COMPACT κ-DEFORMATION AND SPECTRAL TRIPLES

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We construct discrete versions of κ-Minkowski space related to a certain compactness of the time coordinate. We show that these models fit into the framework of noncommutative geometry in the sense of spectral triples. The dynamical system of the underlying discrete groups (which include some Baumslag–Solitar groups) is heavily used in order to construct *finitely summable* spectral triples. This allows to bypass an obstruction to finite-summability appearing when using the common regular representation. The dimension of these spectral triples is unrelated to the number of coordinates defining the κ-deformed Minkowski spaces.

1. Introduction

In 1991, Lukierski, Ruegg, Nowicki and Tolstoy [44, 45] discovered a Hopf algebraic deformation of the Poincaré Lie algebra. The deformation parameter with units of mass is traditionally called κ. Only two years later, the Hopf algebra was represented on the κ-deformation of Minkowski space [60, 47]. This noncommutative space has been used immediately to generalize the notion of a quantum particle [46]. Its use to generalize quantum fields is still in progress [25]. Noncommutative differential calculi on the κ-Minkowski space were investigated [52] as well as the Noether

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theorem [3, 4]. The κ-Minkowski space has been popularized as 'double special relativity' [2] and is also known to appear in a 3-dimensional spinfoam model [30].

It is natural to ask whether κ-Minkowski space is a noncommutative geometry in the sense of Alain Connes [18, 19]. So far, despite some attempts [23], this question remained open.

In the κ-deformation of n-dimensional Minkowski space, the space-time coordinates satisfy the following solvable Lie-algebraic relations:

$$[x^0, x^j] := \frac{i}{\kappa} x^j, \qquad [x^j, x^k] = 0, \qquad j, k = 1, \ldots, n-1. \tag{1}$$

Here we assume $\kappa > 0$.

Using the Baker–Campell–Hausdorff formula, one gets [42, Eq. (2.6)]

$$e^{ic_\mu x^\mu} = e^{ic_0 x^0} e^{ic'_j x^j}$$

where

$$c'_j := \frac{\kappa}{c_0} (1 - e^{-c_0/\kappa}) c_j.$$

Actually, if $[A, B] = sB$, we have the "braiding identity"

$$e^A e^B = e^{(\exp s)B} e^A. \tag{2}$$

If the x^μ's are self-adjoint operators on some Hilbert space, we can define the n unitaries

$$U_\omega := e^{i\omega x^0}, \qquad V_{\vec{k}} := e^{-i\sum_{j=1}^{n-1} k_j x^j}$$

with $\omega, k_j \in \mathbb{R}$, which generate the κ-Minkowski group considered in [1] and an algebra $\mathcal{A}_\kappa$.

If $W(\vec{k}, \omega) := V_{\vec{k}} U_\omega$, one gets as in [1, Eq. (13)]

$$W(\vec{k}, \omega) W(\vec{k}', \omega') = W(e^{-\omega/\kappa}\vec{k}' + \vec{k}, \omega + \omega'). \tag{3}$$

The group law (3) is, for $n = 2$, nothing else but the crossed product

$$G_\kappa := \mathbb{R} \rtimes_\alpha \mathbb{R} \text{ with group isomorphism } \alpha(\omega)k := e^{-\omega/\kappa}k, \qquad k \in \mathbb{R}. \tag{4}$$

$G_\kappa \simeq \mathbb{R} \rtimes \mathbb{R}_+^*$ is the affine group on the real line which is solvable and nonunimodular (left Haar measure is $e^{\omega/\kappa} d\omega\, dk$). Being connected and simply connected, it is uniquely determined by its Lie algebra (1). All its irreducible unitary representations are known: they are either one-dimensional, or fall into two inequivalent classes [1].

It is worthwhile to recall that, when $\kappa \to \infty$, the usual plane $\mathbb{R}^2$ is recovered but with an unpleasant pathology at the origin [24].

An interesting situation occurs when $m := e^{-\omega/\kappa} \in \mathbb{N}^*$ for a given ω, since the group law can be reduced to a commutation relation between unitaries,

$$U_\omega V_{\vec{k}} = (V_{\vec{k}})^m U_\omega, \tag{5}$$

m being independent of k_j.

In this work, we want to investigate different spectral triples $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ associated to the group C^*-algebra of G_κ. To avoid technicalities due to a continuous spectrum of $\mathcal{D}$, we want a unital algebra so that we consider a periodic time x^0 which induces a discrete version G_a of G_κ where a is a real parameter depending on κ. This is motivated and done in Section 2.

In Section 3, we give the main properties of the algebra $\mathcal{U}_a = C^*(G_a)$ and its representations. For C^*-algebras of groups, the left regular representation is the natural one to consider. But due to the structure of G_a (solvable with exponential growth, see Theorem 3.3), there is a well-known obstruction to construct finite-summable spectral triples on $\mathcal{U}_a$ based on this representation. In order to bypass this obstruction, we need to refine our understanding of the structure of G_a in terms of an underlying dynamical system. This structure permits to define in Section 3.5 a particular representation of $\mathcal{U}_a$ using only the periodic points of the dynamical system. At the same time, following Brenken and Jørgensen [11], the topological entropy of this dynamical system is considered.

It is worthwhile to notice that the elementary building blocks G_a given by $a = m \in \mathbb{N}^*$ as in (5) are just some of the amenable Baumslag–Solitar groups, already encountered in wavelet theory [38]. The power of harmonic analysis on groups also justifies a reminder of their main properties in Section 4.

Finally, different spectral triples are exhibited in Section 5. The question of the finite summability of these triples is carefully considered with results by Connes [15] and Voiculescu [56, 57]: using the regular representation of $C^*(G_a)$, an obstruction to finite-summability appears and allows only θ-summability. However, representations of $C^*(G_a)$, not quasi-equivalent to the left regular one and based on the existence of periodic points for dynamical systems, can give rise to arbitrary finite-summable spectral triples. These results are summarized in Theorems 5.2 and 5.3 and they are commented in subsequent remarks.

2. Motivations and models

Relations as in (1) define an abstract noncommutative algebra in the generators x^0, x^j. As a purely algebraic object, this algebra contains elements which are finite sums of monomials in the generators. However, this construction is not suitable to study the noncommutative geometry (in the approach of Connes) of the κ-deformed (noncommutative) Minkowski space.

