κ-DEFORMATION AND SPECTRAL TRIPLES*

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The aim of the paper is to answer the following question: does κ-deformation fit into the framework of noncommutative geometry in the sense of spectral triples? Using a compactification of time, we get a discrete version of κ-Minkowski deformation via C^*-algebras of groups. The dynamical system of the underlying groups (including some Baumslag–Solitar groups) is used in order to construct *finitely summable* spectral triples. This allows to bypass an obstruction to finite-summability appearing when using the common regular representation.

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1. Introduction

Lukierski, Ruegg, Nowicki and Tolstoy discovered a Hopf algebraic deformation of the Poincaré Lie algebra and called κ the deformation parameter (Lukierski, Ruegg, Nowicki 1991; Lukierski, Nowicki, Ruegg 1992). Since this pioneering work, the subject became very active: the Hopf algebra was represented on the κ-deformation of Minkowski space (Zakrzewski 1994; Majid, Ruegg 1994). This has been used to generalize the notion of a quantum particle (Lukierski, Ruegg, Zakrzewski 1995) or in quantum fields (Daszkiewicz, Lukierski, Woronowicz 2009). Algebraic properties like

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differential calculi on the κ-Minkowski space were investigated (Sitarz 1995) as well as the Noether theorem (Amelino-Camelia, Marciano, Pranzetti 2009; Amelino-Camelia *et al.* 2009; Amelino-Camelia *et al.* 2007). The κ-Minkowski space has also been popularized as 'double special relativity' (Amelino-Camelia 2002) and appears in spin foam models (Freidel, Livine 2006).

It is natural to ask whether κ-geometry is a noncommutative one in the sense of Connes (Connes 1996; Connes 2008) (see (D'Andrea 2006) for a first attempt). While the algebraic setting is quite clear, the main difficulty is to overcome the analysis which is an essential part in the definition of spectral triples.

The κ-deformation of n-dimensional Minkowski space is based on the Lie-algebraic relations

$$\left[x^0,\, x^j\right] := \tfrac{i}{\kappa}\, x^j\,, \qquad \left[x^j,\, x^k\right] = 0\,, \qquad j,k = 1,\ldots,n-1\,. \tag{1}$$

Here, we assume $\kappa > 0$. As in (Kosiński, Maślanka, Lukierski, Sitarz 1998, Eq. (2.6)), one gets

$$e^{ic_\mu x^\mu} = e^{ic_0 x^0}\, e^{ic'_j\, x^j}\,, \quad \text{where} \quad c'_j := \tfrac{\kappa}{c_0}\left(1 - e^{-c_0/\kappa}\right) c_j\,.$$

Assuming that the x^μs are selfadjoint operators on some Hilbert space, we define unitaries

$$U_\omega := e^{i\omega x^0} \quad \text{and} \quad V_{\vec{k}} := e^{-i\sum_{j=1}^{n-1} k_j x^j}$$

with $\omega, k_j \in \mathbb{R}$, which generate the κ-Minkowski group considered in (Agostini 2007).

If $W(\vec{k},\omega) := V_{\vec{k}}\, U_\omega$, one gets as in (Agostini 2007, Eq. (13))

$$W\left(\vec{k},\omega\right)\, W\left(\vec{k'},\omega'\right) = W\left(e^{-\omega/\kappa}\vec{k'} + \vec{k}, \omega + \omega'\right)\,. \tag{2}$$

The group law (2) is, for $n = 2$, nothing else but the crossed product

$$G_\kappa := \mathbb{R} \rtimes_\alpha \mathbb{R} \quad \text{with group isomorphism} \quad \alpha(\omega)k := e^{-\omega/\kappa}k\,, \;\; k \in \mathbb{R}\,. \tag{3}$$

Note that $G_\kappa \simeq \mathbb{R} \rtimes \mathbb{R}_+^*$ is the affine group on the real line which is solvable and nonunimodular. The irreducible unitary representations are either one-dimensional, or fall into two nonequivalent classes (Agostini 2007). When $\kappa \to \infty$, the usual plane $\mathbb{R}^2$ is recovered but with an unpleasant pathology at the origin (Dąbrowski, Piacitelli 2009).

For a given ω, a particular case occurs when $m := e^{-\omega/\kappa} \in \mathbb{N}^*$, since

$$U_\omega V_{\vec{k}} = \left(V_{\vec{k}}\right)^m U_\omega\,, \tag{4}$$

m being independent of k_j. This means that for chosen ω and k_j, the presentation of the group is given by two generators and one relation.

Here, we investigate different spectral triples $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ associated to the group C^*-algebra of G_κ. To avoid technicalities due to a continuous spectrum of $\mathcal{D}$, we want a unital algebra so that we consider a periodic time x^0 which induces a discrete version G_a of G_κ where a is a real parameter depending on κ. This is done in Sec. 2.

In Sec. 3, we give the main properties of the algebra $\mathcal{U}_a = C^*(G_a)$ and its representations. For C^*-algebras of groups, the left regular representation is the natural one to consider. But due to the structure of G_a (solvable with exponential growth, see Theorem 3.2), there is a known obstruction to construct finite-summable spectral triples on $\mathcal{U}_a$ based on this representation. In order to bypass this obstruction, we need to refine our understanding of the structure of G_a in terms of an underlying dynamical system. This structure permits to define a particular representation of $\mathcal{U}_a$ using only the periodic points of the dynamical system. Following Brenken and Jørgensen (Brenken, Jørgensen 1991), the topological entropy of this dynamics is considered.

It is worthwhile to notice that the elementary building blocks G_a given by $a = m \in \mathbb{N}^*$ as in (4) are some of the amenable Baumslag–Solitar groups, already encountered in wavelet theory (Jørgensen 2001). The power of harmonic analysis on groups also justifies a reminder of their main properties in Sec. 4.

The question of the finite summability of these triples is carefully considered with results by Connes (Connes 1989) and Voiculescu (Voiculescu 1979; Voiculescu 1990): using the regular representation of $C^*(G_a)$, an obstruction to finite-summability appears and allows only θ-summability. However, faithful representations of $C^*(G_a)$, not quasi-equivalent to the left regular one and based on the existence of periodic points for dynamical systems, can give rise to arbitrary finite-summable spectral triples. These results are summarized in Theorems 5.2 and 5.3.

All proofs will appear elsewhere.

2. Motivations and models

Let us consider the example given by (1) for two hermitian generators x^0 and x^1. For any $(k,\omega) \in \mathbb{R}^2$, one defines $W(k,\omega) := V_k\, U_\omega = e^{-ikx^1} e^{i\omega x^0}$. Then one has (2) which is a representation of the group G_κ defined in (3). The bounded operators $W(f) := \int_{G_\kappa} f(k,\omega)\, W(k,\omega)\, e^{\omega/\kappa}\, dk\, d\omega$ for any $f \in L^1(G_\kappa, e^{\omega/\kappa}\, dk\, d\omega)$ (here $e^{\omega/\kappa} dk\, d\omega$ is the left Haar measure on G_κ) generate a representation of $C^*_{\rm red}(G_\kappa)$. The product of $f, g \in L^1(G_\kappa, e^{\omega/\kappa}\, dk\, d\omega)$ takes the form $(f *_\kappa g)(k,\omega) = \int_{G_\kappa} f(k',\omega') g(e^{\omega'/\kappa}(k-k'), \omega-\omega')\, e^{\omega'/\kappa}\, dk'\, d\omega'$. The advantage of considering the theory of group C^*-algebras is twofold. Many structural properties on groups will turn out to be useful in studying some properties of the corresponding C^*-algebras. Moreover, this allows us to construct in a natural way compact versions of noncommutative spaces as we now explain.

