma \xi^\alpha + C_{\alpha,\mu\sigma} \partial_\nu \xi^\alpha & 0 \\ 0 & \xi^\alpha \partial_\alpha W^\rho_{\nu,\mu\sigma} + W^\rho_{\nu,\alpha\sigma} \partial_\mu \xi^\alpha + W^\rho_{\nu,\mu\alpha} \partial_\sigma \xi^\alpha + W^\rho_{\alpha,\mu\sigma} \partial_\nu \xi^\alpha - \partial_\alpha \xi^\rho W^\alpha_{\nu,\mu\sigma} & * \\ 0 & 0 & 0 \end{pmatrix} dx^\mu \wedge dx^\sigma \\
 &=: \frac{1}{2} \begin{pmatrix} 0 & \mathcal{L}_\xi C_{\nu,\mu\sigma} & 0 \\ 0 & \mathcal{L}_\xi W^\rho_{\nu,\mu\sigma} & * \\ 0 & 0 & 0 \end{pmatrix} dx^\mu \wedge dx^\sigma. \tag{5.13}
 \end{aligned}$$

The entries are the correct infinitesimal transformations under active diffeomorphisms of the Cotton and Weyl tensors. Entry (2,3) is redundant with entry (1,2), and entry (3,2) of (5.12).

The computation in the more general (non-normal) case is just as easy to perform. Beside the nonzero terms in (5.13), it gives as entry (2,1) (and (3,2)) the Lie derivative of the torsion tensor,  $\frac{1}{2}(\mathcal{L}_\xi T^\rho_{\mu\sigma}) dx^\mu \wedge dx^\sigma$ , as in (4.8), and as entry (1,1) (and (3,3)) the transformation of the trace  $f = \frac{1}{2}f_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{1}{2}P_{[\mu\nu]} dx^\mu \wedge dx^\nu$ , which is redundant with entry (1,2) in (5.12).

## VI. CONCLUSION

In this paper, we have briefly summed up the heuristic method that allows to build a BRST algebra describing a mixed gauge + diffeomorphism symmetry,  $\widehat{\text{BRST}}^\xi$ . A process we referred to as “shifting,” according to Ref. 17, of the initial pure gauge algebra BRST.

Then we have shown that the shifting method is compatible with the dressing field approach, the latter was already shown to fit the BRST framework in Ref. 10. Two possibilities were in order: either dressing first then shifting and finding  $\widehat{\text{BRST}}^\xi$  or shifting first then dressing and finding  $\widehat{\text{BRST}}^\xi$ . We highlighted the necessary and sufficient condition the dressing field has to satisfy so as to warrant the commutation of these two fields redefinitions. In the case where this condition is not fulfilled, the treated examples indicate that the correct algebra is always  $\widehat{\text{BRST}}^\xi$  since it is the one taking into account the possible tensorial nature of the dressing field.

Two instances of Cartan geometries illustrate how the general scheme, which consists in shifting first and then dressing, allows to recover efficiently, using a compact matrix formalism, relevant results about mixed (gauge + diffeomorphisms) BRST algebras of gravitational theories.

The first example was concerned with General Relativity. We exhibited the mixed Lorentz +  $\text{Diff}(\mathcal{M})$  BRST algebra satisfied by the Cartan connection and its curvature, viz., Refs. 6 and 14. Applying the dressing field method to this BRST algebra completely gets rid of the Lorentz (gauge) sector, leaving only  $\text{Diff}(\mathcal{M})$  as the residual symmetry to be described, that is, the gauge group for GR. This is consistent with the results of Ref. 8.