A much better approach is to consider a topological C^*-algebra, which will be based on the relations (1), and its representations on a Hilbert space. Certainly, then the operators x^0, x^j can be expected to be unbounded. The usual procedure is to use the Weyl quantization. In the following, we give a brief review of the method illustrated by three examples. This will help to understand the algebra we will consider later as our model in Section 2.2. For simplicity, we restrict ourselves to the two-dimensional case.

The first example is the undeformed plane $\mathbb{R}^2$. In the language of noncommutative geometry, one can (naively) describe its algebra as generated by two commuting

Hermitian generators $\hat{x}$ and $\hat{y}$. We suppose that these generators are represented as (unbounded) operators on a Hilbert space $\mathcal{H}$. To reduce cumbersome notations, we will use the same symbols $\hat{x}$ and $\hat{y}$ for the operators so that the following manipulations take place in the algebra of operators on $\mathcal{H}$. For any $(a, b) \in \mathbb{R}^2$, we introduce the unitary operators $W(a, b) := e^{ia\hat{x}}e^{ib\hat{y}}$. To any function $f \in L^1(\mathbb{R}^2)$, one associates the bounded operator on $\mathcal{H}$

$$W(f) := \int_{\mathbb{R}^2} f(a, b)\, W(a, b)\, da\, db\,.$$

This expression looks very much like a Fourier transform of f to which it is obviously related. The algebra we are looking for is then a topological algebra generated by the $W(f)$'s for f in a convenient space of functions: to be precise, we consider the C^*-algebra generated by the $W(f)$'s for $f \in L^1(\mathbb{R}^2)$.

This construction yields a product $f * g$ of two functions $f, g \in L^1(\mathbb{R}^2)$, which is defined by the relation $W(f * g) := W(f)W(g)$ and in order to understand this product, let us digress and show how this construction can be related to well-known facts about group C^*-algebras.

Let G be a topological group and let W be a unitary representation of G on a Hilbert space $\mathcal{H}$. Then, one has $W(s)W(t) = W(st)$ as unitary operators for any $s, t \in G$. To $f \in L^1(G, d\mu)$ (where $d\mu$ is a left Haar measure on G), one associates $W(f) := \int_G f(s)\, W(s)\, d\mu(s)$ which is a bounded operator on $\mathcal{H}$. Then, the C^*-algebra generated by the $W(f)$ for $f \in L^1(G, d\mu)$ is the image of the representation induced by W of the reduced C^*-algebra $C^*_{\text{red}}(G)$ of G on $\mathcal{H}$. Indeed,

$$W(f)\, W(g) = \int_G \left(\int_G f(t)g(t^{-1}s)\, d\mu(t) \right) W(s)\, d\mu(s),$$

so, if $(f * g)(s) := \int_G f(t)g(t^{-1}s)\, d\mu(t)$ is the usual convolution product in $C^*_{\text{red}}(G)$, then W is an algebra homomorphism.

Let us return to the example of the plane. One has $W(a, b)W(a', b') = W(a + a', b + b')$ so that W is a representation of the abelian group $G = \mathbb{R}^2$ for the additive law on both factors. This implies that the C^*-algebra we are looking for in that case is a representation of $C^*(\mathbb{R}^2) \simeq C_0(\mathbb{R}^2)$, where the explicit isomorphism is obtained by Fourier transforms. Here $C^*(\mathbb{R}^2)$ is an algebra of functions in the variables $(a, b) \in \mathbb{R}^2$ for the (ordinary) convolution product on $\mathbb{R}^2$ while $C_0(\mathbb{R}^2)$ is an algebra of functions (continuous and vanishing at infinity) in the variables (x, y) (the spectrum of the operators representing $\hat{x}$ and $\hat{y}$) for the ordinary product of functions. Notice that as expected the generators $\hat{x}$ and $\hat{y}$ do not belong to this algebra.

The second example is the space with canonical commutation relations. This noncommutative space is generated by two Hermitian elements $\hat{p}$ and $\hat{q}$ such that $[\hat{q}, \hat{p}] = i\hbar$. Then the usual Weyl procedure defines the unitary operators $W(\alpha, \beta) := e^{i\alpha\hat{q}+i\beta\hat{p}}$ for any $(\alpha, \beta) \in \mathbb{R}^2$. This gives immediately

$$W(\alpha, \beta)\, W(\alpha', \beta') = e^{-\frac{i\hbar}{2}(\alpha\beta' - \beta\alpha')}\, W(\alpha + \alpha', \beta + \beta'). \tag{6}$$

The associated C^*-algebra is generated by the bounded operators

$$W(f) := \int_{\mathbb{R}^2} f(\alpha, \beta) W(\alpha, \beta) \, d\alpha \, d\beta$$

for $f \in L^1(\mathbb{R}^2, d\alpha \, d\beta)$. Because (6) is not a group representation, this C^*-algebra is not related to the group C^*-algebra. In fact, (6) expresses the fact that this is a projective representation of the abelian group $\mathbb{R}^2$. One can transform it into a representation of the Heisenberg group H_3 which is the convenient central extension of $\mathbb{R}^2$ to do this end. Then the C^*-algebra associated to the canonical commutation relations is related to the C^*-algebra of the Heisenberg group H_3.

Let us now consider the example given by (1) for two Hermitian generators x^0 and x^1. For any $(k, \omega) \in \mathbb{R}^2$, one defines as before $W(k, \omega) := V_k U_\omega = e^{-ikx^1} e^{i\omega x^0}$. Then one has (3) which is a representation of the nonabelian group G_κ defined in (4). The C^*-algebra generated by the bounded operators

$$W(f) := \int_{G_\kappa} f(k, \omega) W(k, \omega) e^{\omega/\kappa} \, dk \, d\omega$$

for any $f \in L^1(G_\kappa, e^{\omega/\kappa} \, dk \, d\omega)$ (here $e^{\omega/\kappa} dk \, d\omega$ is the left Haar measure on G_κ) is the representation of $C^*_{\mathrm{red}}(G_\kappa)$ by W. The explicit product of two elements $f, g \in L^1(G_\kappa, e^{\omega/\kappa} \, dk \, d\omega)$ takes the form

$$(f *_\kappa g)(k, \omega) = \int_{G_\kappa} f(k', \omega') g(e^{\omega'/\kappa}(k - k'), \omega - \omega') e^{\omega'/\kappa} \, dk' \, d\omega'.$$

As for the example of the plane, the generators x^0 and x^1 cannot belong to this algebra.