For an Abelian topological group G, $C^*_{\rm red}(G)$ is isomorphic to $C_0(\widehat{G})$, where $\widehat{G}$ is the Pontryagin dual of G. Both algebras are defined as spaces of functions. By duality, a discrete subgroup $\varGamma \subset G$ produces the C^*-algebra $C^*_{\rm red}(\varGamma) \simeq C(\widehat{\varGamma})$, where $C(\widehat{\varGamma})$ is the C^*-algebra of continuous functions on the compact space $\widehat{\varGamma}$. Notice that there is a natural dual map $\widehat{G} \to \widehat{\varGamma}$. For the example of the plane, consider the discrete subgroup $\varGamma = \mathbb{Z}^2 \subset \mathbb{R}^2$. Then the resulting C^*-algebra is $C^*(\mathbb{Z}^2) \simeq C(\mathbb{T}^2)$ because $\widehat{\mathbb{Z}} = \mathbb{T}^1$. The dual map $\mathbb{R} \simeq \widehat{\mathbb{R}} \to \widehat{\mathbb{Z}} = \mathbb{T}^1$ is explicitly given by $x \mapsto e^{2\pi i x}$. The choice of the subgroup $\varGamma = \mathbb{Z}^2 \subset \mathbb{R}^2$ corresponds then to the choice of the compact version $\mathbb{T}^2$ of the (dual) space $\widehat{\mathbb{R}}^2 \simeq \mathbb{R}^2$. The compactification takes place in the space of the variables (x, y).

In order to get a compact version of this κ-deformed Minkowski space, one has to choose a discrete subgroup $H_\kappa \subset G_\kappa$. Since H_κ is discrete and non-Abelian, the associated algebra $C^*(H_\kappa)$ is unital and noncommutative, so it can be interpreted as a compact noncommutative space. This point is motivated in Sec. 2.1 and is done in Sec. 2.2.

As a final preliminary remark, let us mention that the groups we will encounter will be decomposed as crossed products with $\mathbb{Z}$, so both the (related) theories of discrete dynamical systems and crossed products of C^*-algebras will be intensively used in many parts of this work.

2.1. Spectral triples

The goal is to study the existence of spectral triples for the κ-deformed space. A spectral triple (or unbounded Fredholm module) $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ (Connes 1995; Connes 1996; Connes, Marcolli 2008) is given by a unital C^*-algebra A with a faithful representation π on a Hilbert space $\mathcal{H}$ and an unbounded self-adjoint operator $\mathcal{D}$ on $\mathcal{H}$ such that

— the set $\mathcal{A} = \{\, a \in \mathcal{A} \,:\, [\mathcal{D}, \pi(a)] \text{ is bounded } \}$ is norm dense in A,

— $(1+\mathcal{D}^2)^{-1}$ has a compact resolvent.

($\mathcal{A}$ is always a *-subalgebra of A.)

Of course, the natural choice of the algebra A is to take the C^*-algebra of the group G_κ, but since $A = C^*(G_\kappa)$ has no unit, we need to replace the second axiom by:

— $\pi(a)(1+\mathcal{D}^2)^{-1}$ has a compact resolvent for any $a \in \mathcal{A}$.

This technical new axiom generates a lot of analytical complexities but is necessary to capture the metric dimension associated to $\mathcal{D}$. For instance, if a Riemannian spin manifold M is non-compact, the usual Dirac operator $\mathcal{D}$ has a continuous spectrum on $\mathcal{H} = L^2(S)$, where S is the spinor bundle on M. Nevertheless, the spectral triple $\big(C^\infty(M), L^2(S), \mathcal{D}\big)$ has a metric dimension which is equal to the dimension of M. A noncommutative example (the Moyal plane) of that kind has been studied in (Gayral *et al.* 2004).

We try to avoid these difficulties here by using a unital algebra A.

2.2. The compact version model as choice of a discrete subgroup

We consider only dimension $n = 2$ but the results can be extended to higher dimensions thanks to (4). To get a unit, we choose a discrete subgroup H_κ of G_κ such that $1 \in C^*(H_\kappa)$.

Since we want also to keep separate the role of the variables x^0 and x^1, we consider the subgroup of the form $H_\kappa = H \rtimes_\alpha \mathbb{Z}$: we first replace the second $\mathbb{R}$ of G_κ in (3) by the lattice $\mathbb{Z}$ which corresponds to unitary periodic functions of a chosen frequency ω_0 (the time x^0 is now periodic). So, given $\kappa > 0$ and $\omega_0 \in \mathbb{R}$, with

$$a := e^{-\omega_0/\kappa} \in \mathbb{R}^+ \,, \tag{5}$$

the group $\mathbb{R} \rtimes_{\alpha_a} \mathbb{Z}$ is a subgroup of G_κ, where $\alpha_a(n)$ is the multiplication by a^n. This subgroup is a non-discrete "$ax+b$" group.

Then, we want a group H to be a discrete (now, not necessarily topological) subgroup of the first $\mathbb{R}$ in $\mathbb{R} \rtimes_{\alpha_a} \mathbb{Z}$, which is invariant by the action α_a. Given $k_0 \in \mathbb{R}$, a natural building block candidate for a discrete H is given by $H = B_a \cdot k_0 \simeq B_a$, where

$$B_a := \left\{ \sum_{i,\,\text{finite}} m_i\, a^{n_i} \;:\; m_i, n_i \in \mathbb{Z} \right\},$$

and more generally, one can take $H \simeq \oplus_{k_0} B_a$.

The search for a discrete subgroup H_κ of G_κ such that $1 \in C^*(H_\kappa)$ leads to $H_{\kappa,a} := B_a \rtimes_{\alpha_a} \mathbb{Z}$ which is isomorphic to a subgroup of G_κ once k_0 is fixed.

This procedure drives us to the analysis of the algebraic nature of a. For instance, when $a = m \in \mathbb{N}^*$ is an integer, this group $H_{\kappa,m}$ is well known since it is the solvable Baumslag–Solitar group $\mathrm{BS}(1,m) = \mathbb{Z}[\frac{1}{m}] \rtimes_{\alpha_m} \mathbb{Z}$ as shown in Sec. 4, where we get $B_m = B_{1/m} = \mathbb{Z}[1/m]$. A broad family of noncommutative spaces appears:

Lemma 2.1. *In two dimensions, there exists a unital subalgebra $C^*(H_{\kappa,m})$ of the κ-deformation algebra, associated to the subgroup $H_{\kappa,m} = \mathbb{Z}[\frac{1}{m}] \rtimes_{\alpha_m} \mathbb{Z}$ of $G_{\kappa,m}$ and which can be seen as generated by two unitaries U, V such that $U = U_{\omega_0}$, $V = V_{k_0}$ and $UV = V^m U$. Here, $\kappa := -\omega_0 \log^{-1}(m) > 0$ for some given integer $m > 1$ and some $\omega_0 \in \mathbb{R}^-$, $k_0 \in \mathbb{R}$.*

3. The algebra $\mathcal{U}_a$ and its representations

We now compute the C^*-algebra $\mathcal{U}_a$ which is our model for a compact version of the 2-dimensional κ-Minkowski space. The structure of this algebra is described through a semi-direct product of two Abelian groups, one of which depends explicitly on the real parameter $a > 0$. This semi-direct structure gives rise to a dynamical system which is heavily used in the following. The classification of the algebras $\mathcal{U}_a$ is performed: the K-groups are not complete invariants, and we use the entropy defined on the underlying dynamical system to complete this classification. Then some representations of $\mathcal{U}_a$ are considered. They strongly depend on the algebraic or transcendental character of a. In the algebraic case, some particular finite dimensional representations are introduced based on periodic points of the dynamical system. This construction will be used in Sec. 5.