The second example dealt with the second-order conformal structure, as a Cartan geometry. It ought to be suited for conformal (Weyl) gravity. The shifted BRST algebra describes the  $H + \text{Diff}(\mathcal{M})$  (see (5.1)) transformations of the conformal Cartan connection  $\varpi$ , and its curvature  $\Omega$ . In that case, two dressing fields were needed. After the first dressing we ended up with a BRST algebra describing the  $CO(1, m - 1) + \text{Diff}(\mathcal{M})$  symmetry of the dressed Cartan connection,  $\varpi_1$ , and its curvature  $\Omega_1$ . This stage illustrated the commutation of the two operations. Then, the second dressing finally provided the final BRST algebra describing the residual Weyl +  $\text{Diff}(\mathcal{M})$  symmetry of the normal conformal Cartan connection in its Riemannian parametrization,  $\varpi_0$ , and its curvature  $\Omega_0$ . In this parametrization,  $\varpi_0$  and  $\Omega_0$  contain, respectively, on the one hand, the metric and Schouten tensors and the Christoffel symbols, and, on the other hand, the Cotton and Weyl tensors. The infinitesimal Weyl and  $\text{Diff}(\mathcal{M})$  transformations of these objects are then easily found from our final mixed BRST algebra whose central object is the ghost  $v'_0$  (5.8). The algebraic connection  $\varpi_0 + v'_0$  provides a geometric framework for the cohomological results given in Ref. 1. The final mixed BRST algebra we have derived is relevant for Weyl gravity and might be useful in the characterization of a mixed Weyl+ $\text{Diff}(\mathcal{M})$  anomaly.

The scheme presented in the paper is robust, handy, and yields relevant results. However, it relies on intertwining ansatz (2.5) for which one sees little mathematical basis apart from the soundness of the outcoming results. This ansatz deserves to be better understood, for instance, thanks to a way to derive the mixed BRST algebra (of the type given in Refs. 6 and 14 which requires the use of a background connection) from a well-grounded geometrical framework. This issue is under study.

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To the memory of Daniel Kastler (1926-2015).

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- <sup>22</sup> As stated in Ref. 43, the Lie algebra of diffeomorphisms is anti-isomorphic to the Lie algebra of vector fields. This explains why the factor 1/2 occurs without a minus sign. Thus upon substituting  $\xi$  by  $-\xi$  one recovers variations obtained in Ref. 6.
- <sup>23</sup> T. Masson and J. C. Wallet, “A remark on the spontaneous symmetry breaking mechanism in the standard model,” e-print [arXiv:1001.1176](https://arxiv.org/abs/1001.1176) (2011).
- <sup>24</sup> This means that  $G'$  is to be suitable for this definition, in particular, it shares the same representations as  $H$  (at least the adjoint representation and  $\rho$ ).
- <sup>25</sup> To some extent,  $H'$  is identified to be a subgroup of  $H$  along which the gauge invariance can be restored. One may also check that the complement  $\mathcal{H} \setminus \mathcal{H}'$  is stable under  $H$ .
- <sup>26</sup> D. Garajeu, R. Grimm, and S. Lazzarini, *J. Math. Phys.* **36**, 7043 (1995).
- <sup>27</sup> S. Lazzarini and C. Tidei, *Lett. Math. Phys.* **85**, 27 (2008).
- <sup>28</sup> Likewise, performing the same substitution starting from *second* presentation (2.15) of BRST $^\xi$  would result in an algebra for the composite fields formally identical to (2.15) but with pure gauge ghost  $(u^{-1}vu + u^{-1}\sigma u) - u^{-1}L\xi u$ .
- <sup>29</sup> The reader is referred to Ref. 17, (Chap.12).
- <sup>30</sup> *Gauge Theories of Gravitation: A Reader with Commentaries*, edited by M. Blagojević and F. Hehl (World Scientific, Imperial College Press, 2013), p. 656.
- <sup>31</sup> R. Sharpe, *Differential Geometry: Cartan’s Generalization of Klein’s Erlangen Program*, Graduate Texts in Mathematics Vol. 166 (Springer, New York, Berlin, Heidelberg, 1997).
- <sup>32</sup> S. Kobayashi, *Transformation Groups in Differential Geometry*, Classics in Mathematics Vol. 70 (Springer-Verlag, Berlin, 1972).
- <sup>33</sup> Indeed this does not depend on the expression of the shifted ghost.
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- <sup>42</sup> Due to the successive dressings, the “hat” symbol is dropped out to the benefit of a lower index.
- <sup>43</sup> J. Milnor, in *Relativity, Groups and Topology II: Les Houches, Session XL, 1983* edited by B. DeWitt and R. Stora (Elsevier Science Publishers, 1984), pp. 1009–1057.