The topological algebra to consider in order to study the noncommutative geometry of the 2-dimensional space defined by (1), is then the C^*-algebra $C^*_{\mathrm{red}}(G_\kappa)$ or some of its dense subalgebras.

The advantage of considering the theory of group C^*-algebras is twofold. Many structural properties on groups will turn out to be useful in studying some properties of the corresponding C^*-algebras. This is why we will focus on groups in the following. Furthermore, this allows us to construct in a natural way compact versions of noncommutative spaces as we shall now explain.

For an abelian topological group G, $C^*_{\mathrm{red}}(G)$ is isomorphic to $C_0(\widehat{G})$ where $\widehat{G}$ is the Pontryagin dual of G. Both algebras are defined as spaces of functions. By duality, a discrete subgroup $\Gamma \subset G$ produces the C^*-algebra $C^*_{\mathrm{red}}(\Gamma) \simeq C(\widehat{\Gamma})$ where $C(\widehat{\Gamma})$ is the C^*-algebra of continuous functions on the compact space $\widehat{\Gamma}$. Notice that there is a natural dual map $\widehat{G} \to \widehat{\Gamma}$. For the example of the plane, consider the discrete subgroup $\Gamma = \mathbb{Z}^2 \subset \mathbb{R}^2$. Then the resulting C^*-algebra is $C^*(\mathbb{Z}^2) \simeq C(\mathbb{T}^2)$ because $\widehat{\mathbb{Z}} = \mathbb{T}^1$. The dual map $\mathbb{R} \simeq \widehat{\mathbb{R}} \to \widehat{\mathbb{Z}} = \mathbb{T}^1$ is explicitly given by $x \mapsto e^{2\pi i x}$. The choice of the subgroup $\Gamma = \mathbb{Z}^2 \subset \mathbb{R}^2$ corresponds then to the choice of the compact version $\mathbb{T}^2$ of the (dual) space $\widehat{\mathbb{R}^2} \simeq \mathbb{R}^2$. The compactification takes place

in the space of the variables (x, y). Notice that not all compactifications of $\mathbb{R}^2$ can be obtained this way. The 2-sphere is a counter-example.

This procedure cannot be applied directly to the canonical commutation relations because its C^*-algebra is not a group C^*-algebra. But, as we will show, this procedure can be applied with success to the κ-deformed Minkowski space. In order to get a compact version of this noncommutative space, one has to choose a discrete subgroup $H_\kappa \subset G_\kappa$. Since H_κ is discrete and nonabelian, the associated algebra $C^*(H_\kappa)$ is unital and noncommutative, so it can be interpreted as a compact noncommutative space. This point is motivated in Section 2.1 and is done in Section 2.2.

As a final preliminary remark, let us mention that the groups we will encounter will be decomposed as crossed products with $\mathbb{Z}$, so both the (related) theories of discrete dynamical systems and crossed products of C^*-algebras will be intensively used in many parts of this work.

2.1. Spectral triples

The goal of this work is to study the existence of spectral triples for the κ-deformed space. Recall that a spectral triple (or unbounded Fredholm module) $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ [17, 18, 20] is given by a unital C^*-algebra A with a faithful representation π on a Hilbert space $\mathcal{H}$ and an unbounded self-adjoint operator $\mathcal{D}$ on $\mathcal{H}$ such that
— the set $\mathcal{A} = \{a \in \mathcal{A} : [\mathcal{D}, \pi(a)]$ is bounded $\}$ is norm dense in A,
— $(1 + \mathcal{D}^2)^{-1}$ has a compact resolvent.
($\mathcal{A}$ is always a *-subalgebra of A.)

Of course, the natural choice of the algebra A is to take the C^*-algebra of the group G_κ, but since $A = C^*(G_\kappa)$ has no unit, we need to replace the second axiom by
— $\pi(a)(1 + \mathcal{D}^2)^{-1}$ has a compact resolvent for any $a \in \mathcal{A}$.

This technical new axiom generates a lot of analytical complexities but is necessary to capture the metric dimension associated to $\mathcal{D}$. For instance, if a Riemannian spin manifold M is noncompact, the usual Dirac operator $\mathcal{D}$ has a continuous spectrum on $\mathcal{H} = L^2(S)$ where S is the spinor bundle on M. Nevertheless, the spectral triple $\left(C^\infty(M), L^2(S), \mathcal{D}\right)$ has a metric dimension which is equal to the dimension of M. A noncommutative example (the Moyal plane) of that kind has been studied in [31].

We try to avoid these difficulties here using a unital algebra A.

2.2. The compact version model as choice of a discrete subgroup

In this paper, we consider only dimension $n = 2$, since many of the results can be extended to higher dimensions thanks to (5).

As explained in detail above, in order to get a unital C^*-algebra, we prefer to work with a discrete subgroup H_κ of G_κ such that $1 \in C^*(H_\kappa)$.

Since we want also to keep separate the role of the two variables x^0 and x^1, it is natural to consider the subgroup of the form $H_\kappa = H \rtimes_\alpha \mathbb{Z}$: we first replace

the second $\mathbb{R}$ of G_κ in (4) by the lattice $\mathbb{Z}$ which corresponds to unitary periodic functions of a chosen frequency ω_0 (the time x^0 is now periodic).

So, given $\kappa > 0$ and $\omega_0 \in \mathbb{R}$, with

$$a := e^{-\omega_0/\kappa} \in \mathbb{R}^+, \tag{7}$$

the group $\mathbb{R} \rtimes_{\alpha_a} \mathbb{Z}$ is a subgroup of G_κ where $\alpha_a(n)$ is the multiplication by a^n. This subgroup is the affine group $\mathrm{Aff}_1(A)$ of a commutative unital ring $A = \mathbb{R}$ (or "$ax + b$" group). It is not discrete.

Then we want a group H to be a discrete (now, not necessarily topological) subgroup of the first $\mathbb{R}$ in $\mathbb{R} \rtimes_{\alpha_a} \mathbb{Z}$, which is invariant by the action α_a. Given $k_0 \in \mathbb{R}$, a natural building block candidate for a discrete H is given by $H = B_a \cdot k_0 \simeq B_a$ where

$$B_a := \{ \sum_{i, \text{ finite}} m_i \, a^{n_i} : m_i, n_i \in \mathbb{Z} \}.$$

More generally, one can take $H \simeq \oplus_{k_0 \in S} B_a$ where S is a discrete subspace of $\mathbb{R}^+$ such that $B_a \cdot k_0 \cap B_a \cdot k_1 = \varnothing$ for any $k_0, k_1 \in S$.