Let $a = e^{-\omega_0/\kappa} \in \mathbb{R}^*_+$ with $a \neq 1$, and let us recall general facts from (Brenken, Jørgensen 1991): Define

$$B_a := \left\{ \sum_i m_i\, a^{n_i} \text{ for finitely many } m_i, n_i \in \mathbb{Z} \right\}.$$

This discrete group is torsion-free so its Pontryagin dual $\widehat{B}_a$ is connected and compact.

Let α_a be the action of $\mathbb{Z}$ on $b \in B_a$ defined by $\alpha_a(n)b := a^n\, b$, let $\widehat{\alpha_a}$ be the associate automorphism on $\widehat{B}_a$ and

$$G_a := B_a \rtimes_{\alpha_a} \mathbb{Z}\,, \quad \mathcal{U}_a := C^*(G_a) = C^*(B_a) \rtimes_{\alpha_a} \mathbb{Z} = C\left(\widehat{B}_a\right) \rtimes_{\widehat{\alpha_a}} \mathbb{Z}\,.$$

This kind of C^*-algebras also appeared in (Carey, Phillips, Putnam, Rennie 2010) for totally different purposes. The group G_a is generated by $u := (0,1)$ and $v := (1,0)$.

Lemma 3.1. *Let $a \in \mathbb{R}_+^*$, then $B_a = B_{1/a}$ and $G_a \simeq G_{1/a}$. Thus the C^*-algebra $\mathcal{U}_a$ and $\mathcal{U}_{1/a}$ are isomorphic.*

The symmetry point $a = 1/a$ corresponds to the commutative case in (4) with $a = 1$ or the undeformed relation (1) with $\kappa = \infty$. In this spirit $\mathcal{U}_a$ can be viewed as a deformation of the two-torus.

The dynamical system $\mathcal{U}_a \simeq C(\widehat{B}_a) \rtimes_\alpha \mathbb{Z}$ has an ergodic action (if $a \neq 1$) (Brenken, Jørgensen 1991; Brenken 1996) and if the set of q-periodic points is

$$\mathrm{Per}_q\left(\widehat{B}_a\right) := \left\{\chi \in \widehat{B}_a \,:\, \widehat{\alpha}^k(\chi) \neq \chi\,,\ \forall k < q\,,\ \widehat{\alpha}^q(\chi) = \chi\right\}$$

then the growth rate $\lim_{q\to\infty} q^{-1} \log\big(\#\mathrm{Per}_q(\widehat{B}_a)\big)$ of this sets is an invariant of $\mathcal{U}_a$ which coincides with the topological entropy $h(\widehat{\alpha_a})$

$$h\left(\widehat{\alpha_a}\right) = \lim_{q\to\infty} q^{-1} \log\left(\#\mathrm{Per}_q\left(\widehat{B}_a\right)\right). \tag{6}$$

This entropy can be finite or infinite, dividing the algebraic properties of a into two cases: a can be a an algebraic or a transcendental number.

3.1. Transcendental case

If a is a transcendental number, then $B_a \simeq \mathbb{Z}[a, a^{-1}]$. Thus, $B_a \simeq \oplus_{\mathbb{Z}}\mathbb{Z}$, $\widehat{B}_a \simeq S_a := \{z = (z_k)_{k=-\infty}^{\infty} \in \mathbb{T}^{\mathbb{Z}}\}$ and $\big(\widehat{\alpha}(z)\big)_k = z_{k+1}$ for $z \in S_a$, $k \in \mathbb{Z}$ so that $\widehat{\alpha}$ is just the shift σ on $\mathbb{T}^{\mathbb{Z}}$. Thus $\mathcal{U}_a \simeq C(\mathbb{T}^{\mathbb{Z}}) \rtimes_\sigma \mathbb{Z}$ and $h(\widehat{\alpha}) = \infty$. Note that the wreath product $\wr$ appears with its known presentation:

$$G_a = B_a \rtimes_\alpha \mathbb{Z} \simeq \mathbb{Z} \wr \mathbb{Z} \simeq \left\langle u, v \,:\, \left[u^i v u^{-i}, v\right] = 1 \text{ for all } i \geq 1\right\rangle.$$

This group is amenable (solvable), torsion-free, finitely generated (but not finitely presented), residually finite with exponential growth.

S_a contains a lot of q-periodic points: they are obtained by repeating any sequence $(z_k)_{k=0}^{q-1}$ of arbitrary elements in $\mathbb{T}$. Aperiodic points are also easily constructed. Moreover, S_a is the Bohr compactification $b_{B_a}\mathbb{R}$ of $\mathbb{R}$.

3.2. Algebraic case

Assume now that a is algebraic. Let $P \in \mathbb{Q}[x]$ be the monic irreducible polynomial such that $P(a) = 0$ and let $P = c\, Q_a$, where $Q_a \in \mathbb{Z}[x]$. If d is the degree of Q_a, we get the ring isomorphism $B_a \simeq \mathbb{Z}[x, x^{-1}]/(Q_a)$ (Brenken, Jørgensen 1991). Moreover, B_a has a torsion-free rank d. If $Q_a(x) = \sum_{j=0}^d q_j\, x^j$ (so Q_a has leading coefficient $q_d \in \mathbb{N}^*$), let $A_a \in M_{d\times d}(\mathbb{Z})$ be the $d \times d$-matrix defined by $(A_a)_{i,j} := q_d\, \delta_{i,j-1}$ for $1 \leq j \leq d$ and $(A_a)_{d,j} = -q_{j-1}$. Then $q_d\, a^j = \sum_{k=1}^d (A_a)_{j,k}\, a^{k-1}$.

For instance, if $a = 1/m$ for $m \in \mathbb{N}^*$ then $P(x) = x - 1/m$, $Q_a(x) = mx - 1$, $d = 1$, so $B_a = \mathbb{Z}[\frac{1}{m}]$ and A_a is just the number 1.

Let σ be the shift on the group $\left(\mathbb{T}^d\right)^{\mathbb{Z}}$ and consider its σ-invariant subgroup $K_a := \{z = (z_k)_{k=-\infty}^{\infty} \in \left(\mathbb{T}^d\right)^{\mathbb{Z}} \,:\, q_d\, z_{k+1} = A_a\, z_k\}$ (use $\mathbb{T} \simeq \mathbb{R}/\mathbb{Z}$).

If S_a is the connected component of the identity of K_a, then there exists a topological group isomorphism $\psi : \widehat{B}_a \to S_a$ such that $\sigma_{|S_a} \circ \psi = \psi \circ \widehat{\alpha}$ (Lawton 1973, Theorem 19). For any $\chi \in \widehat{B}_a$, the associated $z = (z_k)_{k=-\infty}^{\infty}$ is given by $z_k^{(i)} = \chi(a^{k+i-1})$, where $z_k = (z_k^{(i)})_{i=1}^d \in \mathbb{T}^d$. In particular, z_0 is given by $\big(\chi(1), \chi(a), \ldots, \chi(a^{d-1})\big)$. This map is only surjective on the connected component of the identity.

When $a = 1/m$ with $m \in \mathbb{N}^*$, we will recover $S_{1/m} = S_m$ in (13).

There is a morphism of groups $\hat{\imath} : \mathbb{R}^d \to S_a$ defined as follows: to any $\phi = (\phi^{(i)})_{i=1,\ldots,d} \in \mathbb{R}^d$, one associates $\hat{\imath}(\phi) = z = (z_k)_{k=-\infty}^{\infty} \in \left(\mathbb{T}^d\right)^{\mathbb{Z}}$ with

$$z_k^{(i)} = \exp\left(2i\pi q_d^{-k} \sum_{j=1}^{d} \left(A_a^k\right)_{i,j} \phi^{(j)}\right). \tag{7}$$

This shows that S_a is a Bohr compactification of $\mathbb{R}^d$ (Brenken 1996, Proposition 2.4). Then $\widetilde{\alpha}(\phi) = q_d^{-1} A_a \phi$ defines an action of $\mathbb{Z}$ on $\mathbb{R}^d$ which satisfies $\hat{\imath} \circ \widetilde{\alpha} = \widehat{\alpha} \circ \hat{\imath}$.

If r_i, $i = 1, \cdots d$ are the roots of P, then by (Brenken, Jørgensen 1991, Proposition 3, Corollary 1)

$$c_q(a) := \#\ \mathrm{Per}_q(S_a) = \prod_{k=1}^{q} \left| Q_a\left(e^{i2\pi k/q}\right) \right| = |q_d|^q \prod_{k=1}^{d} |1 - {r_k}^q|\,. \tag{8}$$

Thus by (6), the topological entropy is

$$h(\widehat{\alpha_a}) = \log|q_d| + \sum_{i,\, |r_i|>1} \log|r_i|\,. \tag{9}$$

In the case of $a = m$ or $a = 1/m$, (9) gives $h(\widehat{\alpha_m}) = h(\widehat{\alpha_{1/m}}) = \log(m)$.

Aperiodic points in S_a can be easily constructed using the map $\hat{\imath}$ defined by (7): any $\phi \in (\mathbb{R}\backslash\mathbb{Q})^d$ defines an aperiodic point $\hat{\imath}(\phi) \in S_a$.

3.3. On the structure and classification of algebras $\mathcal{U}_a$

We can now give the main properties of the algebras $\mathcal{U}_a$:

Theorem 3.2. *Let $a \in \mathbb{R}_+^*$ and $a \neq 1$. Then*
(i) The group $G_a = B_a \rtimes_\alpha \mathbb{Z}$ is a torsion-free discrete solvable group with exponential growth and $\widehat{B_a}$ is a compact set isomorphic to a solenoid S_a.

(ii) $\mathcal{U}_a = C^*_{\rm red}(B_a \rtimes_\alpha \mathbb{Z}) \simeq C(S_a) \rtimes_{\widehat{\alpha}} \mathbb{Z}$ *is a NGCR*[1]*, AF-embeddable, non-simple, residually finite dimensional* C^**-algebra and its generated von Neumann algebra for the left regular representation is a type* II_1*- factor.*

A main point of this theorem, crucial for the sequel is that the algebra $\mathcal{U}_a$ is residually finite and its proof is based on properties of the underlying dynamical system:

Proposition 3.3. *Let* $a \in \mathbb{R}^*_+$ *and* $a \neq 1$*. The subgroup of periodic points and the set of aperiodic points of* S_a *under* $\widehat{\alpha}$ *are dense. The space of orbits of* S_a *is not a* T_0*-space.*

The classification of algebras $\mathcal{U}_a$ is also based on the dynamical system:

Theorem 3.4. *Let* $\omega_0 \in \mathbb{R}$ *and* $\kappa \in \mathbb{R}^*_+$ *defining* $a \neq 1$ *in* (5).
(i) $\mathcal{U}_a \simeq \mathcal{U}_{a'}$ *yields* $c_q(a) = c_q(a')$, $\forall q \in \mathbb{N}^*$.
(ii) The entropy $h(\widehat{\alpha})$ *is also an isomorphism-invariant of* $\mathcal{U}_a$.

This result has important physical consequences since a full Lebesgue measure dense set of different parameters a (namely the transcendental ones) generates the same algebra or κ-deformed space, while in the rational case, these spaces are different:

Corollary 3.5. *As already seen,* $\mathcal{U}_a \simeq \mathcal{U}_{1/a}$*. Moreover,*
(i) All transcendental numbers a *generate isomorphic algebras* $\mathcal{U}_a$.
(ii) If $\mathcal{U}_a \simeq \mathcal{U}_{a'}$*, then* a *and* a' *are both simultaneously algebraic or transcendental numbers.*
(iii) If $\mathcal{U}_a \simeq \mathcal{U}_{a'}$*, then* $a' = a$ *or* $a' = a^{-1}$ *in the following cases:* a*,* a' *or their inverses are in* $\mathbb{Q}^*$ *or are quadratic algebraic numbers.*
(iv) If $a = m/l \in \mathbb{Q}^*_+$*,* $K_0(\mathcal{U}_a) \simeq \mathbb{Z}$ *and* $K_1(\mathcal{U}_a) \simeq \mathbb{Z} \oplus \mathbb{Z}_{l-m}$.

Using (Brenken 1995), we can show that the K-groups do not give a complete classification even in this algebraic case.

3.4. On some representations of $\mathcal{U}_a$ *for algebraic* a

We will concentrate on $a \neq 1$ algebraic and follow the construction of finite dimensional representations (Svensson, Tomiyama 2009; Yamashita 2008): Let $z_q \in \mathrm{Per}_q(S_a)$ be a q-periodic point of $\widehat{\alpha}$. Let $\rho_{z_q} : C(S_a) \to M_q(\mathbb{C})$ be a representation of $C(S_a)$ defined by

$$\rho_{z_q}(f) := \mathrm{Diag}\left(f(z_q), \ldots, f\left(\widehat{\alpha}^{q-1}(z_q)\right) \right) \in M_q(\mathbb{C})$$

[1] A C^*-algebra A is said to be CCR or liminal if $\pi(A)$ is equal to the set of compact operators on the Hilbert space $\mathcal{H}_\pi$ for every irreducible representation π. The algebra A is called NGCR if it has no nonzero CCR ideals. An AF-algebra is a inductive limit of sequences of finite-dimensional C^*-algebras. The algebra A is residually finite-dimensional if it has a separating family of finite-dimensional representations.

and for $x \in \mathbb{T}$, let $u_{x,z_q} := \begin{pmatrix} 0 & x \\ 1_{q-1} & 0 \end{pmatrix} \in M_q(\mathbb{C})$. It is a unitary which satisfies the covariance relation $u^*_{x,z_q}\, \rho_{z_q}(f)\, u_{x,z_q} = \rho_{z_q}(f \circ \widehat{\alpha})$ thus $\pi_{x,z_q} := \rho_{z_q} \rtimes u_{x,z_q}$ is a representation of $\mathcal{U}_a$ on $M_q(\mathbb{C})$. Again, $\pi_x := \oplus_{q=1}^{\infty} \oplus_{z_q \in \mathrm{Per}_q(S_a)}$ is a representation of $\mathcal{U}_a$. So, for a dense family $\{x_l\}_{l=1}^{\infty}$ in $\mathbb{T}$, using the canonical faithful conditional expectation $C(S_a) \rtimes_{\widehat{\alpha}} \mathbb{Z} \to C(S_a)$ and the density of periodic points, one shows that $\pi := \oplus_{l=1}^{\infty} \pi_{x_l}$ is a faithful representation of $\mathcal{U}_a$ such that $\pi(\mathcal{U}_a) \subset \oplus_{l=1}^{\infty} \oplus_{q=1}^{\infty} \oplus_{z \in \mathrm{Per}_q(S_a)} M_q(\mathbb{C})$.