In some way, the search for a discrete subgroup H_κ of G_κ such that $1 \in C^*(H_\kappa)$ leads us to the group $H_{\kappa,a} := B_a \rtimes_{\alpha_a} \mathbb{Z}$ which is isomorphic to a subgroup of G_κ once k_0 is fixed.

This procedure drives us to the following situation which depends in particular upon the algebraic nature of a. Even if the details are given in Section 3, it is interesting to note here that for a transcendental, $B_a = \oplus_{\mathbb{Z}} \mathbb{Z}$, so that the groups $H_{\kappa,a}$ are all isomorphic. When a is algebraic, $\widehat{B}_a$ is a solenoid, isomorphic to a subgroup of $(\mathbb{T}^d)^{\mathbb{Z}}$ where d is related to the algebraic nature of the real number a. In that case, the structure of the groups $H_{\kappa,a}$ relies heavily on a.

Finally, simplifications occur when $a = m \in \mathbb{N}^*$ is an integer (which requires $\omega_0 < 0$, but see the remark after Lemma 3.1): this group $H_{\kappa,m}$ is well known since it is the solvable Baumslag–Solitar group $BS(1, m) = \mathbb{Z}[\frac{1}{m}] \rtimes_{\alpha_m} \mathbb{Z}$ as shown in Section 4 where we get $B_m = B_{1/m} = \mathbb{Z}[1/m]$. Moreover, in that situation we recover the two unitaries U_{ω_0} and V_{k_0} satisfying (5) and we may assume $H_{\kappa,m} = B_m \cdot k_0 \rtimes \mathbb{Z}$.

To conclude, we see that for different values of the (dimensionless) parameter a, we obtain a broad range of noncommutative spaces. In some cases, the algebras simplify significantly, as summarized in the following lemma.

LEMMA 2.1. *In two dimensions, there exists a unital subalgebra $C^*(H_{\kappa,m})$ of the κ-deformation algebra which is associated to the subgroup $H_{\kappa,m} = \mathbb{Z}[\frac{1}{m}] \rtimes_{\alpha_m} \mathbb{Z}$ of $G_{\kappa,m}$ and can be described as the algebra generated by two unitaries U, V satisfying the constraints $U = U_{\omega_0}$, $V = V_{k_0}$ and $UV = V^m U$. Here, $\kappa := -\omega_0 \log^{-1}(m) > 0$ for some given integer $m > 1$ and some $\omega_0 \in \mathbb{R}^-$, $k_0 \in \mathbb{R}$.*

3. The algebra $\mathcal{U}_a$ and its representations

In this section we define precisely and study in detail the C^*-algebra $\mathcal{U}_a$ which is our model for a compact version of the 2-dimensional κ-Minkowski space. The

structure of this algebra is described through a semi-direct product of two abelian groups, one of which depends explicitly on the real parameter $a > 0$. This semi-direct structure gives rise to a dynamical system which is heavily used in the following. The classification of the algebras $\mathcal{U}_a$ is performed: the K-groups are not complete invariants, and we use the entropy defined on the underlying dynamical system to complete this classification. Then some representations of $\mathcal{U}_a$ are considered. They depend on the algebraic or transcendental character of a. In the algebraic case, some particular finite-dimensional representations are introduced based on periodic points of the dynamical system. This construction will be used in Section 5.

Let $a = e^{-\omega_0/\kappa} \in \mathbb{R}_+^*$ with $a \neq 1$, and let us recall general facts from [11].

Define

$$B_a := \left\{ \sum_i m_i \, a^{n_i} \ \text{for finitely many} \ m_i, n_i \in \mathbb{Z} \right\}.$$

This discrete group is torsion-free so its Pontryagin dual $\widehat{B}_a$ is connected and compact.

Let α_a be the action of $\mathbb{Z}$ on $b \in B_a$ defined by $\alpha_a(n)b := a^n \, b$, let $\widehat{\alpha}_a$ be the associate automorphism on $\widehat{B}_a$ and

$$G_a := B_a \rtimes_{\alpha_a} \mathbb{Z}, \qquad \mathcal{U}_a := C^*(G_a) = C^*(B_a) \rtimes_{\alpha_a} \mathbb{Z} = C(\widehat{B}_a) \rtimes_{\widehat{\alpha}_a} \mathbb{Z}.$$

This type of C^*-algebras also appeared in [13] for totally different purposes.

We will write α instead of α_a when there is no confusion. The group G_a is generated by the two generators

$$u := (0, 1) \quad \text{and} \quad v := (1, 0).$$

LEMMA 3.1. *Let* $a \in \mathbb{R}_+^*$, *then* $B_a = B_{1/a}$ *and* $G_a \simeq G_{1/a}$. *Thus the* C^*-*algebra* $\mathcal{U}_a$ *and* $\mathcal{U}_{1/a}$ *are isomorphic.*

Proof: Clearly $B_a = B_{1/a}$ as subset of $\mathbb{R}$. The map : $(b, n) \in B_a \rtimes_{\alpha_a} \mathbb{Z} \mapsto (b, -n) \in B_{1/a} \rtimes_{\alpha_{1/a}} \mathbb{Z}$ defines a group isomorphism between G_a and $G_{1/a}$. $\quad\square$

As a consequence, the restriction on the sign of the frequency ω_0 in Lemma 2.1 is just an artifact since the cases $a = m$ and $a = 1/m$ are the same.

The symmetry point $a = 1/a$ corresponds to the commutative case in (5) with $a = 1$ or the undeformed relation (1) with $\kappa = \infty$. In this spirit $\mathcal{U}_a$ can be viewed as a deformation of the two-torus.

To analyze $\mathcal{U}_a$, we essentially follow [11, 10]. The dynamical system $\mathcal{U}_a \simeq C(\widehat{B}_a) \rtimes_\alpha \mathbb{Z}$ has an ergodic action (if $a \neq 1$) and a lot of periodic points. If

$$\mathrm{Per}_q(\widehat{B}_a) := \{ \chi \in \widehat{B}_a : \widehat{\alpha}^k(\chi) \neq \chi, \ \forall k < q, \ \widehat{\alpha}^q(\chi) = \chi \}$$

is the set of q-periodic points, then the growth rate $\lim_{q \to \infty} q^{-1} \log \left(\#\mathrm{Per}_q(\widehat{B}_a) \right)$ of this set is an invariant of $\mathcal{U}_a$ which coincides with the topological entropy $h(\widehat{\alpha}_a)$ by Yuzvinskii's formula (the topological entropy $h(\gamma)$ is a good invariant for

a homeomorphism γ of a compact topological space onto itself),

$$h(\widehat{\alpha_a}) = \lim_{q \to \infty} q^{-1} \log \left(\#\mathrm{Per}_q(\widehat{B}_a) \right). \tag{8}$$

This entropy can be finite or infinite, dividing the algebraic properties of a into two cases: a can be an algebraic or a transcendental number.