The representation $\pi_{x,\chi}$, where $\chi = z_q \in \widehat{B_a} = S_a$, can be extended to a representation π_χ on the Hilbert space $\mathcal{H}_\chi = L^2(\mathbb{T}) \otimes \mathbb{C}^q$ by

$$\pi_\chi = \int_{\mathbb{T}} \pi_{x,\chi}\, dx\,. \tag{10}$$

Denote by $\{e_s^{(q)}\}_{s=1,\dots q}$ the canonical basis of $\mathbb{C}^q$ and by $e_n : \theta \mapsto e^{in\theta}$, for $n \in \mathbb{Z}$, the basis of $L^2(\mathbb{T}) \simeq \ell^2(\mathbb{Z})$. Then let, for $f \in C^*(B_a) = C(\widehat{B}_a)$:

$$\pi_\chi(f)\left(e_n \otimes e_s^{(q)}\right) := f \circ \widehat{\alpha}^{s-1}(\chi)\, e_n \otimes e_s^{(q)}\,,$$
$$U\left(e_n \otimes e_s^{(q)}\right) := \begin{cases} e_n \otimes e_s^{(q)} & \text{for } 1 \le s < q\,, \\ e_{n+1} \otimes e_1^{(q)} & \text{for } s = q\,, \end{cases}$$

where U is the generator of $\mathbb{Z}$. This representation is constructed from the representation of $G_a = B_a \rtimes_\alpha \mathbb{Z}$ given by $\pi_\chi(b)(e_n \otimes e_s^{(q)}) = \chi \circ \alpha^{s-1}(b)\, e_n \otimes e_s^{(q)}$ for any $b \in B_a$ (the generator U of the action of $\mathbb{Z}$ is the same).

Another natural representation to consider is the representation of $\mathcal{U}_a$ obtained from the left regular representation of G_a.

In (Lim, Packer, Taylor 2001) and more systematically in (Dutkay, Jørgensen 2008), the induced representations *à la* Mackey of G_a for algebraic a have been investigated. The main results are the following.

For any $\chi \in \widehat{B}_a$, let the space of functions $\varphi : B_a \rtimes_\alpha \mathbb{Z} \to \mathbb{C}$ such that $\varphi(b,k) = \chi(b)\varphi(0,k)$ for any $b \in B_a$ and $k \in \mathbb{Z}$ be endowed with the norm $\|\varphi\|_\chi^2 = \sum_{k\in\mathbb{Z}} |\varphi(0,k)|^2$. This defines a Hilbert space denoted by $\mathcal{H}_\chi^{\mathrm{Ind}}$. The induced representation of G_a on $\mathcal{H}_\chi^{\mathrm{Ind}}$ is given by $(\pi_\chi^{\mathrm{Ind}}(g)\varphi)(h) = \varphi(hg)$ for any $g, h \in G_a$.

This representation is unitarily equivalent to the following π'_χ (Dutkay, Jørgensen 2008, Theorem 4.2): the Hilbert space is $\ell^2(\mathbb{Z})$ and for any $\xi = (\xi_k)_{k\in\mathbb{Z}}$, one takes $(\pi'_\chi(b)\xi)_k := \chi \circ \alpha^k(b)\xi_k$ and the generator of $\mathbb{Z}$ is $(U\xi)_k = \xi_{k+1}$. As a representation of $\mathcal{U}_a$, one has $(\pi'_\chi(f)\xi)_k = f \circ \widehat{\alpha}^k(\chi)\xi_k$ for any $f \in C^*(B_a)$.

Theorem 3.6. *Assume $a \neq 1$ is algebraic.*
(i) There is a natural bijection between the set of orbits of $\widehat{\alpha}$ in S_a and the set of all equivalence classes of induced representations of $\mathcal{U}_a = C^*(G_a)$. This bijection is realized by $\chi \mapsto \pi_\chi^{\mathrm{Ind}}$.
(ii) The representation π_χ^{Ind} is irreducible if and only if χ is aperiodic.
(iii) The commutant of π_χ^{Ind} for a q-periodic point χ is the commutative algebra $C(\mathbb{T})$.
(iv) The right regular representation R of $\mathcal{U}_a = C^*(G_a)$ is unitarily equivalent to the representation $\int_{B_a}^{\oplus} \pi_\chi^{\mathrm{Ind}}\, d\mu(\chi)$.

For a q-periodic χ, the representation π_χ^{Ind} is reducible. Explicitly one has:

Proposition 3.7. *If χ is q-periodic, π_χ^{Ind} is unitarily equivalent to the representation π_χ on $\mathcal{H}_\chi$, so that its continuous decomposition into irreducible finite dimensional representations on $\mathbb{C}^q$ is realized by (10) along $\mathbb{T}$.*

This proposition states that, while the finite dimensional representations of $\mathcal{U}_a$ are not obtained as induced representations, they are nevertheless reductions of induced representations. The right regular representation R contains the infinite dimensional irreducible induced representations which are only accessible using aperiodic points. The representations R and π_χ are not quasi-equivalent: this difference will play a crucial role in the construction of different spectral triples, see Remark 5.4.

While the π_χ^{Ind}s yield a von Neumann factor of type I, R gives a type II_1 factor because of the integral. So the group G_a is non-type I.

4. The particular case $a = m \in \mathbb{N}^*$

According to Lemma 3.1, the case $a = m \in \mathbb{N}^*$ also covers the case $a = 1/m$. We do insist on this κ-deformed space since the algebra is then generated by two unitaries related by one relation (see (11) below) in the spirit of the noncommutative two-torus: $G_m = \mathrm{BS}(1, m)$ is the Baumslag–Solitar group which is generated by two elements and one-relator while, when a and a^{-1} are not integers, G_a is not a finitely presented group (still with two generators). This simplifies the computations of Sec. 3.2.

Moreover, the results described now rely more on some properties of the Baumslag–Solitar group than on the dynamical system. Thus, these results (which for the most are already valid and exposed for generic values of a) are presented independently. These structures appear also naturally in wavelet theory, which could benefit from our analysis.

4.1. The algebra

Definition 4.1. *Let $\mathcal{U}_m$ be the universal C^*-algebra labelled by $m \in \mathbb{Z}^*$ (restricted to $\mathbb{N}^*$ later) and generated by two unitaries U and V such that*

$$UVU^{-1} = V^m\,. \tag{11}$$

This universal C^*-algebra $\mathcal{U}_m$ is denoted by $\mathcal{O}(E_{1,m})$ in (Katsura 2008), and also $\mathcal{O}_{m,1}(\mathbb{T})$ in (Yamashita 2008) (where only $m \in \mathbb{N}^*$ is considered.) These algebras are topological graph C^*-algebras which can be seen as transformation group C^*-algebras on solenoid groups as already noticed in (Brenken, Jørgensen 1991; Jørgensen 2001; Brenken 1996). They have been used in wavelets and coding theory (Dutkay, Han, Picioroaga, Sun 2008; Dutkay, Jørgensen 2008; Dutkay, Jørgensen, Picioroaga 2009).