3.1. Transcendental case

If a is transcendental, then $B_a \simeq \mathbb{Z}[a, a^{-1}]$. Thus, $B_a \simeq \oplus_{\mathbb{Z}}\mathbb{Z}$, $\widehat{B}_a \simeq S_a := \{z = (z_k)_{k=-\infty}^{\infty} \in \mathbb{T}^{\mathbb{Z}}\}$ and $\left(\widehat{\alpha}(z) \right)_k = z_{k+1}$ for $z \in S_a$, $k \in \mathbb{Z}$, so that $\widehat{\alpha}$ is just the shift σ on $\mathbb{T}^{\mathbb{Z}}$. Thus

$$\mathcal{U}_a \simeq C(\mathbb{T}^{\mathbb{Z}}) \rtimes_{\sigma} \mathbb{Z} \quad \text{and} \quad h(\widehat{\alpha}) = \infty.$$

Note that the wreath product $\wr$ appears with its known presentation

$$G_a = B_a \rtimes_{\alpha} \mathbb{Z} \simeq \mathbb{Z} \wr \mathbb{Z} \simeq \langle u, v \ : \ [u^i v u^{-i}, v] = 1 \ \text{for all} \ i \geq 1 \rangle.$$

This group is amenable (solvable), torsion-free, finitely generated (but not finitely presented), residually finite with exponential growth.

S_a contains a lot of periodic points: q-periodic points are simply obtained by repeating any finite sequence $(z_k)_{k=0}^{q-1}$ of arbitrary elements in $\mathbb{T}$. Aperiodic points are easily constructed also.

Moreover, S_a is the Bohr compactification $b_{B_a}\mathbb{R}$ of $\mathbb{R}$: let $\hat{\iota}$ be the dual map of the natural group embedding ι of B_a into $\mathbb{R}$, so $\hat{\iota}$ is a continuous group homomorphism from $\mathbb{R}$ to S_a with dense image since we have $\hat{\iota} : x \in \mathbb{R} \mapsto (z_k)_{k=-\infty}^{\infty} \in S_a$ with $z_k = e^{i 2\pi a^{-k} x}$. Thus $\hat{\iota}$ is injective and surjective on elements $(z_k)_{k=-\infty}^{\infty} \in S_a$ such that $\lim_{k \to \infty} z_k = 1$ and it satisfies $\hat{\iota}(ax) = \widehat{\alpha}\left(\iota(x) \right)$. $\hat{\iota}$ is an infinitely winding curve on the solenoid S_a.

3.2. Algebraic case

Assume now that a is algebraic. Let $P \in \mathbb{Q}[x]$ be the monic irreducible polynomial such that $P(a) = 0$ and let $P = c \, Q_a$ where c^{-1} is the least common multiple of the denominators of coefficients of P (also called content), so $Q_a \in \mathbb{Z}[x]$. Since the coefficients of Q_a have greatest common divisor 1, the ideal (Q_a) generated by Q_a in $\mathbb{Z}[x]$ is prime. If d is the degree of Q_a, note the following ring isomorphism [11]

$$B_a \simeq \mathbb{Z}[x, x^{-1}]/(Q_a).$$

Moreover, B_a has a torsion-free rank d since $\{1, a, a^2, \cdots, a^{d-1}\}$ is a maximal set of independent elements in B_a, and $\widehat{B}_a$ has dimension d.

If

$$Q_a(x) = \sum_{j=0}^{d} q_j \, x^j$$

(so Q_a has leading coefficient $q_d \in \mathbb{N}^*$), let $A_a \in M_{d \times d}(\mathbb{Z})$ be the $d \times d$-matrix defined by $(A_a)_{i,j} := q_d \, \delta_{i,j-1}$ for $1 \leq j \leq d$ and $(A_a)_{d,j} = -q_{j-1}$. Then $q_d \, a^j = \sum_{k=1}^{d} (A_a)_{j,k} \, a^{k-1}$.

For instance, if $a = 1/m$ for $m \in \mathbb{N}^*$ then $P(x) = x - 1/m$, $Q_a(x) = mx - 1$, $d = 1$, so $B_a = \mathbb{Z}[\frac{1}{m}]$ and A_a is just the number 1.

Let σ be the shift on the group $(\mathbb{T}^d)^{\mathbb{Z}}$ and consider its σ-invariant subgroup

$$K_a := \{z = (z_k)_{k=-\infty}^{\infty} \in (\mathbb{T}^d)^{\mathbb{Z}} : q_d z_{k+1} = A_a z_k\}$$

where we use the identification $\mathbb{T} \simeq \mathbb{R}/\mathbb{Z}$. This group K_a is connected if and only if the greatest common divisor of q_d and $\det(A)$ is 1 by[10, Proposition 3.12]. This is automatic for $d = 1$ and for $d \geq 2$, this means $q_d = 1$ since $\det(A_a) = (-1)^d q_0 q_d^{d-1}$.

If S_a is the connected component of the identity of K_a, then there exists a topological group isomorphism $\psi : \widehat{B_a} \to S_a$ such that $\sigma_{|S_a} \circ \psi = \psi \circ \widehat{\alpha}$ [41, Theorem 19]. For any $\chi \in \widehat{B_a}$, the associated $z = (z_k)_{k=-\infty}^{\infty}$ is explicitly given by

$$z_k^{(i)} = \chi(a^{k+i-1})$$

where $z_k = (z_k^{(i)})_{i=1}^d \in \mathbb{T}^d$. In particular z_0 is given by $\left(\chi(1), \chi(a), \dots, \chi(a^{d-1})\right)$. This map is only surjective on the connected component of the identity.

When $a = 1/m$ with $m \in \mathbb{N}^*$, we will recover $S_{1/m} = S_m$ in (16).