Relation (11) also appeared in the Baumslag–Solitar group $\mathrm{BS}(1,m)$ introduced in (Baumslag, Solitar 1962) as the group generated by u, v with a one-relator

$$\mathrm{BS}(1,m) := \left\langle u,\, v \,|\, uvu^{-1} = v^m \right\rangle\,.$$

This group plays a role in combinatorial and geometric group theory. It is a finitely generated, meta-Abelian, residually finite, Hopfian, torsion-free, amenable (solvable non-nilpotent) group. It has infinite conjugacy classes, a uniformly exponential growth (for $m \neq 1$) but is not Gromov hyperbolic (de la Harpe 2000). Note that $\mathrm{BS}(1,1)$ is the free Abelian group on two generators and $\mathrm{BS}(1,-1)$ is the Klein bottle group. As for the $\mathrm{BS}(1,m)$ groups, within the algebras $\mathcal{U}_m$, we remark that $\mathcal{U}_1$ and $\mathcal{U}_{-1}$ play a particular role: $\mathcal{U}_1 = C(\mathbb{T}^2)$ and $\mathcal{U}_{-1} \supset C(\mathbb{T}^2)$ will not be considered here since we need $a = m > 0$.

For $m \geq 2$, a solenoid appears as in Sec. 3.2, as well as a crossed product structure, a fact that we recall now in this particular context.

Assume $2 \leq m \in \mathbb{N}$ and let the subring of $\mathbb{Q}$ generated over $\mathbb{Z}$ by $\frac{1}{m}$

$$B_m = B_{1/m} := \mathbb{Z}\left[\tfrac{1}{m}\right] := \bigcup_{l \in \mathbb{N}} m^{-l}\mathbb{Z} \subset \mathbb{Q}\,.$$

It is the additive subgroup of $\mathbb{Q}$ which is an inductive limit of the rank-one groups $m^{-l}\mathbb{Z}$, for $l = 0, 1, 2, \ldots$ and B_m has a natural automorphism α defined by $\alpha(b) := mb$. Note that the Abelian group B_m is not finitely generated. When $m \to \infty$, $\mathrm{BS}(1,m) \to \mathbb{Z} \wr \mathbb{Z}$ (in the space of marked groups on two generators) (Stalder 2006). This group also appears when $m = e^{-\omega_0/\kappa}$ is replaced by a transcendental number $a \in \mathbb{R}^*_+$ as seen in Sec. 3.

B_m can be identified with the subgroup of the affine group $\mathrm{Aff}_1(\mathbb{Q})$ generated by the dilatation $u : x \to mx$ and the translation $v : x \to x+1$. It is the subgroup normally generated in $\mathrm{BS}(1,m)$ by $\langle v, u^{-1}vu, u^{-2}vu^2, \ldots\rangle$. The Baumslag–Solitar group $\mathrm{BS}(1,m)$ is then isomorphic to the crossed product $\mathrm{BS}(1,m) \simeq B_m \rtimes_\alpha \mathbb{Z}$ so that one has the group extension

$$1 \to \mathbb{Z}\left[\tfrac{1}{m}\right] \to B(1,m) \to \mathbb{Z} \to 1\,.$$

Using this crossed product decomposition, the group $\mathrm{BS}(1,m)$ has the following explicit law: $(b,l)(b',l') = (b+\alpha^l(b'), l+l')$ for $l,l' \in \mathbb{Z}$ and $b,b' \in B_m$. It is of course generated by the elements $u := (0,1)$ and $f_b := (b,0)$ with $b \in B_m$. Thus for $j,l \in \mathbb{Z}, n \in \mathbb{N}$, $uf_bu^{-1} = f_{\alpha(b)}$, and

$$\text{if } f_{\alpha^{-n}j} := \big(\alpha^{-n}j, 0\big) = u^{-n} f_j u^n\,, \text{ then } \big(\big(\tfrac{1}{m}\big)^n j, l\big) = f_{\left(\frac{1}{m}\right)^n j}\, u^l\,. \tag{12}$$

$\mathrm{BS}(1,m)$ is a subgroup of the "$ax+b$" group (endowed with the law $(b,a)(b',a') := (b+ab', aa')$) and can be viewed as the following subgroup of two-by-two matrices $\{\left(\begin{smallmatrix} m^l & b \\ 0 & 1\end{smallmatrix}\right) : l \in \mathbb{Z},\, b \in B_m\}$. $\mathrm{BS}(1,m) \simeq B(1,m')$ is equivalent to $m = m'$ (Moldavanskii 1991).

Let $\widehat{B_m}$ be the Pontryagin dual of B_m endowed with the discrete topology. It is isomorphic to the solenoid

$$S_m = S_{1/m} \simeq \left\{ (z_k)_{k=0}^\infty \in \prod_{i=0}^\infty \mathbb{T} \,:\, z_{k+1}^m = z_k,\ \forall k \in \mathbb{N}_0 \right\} \tag{13}$$

using $z_k := \chi\big((\tfrac{1}{m})^k\big)$ for any $\chi \in \widehat{B}_m$. The group S_m is compact connected and Abelian. Notice that (see Sec. 3.2)

$S_m \simeq \{(z_k)_{k=-\infty}^\infty \in \mathbb{T}^\mathbb{Z} \,:\, z_{k+1}^m = z_k,\ k \in \mathbb{Z}\}$ defining $z_{-k} := z_0^{mk}$ for $k > 0$.

The embedding $\hat\imath : \theta \in \mathbb{R} \mapsto \chi_\theta \in S_m$ where $\chi_\theta(b) := e^{i2\pi\theta b} \in \mathbb{T}$ for $b \in B_m$ identifies S_m as the Bohr compactification $b_{B_m}\mathbb{R}$ of $\mathbb{R}$.

S_m is endowed with a natural group automorphism $\widehat{\alpha}$ given by

$$\widehat{\alpha}(z_0, z_1, z_2, \ldots) = (z_0^m, z_0, z_1, \ldots)\,, \qquad \widehat{\alpha}^{-1}(z_0, z_1, z_2, \ldots) = (z_1, z_2, \ldots)\,.$$

All q-periodic points in S_m are of the following form: if z_0 is a solution of $z^{m^q-1} = 1$, then $(z_0, z_0^{m^{q-1}}, \ldots, z_0^m, z_0, \ldots) \in S_m$; so there are only finitely many periodic points, namely $c_q(m) = m^q - 1$ such points.

The C^*-algebra $C(S_m) \simeq C^*(B_m)$ is precisely the algebra of almost periodic functions on $\mathbb{R}$, with frequencies in B_m and the isomorphism is the map $f \mapsto f \circ \hat\imath$. Thus

$$\mathcal{U}_m = C^*(\mathrm{BS}(1,m)) \simeq C^*(B_m) \rtimes_\alpha \mathbb{Z} \simeq C(S_m) \rtimes_{\widehat{\alpha}} \mathbb{Z}\,.$$

The unitary element U of Definition 4.1 is precisely the generator of the action α of $\mathbb{Z}$ on $C^*(B_m)$ while V is one of the generators $\{U^{-\ell}VU^{\ell} : \ell \in \mathbb{Z}\}$ of the Abelian algebra $C^*(B_m)$. As a continuous function on S_m, $U^{-\ell}VU^{\ell}$ is the function $(z_k)_{k=0}^{\infty} \mapsto z_\ell$ and in particular, $V : (z_k)_{k=0}^{\infty} \mapsto z_0$.

The subgroup $\{z := (z_k)_{k=0}^{\infty} \in S_m : \widehat{\alpha}^q(z) = z \text{ for some } q \in \mathbb{N}^*\}$ of periodic points is dense in S_m and $\widehat{\alpha}$ is ergodic on S_m for $m \geq 2$ as previously seen (Brenken, Jørgensen 1991, Proposition 1).