There is a natural morphism of groups $\hat{\iota} : \mathbb{R}^d \to S_a$ defined as follows:

to any $\phi = (\phi^{(i)})_{i=1,\dots,d} \in \mathbb{R}^d$, one associates $\hat{\iota}(\phi) = z = (z_k)_{k=-\infty}^{\infty} \in (\mathbb{T}^d)^{\mathbb{Z}}$ with

$$z_k^{(i)} = \exp\left(2i\pi q_d^{-k} \sum_{j=1}^d (A_a^k)_{i,j} \phi^{(j)}\right). \tag{9}$$

This morphism shows that S_a is a Bohr compactification of $\mathbb{R}^d$ [10, Proposition 2.4].

Then $\widetilde{\alpha}(\phi) = q_d^{-1} A_a \phi$ defines a natural action of $\mathbb{Z}$ on $\mathbb{R}^d$ which satisfies $\hat{\iota} \circ \widetilde{\alpha} = \widehat{\alpha} \circ \hat{\iota}$.

If r_i, $i = 1, \cdots d$, are the complex roots of P, then by [11, Proposition 3 and Corollary 1],

$$c_q(a) := \# \operatorname{Per}_q(S_a) = \Pi_{k=1}^q |Q_a(e^{i2\pi k/q})| = |q_d|^q \Pi_{k=1}^d |1 - r_k^q|. \tag{10}$$

Thus by (8), the topological entropy is

$$h(\widehat{\alpha_a}) = \log|q_d| + \sum_{i, |r_i|>1} \log|r_i|. \tag{11}$$

In case $a = m$ or $a = 1/m$, (11) gives $h(\widehat{\alpha_m}) = h(\widehat{\alpha_{1/m}}) = \log(m)$.

Remark that $c_q(a) \neq 0$ or $\widehat{\alpha_a}$ is expansive or $h(\widehat{\alpha_a}) \neq 0$ if and only if $a \neq 1$ or $|r_k| \neq 1$ for all k.

Aperiodic points in S_a can be easily constructed using the map $\hat{\iota}$ defined by (9): any $\phi \in (\mathbb{R} \backslash \mathbb{Q})^d$ defines an aperiodic point $\hat{\iota}(\phi) \in S_a$.

3.3. The structure of algebras $\mathcal{U}_a$

In order to better understand the structure of the algebra $\mathcal{U}_a$, we obtain some results about the underlying dynamical system.

PROPOSITION 3.2. *Let $a \in \mathbb{R}_+^*$ and $a \neq 1$.*

(i) *The subgroup of periodic points and the set of aperiodic points of S_a under $\widehat{\alpha}$ are dense.*

(ii) *The space of orbits of S_a is not a T_0-space.*

Proof: (i) By [11, Proposition 4], the periodic points are dense.

When a is transcendental, the aperiodic points are clearly dense.

For $a \neq 1$ algebraic, we use the fact that the set of aperiodic points is dense if and only if $\mathrm{Per}_q(S_a)$ has empty interior for any positive integers q by [55, Lemma 3.1]. When a is an algebraic number, we already saw in (10) that $\mathrm{Per}_q(S_a)$ has only a finite number of points $c_q(a)$, so has empty interior.

(ii) It is enough to find two points z and z' with different orbits $\mathcal{O}(z) \neq \mathcal{O}(z')$ such that $\overline{\mathcal{O}(z)} = \overline{\mathcal{O}(z')} = S_a$, i.e. to find two distinct points in the space of orbits whose closures are the same. We will use the following criteria: a subset $X \subset \widehat{B_a}$ is dense in $\widehat{B_a}$ if and only if $\{b \in B_a : \chi(b) = 1, \chi \in X\} = \{0\}$.

Consider first the case $a \neq 1$ algebraic with associated irreducible polynomial $Q_a \in \mathbb{Z}[x]$. Let $\phi \in \mathbb{R}^d$ and define $z = \hat{\iota}(\phi)$. Using the relations induced by Q_a on the monomials a^r for $r \in \mathbb{Z}$, any element $b = \sum_{i \in I} m_i a^i$ with $I \subset \mathbb{Z}$ finite and $m_i \in \mathbb{Z}$, can be written $b = \sum_{j=1}^d r_j a^{j-1}$ with now $r_j \in \mathbb{Q}$. Then $z(b) = \exp\left(2i\pi q_d^{-k} \sum_{j=1}^d r_j \phi^{(j)}\right)$. For the particular points $z = \hat{\iota}(\phi)$ with $\phi^{(j)} = \phi^{(0)} \in \mathbb{R} \backslash \mathbb{Q}$, the relation $z(b) = 1$ admits no nontrivial solution in b, so that $b = 0$.

When a is transcendental, the situation is simpler. Consider the aperiodic point z defined by $z_k = e^{2i\pi k\alpha}$ for an irrational $\alpha \in \mathbb{R} \backslash \mathbb{Q}$. Then one has $z(b) = \exp\left(2i\pi\alpha \sum_{k \in I} k\, m_k\right)$ and only $b = 0$ is solution to $z(b) = 1$.

In both cases, the orbits of all given points z are dense in S_a, and there are a lot of such points with different orbits, so the space of orbits of S_a is not a T_0-space. $\qquad\square$

In this proof we saw that many points $z \in S_a$ have dense orbits in S_a.

We can now give the main properties of the algebras $\mathcal{U}_a$.

THEOREM 3.3. *Let $a \in \mathbb{R}_+^*$ and $a \neq 1$. Then:*

(i) *The group $G_a = B_a \rtimes_\alpha \mathbb{Z}$ is a torsion-free discrete solvable group with exponential growth and $\widehat{B_a}$ is a compact set isomorphic to a solenoid S_a.*

(ii) *$\mathcal{U}_a = C_{\mathrm{red}}^*(B_a \rtimes_\alpha \mathbb{Z}) \simeq C(S_a) \rtimes_{\widehat{\alpha}} \mathbb{Z}$ is an NGCR [§], AF-embeddable, nonsimple, residually finite-dimensional C^*-algebra and its generated von Neumann algebra for the left regular representation is a type II_1- factor.*

[§]A C^*-algebra A is said to be CCR or liminal if $\pi(A)$ is equal to the set of compact operators on the Hilbert space $\mathcal{H}_\pi$ for every irreducible representation π. The algebra A is called NGCR if it has no nonzero CCR ideals.

An AF-algebra is a inductive limit of sequences of finite-dimensional C^*-algebras.

The algebra A is residually finite-dimensional if it has a separating family of finite-dimensional representations.

(iii) *Every element in* $\mathcal{U}_a$ *has a connected spectrum. In particular, the only idempotents in* $\mathcal{U}_a$ *are* 0 *and* 1.