4.2. The representations

The knowledge of *-representations of $\mathcal{U}_m$ is essential in the context of spectral triples (see Definition 5.1). According to (12), any unitary representation of $\mathrm{BS}(1,m)$ is given by a unitary operator U and a family of unitaries T_k, $k \in \mathbb{Z}$, with the constraint $UT_kU^{-1} = T_{mk}$, so there is a bijection between the *-representations of $\mathcal{U}_m$ on some Hilbert space $\mathcal{H}$ and the corresponding unitary representations of $\mathrm{BS}(1,m)$. This is rephrased usefully in the following lemma (Jørgensen 2001):

Lemma 4.2. *The algebra $\mathcal{U}_m$ is the C^*-algebra generated by $L^\infty(\mathbb{T})$ and a unitary symbol $\tilde{U}$ with commutation relations, where $e_n(z) := z^n$*

$$\tilde{U}\, f\, \tilde{U}^{-1} = f \circ e_m\,, \qquad \forall f \in L^\infty(\mathbb{T})\,. \tag{14}$$

$\mathcal{U}_m$ contains a family of Abelian subalgebras $\mathcal{A}_n := \tilde{U}^{-n}\, L^\infty(\mathbb{T})\, \tilde{U}^n$ for $n \in \mathbb{N}$, which is increasing since $\tilde{U}^{-n}\, f\, \tilde{U}^n = \tilde{U}^{-(n+1)}\, f \circ e_m\, \tilde{U}^{(n+1)}$.

If we choose the Hilbert space $\mathcal{H} := L^2(\mathbb{R})$, the scaling and shift operators give rise to a representation π of $\mathcal{U}_m$ on $\mathcal{H}$ by $\pi(U) : \psi(x) \mapsto \frac{1}{\sqrt{m}}\, \psi(\frac{x}{m})$ and $\pi(V) : \psi(x) \mapsto \psi(x-1)$.

The Haar measure ν on $\widehat{B}_m$ gives rise to a faithful trace on $\mathcal{U}_m$ and, since there are many finite dimensional representations of $\mathcal{U}_m$ (see proof of Theorem 3.2), there are many traces on it.

With $\mathcal{H} := L^2(S_m,\nu) \simeq \ell^2(B_m)$ and $U : \psi \in \mathcal{H} \mapsto \psi \circ \widehat{\alpha} \in \mathcal{H}$, $C(S_m)$ acts on $\mathcal{H}$ by left multiplication, we get a covariant representation for (S_m, U) of the dynamical system $(C(S_m), \widehat{\alpha}, \mathbb{Z})$, so a representation of $\mathcal{U}_m$ on $\mathcal{H}$. Since $\widehat{\alpha}$ is ergodic, this representation is irreducible and faithful (Brenken, Jørgensen 1991, Theorem 1).

If we choose $\mathcal{H} := \ell^2(\mathbb{Z})$, for each $\theta \in \mathbb{R}$ we get an induced representation of $\mathcal{U}_m$ by $\pi_\theta(U)\psi(k) := \psi(k-1)$ and $\pi_\theta(V)\psi(k) := \chi_\theta(m^{-k})\,\psi(k)$, $k \in \mathbb{Z}$.

5. On the existence of spectral triples

Since we want to construct spectral triples on $\mathcal{U}_a$, it is worthwhile to know the heat decay of $G_a = B_a \rtimes_\alpha \mathbb{Z}$ via a random walk on the Cayley graph of G_a with generators $S = \{x, x^{-1}, y, y^{-1}\}$, where $x = (0,1)$ and $y = (1,0)$, and with a constant weight and standard Laplacian.

The decay of the heat kernel p_t, with $t \in \mathbb{N}$, has been computed on the diagonal in (Pittet, Saloff-Coste 2002, Theorem 1.1), (Coulhon, Grigor'yan, Pittet 2001, Theorem 5.2): when $t \to \infty$, we get $p_{2t} \sim e^{-t^{1/3}\,(\log t)^{2/3}}$ if a is transcendental while $p_{2t} \sim e^{-t^{1/3}}$ if a is algebraic. This is related to the fact that G_a has exponential volume growth.

However, for a finite dimensional connected non-compact Lie group, the behaviour of the heat kernel p_t depends on $t \in \mathbb{R}_+^*$ and can diverge for the short time behaviour when $t \to 0$. Let us explain how this point is related to the dimension: In noncommutative geometry, a (regular simple) spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ has a (spectral) dimension which is given by $\max\{n \in \mathbb{N} \,:\, n \text{ is a pole of } \zeta_{\mathcal{D}} : s \in \mathbb{C} \to \operatorname{Tr}(|\mathcal{D}|^{-s})\}$ (here $\mathcal{D}$ is assumed invertible). In particular, when M is a n-dimensional compact Riemannian spin manifold, and $\mathcal{A} = C^\infty(M)$, $\mathcal{H} = L^2(S)$, where S is the spinor bundle and $\mathcal{D}$ is the canonical Dirac operator, the spectral dimension coincides with n. Via the Wodzicki residue, an integral $⨍ X := \operatorname{Res}_{s=0} \operatorname{Tr}(X|\mathcal{D}|^{-s})$ is defined on (classical) pseudodifferential operators X acting on smooth sections of S. For instance, $⨍ |\mathcal{D}|^{-n}$ coincides (up to a universal constant) with the Dixmier trace $\operatorname{Tr}_{\text{Dix}}(|\mathcal{D}|^{-n}) = \lim_{N\to\infty} \log(N)^{-1} \sum_{k=1}^N |\lambda_k|^{-n}$, where the λ_k are the singular values of $\mathcal{D}$. The dimension of M appears in $\operatorname{Tr}(e^{-t\mathcal{D}^2}) \sim \sum_{N\geq 0} \frac{1}{t^{(n-N)/2}}\, a_N(\mathcal{D})$ when $t \to 0$. In particular, when $M = \mathbb{R}^n$ with Lebesgue measure and $\mathcal{D}^2 = -\triangle$ is the standard Laplacian (non-compactness is not a problem), the heat kernel is $p_t(x,x) = \frac{1}{(4\pi t)^{n/2}}$ for all $x \in M$ (see (Connes 1994; Connes, Marcolli 2008; Gracía-Bondía, Várilly, Figueroa 2001). As a consequence, $\operatorname{Tr}(|\mathcal{D}|^{-(n+\epsilon)}) < \infty$ for all $\epsilon > 0$.

We will see that, depending on the chosen representation of $\mathcal{U}_a$, such an n does not always exist, meaning that the "dimension is infinite".

Definition 5.1. *A spectral triple (also called unbounded Fredholm module) $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is given by a unital C^*-algebra A with a faithful representation π on a Hilbert space $\mathcal{H}$ and an unbounded self-adjoint operator $\mathcal{D}$ such that*

- *the set $\mathcal{A} = \{a \in \mathcal{A} \,:\, [\mathcal{D}, \pi(a)]$ is bounded $\}$ is norm dense in A,*
- *$(1+\mathcal{D}^2)^{-1} \in J$, where J is a symmetrically-normed ideal of the compact operators $\mathcal{K}(\mathcal{H})$ on $\mathcal{H}$.*

The triple is p-summable if $J = \mathcal{L}^p(\mathcal{H})$ for $1 \leq p < \infty$ which means $\operatorname{Tr}\left((1+\mathcal{D}^2)^{-p/2}\right) < \infty$. It is p^+-summable if $J = \mathcal{L}^{p+}(\mathcal{H})$.

It is finitely summable if it is p-summable for some p.

It is θ-summable if there exists $t_0 \geq 0$ such that $\operatorname{Tr}\left(e^{-t\mathcal{D}^2}\right) < \infty$ for all $t > t_0$ (thus $J = \mathcal{K}(H)$).

Note that $\mathcal{A}$ is a *-subalgebra of A and p-summability implies θ-summability.