(iv) *The* C^*-*subalgebra* $C(S_a)$ *is maximal abelian in* $\mathcal{U}_a$.

Proof: (i) It remains to control the growth of the group. We apply a method inspired by the case of Baumslag–Solitar groups, using the fact that G_a is generated by the two elements $u = (0, 1)$ and $v = (1, 0)$. Define the map $\pi : G_a \to GL(2, \mathbb{R})$ with

$$\pi(u) := \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \qquad \pi(v) := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{12}$$

Then, π is a morphism of groups and $\pi(G_a)$ has an exponential growth as in [33, Example 3, p. 188]. Since the growth of a quotient of a group is less than the growth of the group itself, we get the result.

(ii) To show that $\mathcal{U}_a$ is NGCR, it is sufficient, using [58, Theorem 8.43], to show that the orbit space of S_a is not a T_0-space. This is done in (ii) of Proposition 3.2.

The set of periodic points z in S_a is dense by Proposition 3.2 and the nonwandering set of $(S_a, \widehat{\alpha})$ is S_a [11]. Thus $\mathcal{U}_a = C(S_a) \rtimes_{\widehat{\alpha}} \mathbb{Z}$ is AF-embeddable by [50] and so $\mathcal{U}_a$ is quasidiagonal.

The algebra $\mathcal{U}_a$ is nonsimple since the existence of periodic points in S_a implies that $\widehat{\alpha}$ is not minimal.

The algebra $\mathcal{U}_a$ is residually finite:

When $a \neq 1$ is algebraic, we follow the construction of finite-dimensional representations [55, 59]: Let $z_q \in \mathrm{Per}_q(S_a)$ be a q-periodic point of $\widehat{\alpha}$. Let $\rho_{z_q} : C(S_a) \to M_q(\mathbb{C})$ be a representation of $C(S_a)$ defined by

$$\rho_{z_q}(f) := \mathrm{Diag}\big(f(z_q), \cdots , f\big(\widehat{\alpha}^{q-1}(z_q)\big)\big) \in M_q(\mathbb{C}),$$

and for $x \in \mathbb{T}$, let

$$u_{x,z_q} := \begin{pmatrix} 0 & x \\ 1_{q-1} & 0 \end{pmatrix} \in M_q(\mathbb{C}).$$

This is a unitary which satisfies the covariance relation $u_{x,z_q}^* \rho_{z_q}(f) u_{x,z_q} = \rho_{z_q}(f \circ \widehat{\alpha})$ thus $\pi_{x,z_q} := \rho_{z_q} \rtimes u_{x,z_q}$ is a representation of $\mathcal{U}_a$ on $M_q(\mathbb{C})$. Again, $\pi_x := \oplus_{q=1}^{\infty} \oplus_{z_q \in \mathrm{Per}_q(S_a)} \pi_{x,z_q}$ is a representation of $\mathcal{U}_a$. So, for a dense family $\{x_l\}_{l=1}^{\infty}$ in $\mathbb{T}$, using the canonical faithful conditional expectation $C(S_a) \rtimes_{\widehat{\alpha}} \mathbb{Z} \to C(S_a)$ and the density of periodic points, one shows that $\pi := \oplus_{l=1}^{\infty} \pi_{x_l}$ is a faithful representation of $\mathcal{U}_a$ such that $\pi(\mathcal{U}_a) \subset \oplus_{l=1}^{\infty} \oplus_{q=1}^{\infty} \oplus_{z \in \mathrm{Per}_q(S_a)} M_q(\mathbb{C})$.

When a is transcendental, $B_a = \mathbb{Z} \wr \mathbb{Z}$ is an amenable group which has a separating family of finite-dimensional unitary representations (it is a maximally almost periodic group) since it is residually finite and finitely generated, so $\mathcal{U}_a$ is residually finite by [6, Theorem 4.3].

Since the conjugacy classes of B_a are infinite, the von Neumann algebra generated by its left regular representation is a type II_1 factor ([26, 13.11.14.e] or [7, III.3.3.7 Prop]).

(iii) As in [48, Lemma 2.1], the claim on the connected spectrum is equivalent to the conjecture of idempotents for $\mathcal{U}_a$. This in turn is a consequence of the Baum–Connes conjecture which has been proved by Higson–Kasparov [34] for torsion-free solvable groups like $G_a = B_a \rtimes_\alpha \mathbb{Z}$.

(iv) By [55, Theorem 1.1], this property is equivalent to the fact that $\widehat{\alpha}$ is topologically free, namely the aperiodic points in S_a are dense, property already proved in Proposition 3.2. □

A main point of this theorem which is crucial for the sequel is that the algebra $\mathcal{U}_a$ is residually finite.

3.4. On the classification of algebras $\mathcal{U}_a$

The algebras $\mathcal{U}_a$ are characterized by their underlying dynamical system.

THEOREM 3.4. *Let* $\omega_0 \in \mathbb{R}$ *and* $\kappa \in \mathbb{R}_+^*$ *defining* $a \neq 1$ *in* (7).
(i) $\mathcal{U}_a \simeq \mathcal{U}_{a'}$ *yields* $c_q(a) = c_q(a')$, $\forall q \in \mathbb{N}^*$.
(ii) *The entropy* $h(\widehat{\alpha})$ *is also an isomorphism-invariant of* $\mathcal{U}_a$.

Proof: (i) The set $\{c_q(a) : q \in \mathbb{N}^*\}$ defined in (10) is an invariant of $C(S_a) \rtimes_{\widehat{\alpha}} \mathbb{Z}$ by [40]. This can be also seen concretely in the construction of the above residually finite representation π since the π_{x,z_q} depends only on c_q.

(ii) This is a consequence of (i); see [11, Theorem 2]. □

This result has important physical consequences since a full Lebesgue measure dense set of different parameter a (namely the transcendental ones) generates the same algebra or κ-deformed space, while in the rational case, these spaces are different.