Connes proved in (Connes 1989) that, for an infinite, discrete, non-amenable group G, there exist no finitely summable spectral triples on $A = C^*_{\mathrm{red}}(G)$. However, in this case, there always exist θ-summable spectral triples on A (even with $\mathcal{D} > 0$). Using a computable obstruction to the existence of quasicentral approximate units relative to J for A, Voiculescu was able to derive, for solvable groups with exponential growth, the non-existence result for unbounded (generalized) Fredholm modules using the Macaev ideal $J = \mathcal{L}^{\infty,1}(\mathcal{H})$ (Voiculescu 1990). We use these results in the following:

Theorem 5.2. *Non-existence of finite-summable spectral triples.*
Let $A = \mathcal{U}_a$, $G_a = B_a \rtimes_\alpha \mathbb{Z}$ and $\mathcal{A} = \mathbb{C}[G_a]$.
(i) There is no finitely summable spectral triple $\big(\pi(\mathcal{A}), \mathcal{H}_\pi, \mathcal{D}\big)$ when the representation π is quasiequivalent to the left regular one.
(ii) There exist θ-summable spectral triples $\big(\pi(\mathcal{A}), \mathcal{H}_\pi, \mathcal{D}\big)$ with $t_0 = 0$, where the representation π is quasiequivalent to the left regular one.

Despite the previous result, we add a few explicit examples of spectral triples using the fact that the algebra $\mathcal{U}_a$ is residually finite. Clearly, these triples deal with a restrictive part of the geometry of the κ-deformation based on $\mathcal{U}_a$, namely the dynamical system which is behind. The residually finite property is seen via the periodic points of this dynamics.

Theorem 5.3. *Existence of finite-summable spectral triples.*
Let $A = \mathcal{U}_a$ and $\mathcal{A} = \mathbb{C}[G_a]$.
(i) There exist spectral triples $\big(\pi(\mathcal{A}), \mathcal{H}_\pi, \mathcal{D}\big)$ which are compact, i.e. $[\mathcal{D}, \pi(x)]$ is compact for all $x \in \mathcal{A}$.
(ii) There exist spectral triples $\big(\pi(\mathcal{A}), \mathcal{H}_\pi, \mathcal{D}\big)$ such that $[\mathcal{D}, \pi(x)] = 0$, $\forall x \in \mathcal{U}_a$, and with arbitrary summability.
(iii) When a is algebraic, there exist spectral triples $\big(\pi(\mathcal{A}), \mathcal{H}_\pi, \mathcal{D}\big)$ such that $[\mathcal{D}, \pi(v)] = 0$, $[\mathcal{D}, \pi(u)] \neq 0$ and with arbitrary summability $p \geq 2$.

In the case of (i), $[\mathcal{D}, \pi(x)]$ is not necessarily zero but the summability is not controlled while for the case (ii), the condition $[\mathcal{D}, \pi(x)] = 0$ enables us to control summability. In a sense, case (iii) is a mixed situation requiring that a be algebraic. In that situation, we have an explicit representation π so that formulae for Dirac operators can be proposed.

Remark 5.4. *There is no contradiction between Theorems 5.2 and 5.3 since the faithful quasidiagonal representation (or residually finite one) π of $\mathcal{U}_a$ used above to construct $\mathcal{D}$ is not quasiequivalent to the left regular one: actually, as already mentioned, the von Neumann algebra generated by $\pi(\mathcal{U}_a)$ is a II_1 factor when π is the left regular representation, while it is of type* I *when π is the quasidiagonal or residually finite one (Dixmier 1964, 5.4.3.).*

A more direct way to confirm that the representation π used in the proof of point (iii) of Theorems 5.3 is not quasiequivalent to the left regular representation (or to the right regular representation which is unitarily equivalent to the left one) is to notice that π is the direct integral $\pi = \int_{\rm Per}^{\oplus} \pi_\chi \, d\mu(\chi)$ of the finite dimensional *representations π_χ defined in (10). As such, this representation is strictly contained in the right regular representation R as can be checked using (iv) of Theorem 3.6. The part of R which is not in π is given by the induced* infinite dimensional *irreducible representations constructed on aperiodic χs.*

As noticed in (Skalski, Zacharias 2009), if $(\mathcal{A}, \mathcal{H}_\pi, \mathcal{D})$ is a spectral triple with $[\mathcal{D}, \pi(x)] = 0$, $\forall x \in A$, then A is a residually finite C^*-algebra.

Remark 5.5. *Theorem 5.2 says that the 2-dimensional κ-deformed space reflected by the algebra $\mathcal{U}_a$ with $\kappa = -\omega_0 \log^{-1}(a)$ is in fact "infinite dimensional" as a metric noncommutative space. Theorem 5.3 is a tentative to restore a metric. For instance, the distances on the state space $\mathcal{S}(\mathcal{U}_a)$ generated by Connes' formula*

$$d(\omega, \omega') := \sup \left\{ |\omega(a) - \omega'(a)| \, : \, a \in \mathcal{A}, \, ||[\mathcal{D}, a]|| \leq 1 \right\} , \qquad \omega, \omega' \in \mathcal{S}(\mathcal{U}_a)$$

are infinite in the case (ii) of Theorem 5.3, while in the case (iii) some states can be at finite distances.

Remark 5.6. *The operator $\mathcal{D}$ given in Theorem 5.3 (iii) is not directly related to the group structure of G_a but rather connected to the underlying dynamical system associated to the algebraic nature of a: it depends explicitly of the isomorphism-invariant $\{c_q(a) \, : \, q \in \mathbb{N}^*\}$.*

6. Conclusion

We have shown that κ-Minkowski space defined by (1) can be reduced to a compact or discrete version. Depending on κ, or on a defined in (5), this involves discrete amenable groups G_a, in particular the well-known Baumslag–Solitar ones. The associated C^*-algebras $\mathcal{U}_a$ can be viewed as a deformation of the two-torus. They are different when a varies within the rational numbers (of zero Lebesgue measure) because of the structure of the underlying dynamical system. For transcendental values of a, which are dense in $\mathbb{R}_+$ and of full Lebesgue measure, all these algebras are isomorphic to each other.

Due to the exponential growth of G_a, we have proved that the algebras $\mathcal{U}_a$ are not only quasidiagonal but also residually finite dimensional. They admit different spectral triples: the ones which are quasi-equivalent to the left regular representation and are never p-summable but only θ-summable, *i.e.* they are of "infinite metric dimension". This situation reminds us of the passage from non-relativistic to relativistic quantum mechanics: in quantum field theory, the θ-summability (and not the p-summability) naturally occurs due to the behaviour of $\mathrm{Tr}(e^{-tH})$ (when $t \to 0$), where H is the Hamiltonian (or $\mathcal{D}^2$), see for instance (Carpi, Hillier, Kawahigashi, Longo 2010).

The other faithful representations can generate fancy spectral triples which can have arbitrary summability (or "dimension") depending on the algebraic properties of the real parameter a, but are in fact degenerate to some extent. It is also not entirely clear what the topological content of these unbounded Fredholm modules is (*i.e.* whether they correspond to nontrivial elements of K-homology). The dimension of these spectral triples is unrelated to the number of coordinates defining the κ-deformed Minkowski spaces.

The nonexistence theorem, though powerful, does not preclude the possible existence of a genuine, non-degenerate, nontrivial spectral geometry on the κ-deformation spaces presented here, they only restrict the possible algebra representations that could be used in the construction.

This shows how delicate the notion of spectral or metric dimensions of κ-Minkowski space is, and how subtle its analysis through noncommutative geometry.

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