COROLLARY 3.5. *As already seen,* $\mathcal{U}_a \simeq \mathcal{U}_{1/a}$. *Moreover:*
(i) *All transcendental numbers* a *generate isomorphic algebras* $\mathcal{U}_a$.
(ii) *If* $\mathcal{U}_a \simeq \mathcal{U}_{a'}$, *then* a *and* a' *are both simultaneously algebraic or transcendental numbers.*
(iii) *If* $\mathcal{U}_a \simeq \mathcal{U}_{a'}$, *then* $a' = a$ *or* $a' = a^{-1}$ *in the following cases:* a, a' *or their inverses are in* $\mathbb{Q}^*$ *or are quadratic algebraic numbers.*
(iv) *If* $a = m/l \in \mathbb{Q}_+^*$, $K_0(\mathcal{U}_a) \simeq \mathbb{Z}$ *and* $K_1(\mathcal{U}_a) \simeq \mathbb{Z} \oplus \mathbb{Z}_{l-m}$.
(v) *Let* a *be a quadratic algebraic number with* $Q_a = \sum_{j=0}^2 q_j x^j$ *and let* l *be the least common multiple of* q_0 *and* q_2. *For* n *integer, define* $n{:}l = n(n, l^{m_0})^{-1}$ *where* m_0 *is the maximum multiplicity of any prime dividing* n *and* (n, m) *is the greatest common divisor of* n *and* m *(so* $n{:}l$ *is obtained by removing from* n *any prime also dividing* l).

If $q_2 \neq q_0$, *then* $K_0(\mathcal{U}_a) \simeq \mathbb{Z} \oplus \mathbb{Z}_{(q_2-q_0){:}l}$ *and* $K_1(\mathcal{U}_a) \simeq \mathbb{Z} \oplus \mathbb{Z}_{q_0+q_1+q_2}$.
If $q_2 = q_0$, *then* $K_0(\mathcal{U}_a) \simeq \mathbb{Z} \oplus \mathbb{Z}[\frac{1}{l}]$ *and* $K_1(\mathcal{U}_a) \simeq \mathbb{Z} \oplus \mathbb{Z}[\frac{1}{l}] \oplus \mathbb{Z}_{q_0+q_1+q_2}$.

Proof: (i) This is proved in Section 3.1.
(ii) We saw that the entropy $h(\widehat{\alpha_a})$ is finite if and only if a is algebraic.

(iii) The proof follows from [8], but let us give an easy argument in the first case: we may assume $a = m/l$ and $a' = m'/l'$ are irreducible rational numbers with $0 < m < l$ and $0 < m' < l'$. For $a = m/l$, $Q_a(x) = mx - l$, so $h(\widehat{\alpha_{m/l}}) = \log l$ and $c_1(\widehat{\alpha_{m/l}}) = l - m$. Using Theorem 3.4(i), we obtain $l = l'$ and $m = m'$.

(iv) and (v) see [9]. □

A natural question, turning up in the algebraic case is whether K-theory can be used to classify the corresponding algebras. However, using the results of [9], where algebraic numbers of degree less than 3 were considered, we already know that the K-groups do not give a complete classification: if $a = \sqrt{3}$ and $a' = \sqrt{7} - 2$, then $\mathcal{U}_a$ and $\mathcal{U}_{a'}$ are not isomorphic by (iii), but yield the same K-groups by (v) since $Q_a(x) = x^2 - 3$ and $Q_{a'}(x) = x^2 + 4x - 3$.

Moreover, Brenken [9, Proposition 3.1] also proved the following.

PROPOSITION 3.6. *If τ is a tracial state on $\mathcal{U}_a$, then the range of τ on $K_0(\mathcal{U}_a)$ is $\mathbb{Z}$.*

3.5. On some representations of $\mathcal{U}_a$ for algebraic a

In order to construct spectral triples on $\mathcal{U}_a$ in Section 5, it is convenient to get a clear picture of some of the essential representations of algebras $\mathcal{U}_a$. We will concentrate on the algebraic case.

In the proof of Theorem 3.3, we introduced an irreducible finite-dimensional representation $\pi_{x,\chi} = \rho_\chi \rtimes u_{x,\chi}$ of $\mathcal{U}_a$ on $\mathbb{C}^q$ associated to a q-periodic point $\chi \in \widehat{B}_a (= S_a)$ and a $x \in \mathbb{T}$. This representation can be extended to a representation π_χ on the Hilbert space $\mathcal{H}_\chi = L^2(\mathbb{T}) \otimes \mathbb{C}^q$ by

$$\pi_\chi = \int_{\mathbb{T}} \pi_{x,\chi} \, dx. \qquad (13)$$

In order to get an explicit formula for this representation, denote by $\{e_s^{(q)}\}_{s=1,\dots q}$ the canonical basis of $\mathbb{C}^q$ and by $e_n : \theta \mapsto e^{in\theta}$, for $n \in \mathbb{Z}$, the natural basis of $L^2(\mathbb{T}) \simeq \ell^2(\mathbb{Z})$. Then one defines, for any $f \in C^*(B_a) = C(\widehat{B}_a)$:

$$\pi_\chi(f)(e_n \otimes e_s^{(q)}) = f \circ \widehat{\alpha}^{s-1}(\chi) \, e_n \otimes e_s^{(q)},$$

$$U(e_n \otimes e_s^{(q)}) = \begin{cases} e_n \otimes e_{s+1}^{(q)} & \text{for } 1 \leq s < q, \\ e_{n+1} \otimes e_1^{(q)} & \text{for } s = q, \end{cases}$$

where U is the generator of $\mathbb{Z}$. This representation is obviously constructed from the representation of $G_a = B_a \rtimes_\alpha \mathbb{Z}$ given by $\pi_\chi(b)(e_n \otimes e_s^{(q)}) = \chi \circ \alpha^{s-1}(b) \, e_n \otimes e_s^{(q)}$ for any $b \in B_a$ (the generator U of the action of $\mathbb{Z}$ is the same).

Another natural representation to consider, is the representation of $\mathcal{U}_a$ obtained from the left regular representation of G_a.

In [43] and more systematically in [29], the induced representations *à la* Mackey of G_a for algebraic a have been investigated. The main results are the following.

For any $\chi \in \widehat{B}_a$, let the space of functions $\varphi : B_a \rtimes_\alpha \mathbb{Z} \to \mathbb{C}$ such that $\varphi(b, k) = \chi(b)\varphi(0, k)$ for any $b \in B_a$ and $k \in \mathbb{Z}$ be endowed with the norm $\|\varphi\|^2_\chi = \sum_{k \in \mathbb{Z}} |\varphi(0, k)|^2$. This defines a Hilbert space denoted by $\mathcal{H}^{\mathrm{Ind}}_\chi$. The induced representation of G_a on $\mathcal{H}^{\mathrm{Ind}}_\chi$ is given by $(\pi^{\mathrm{Ind}}_\chi(g)\varphi)(h) = \varphi(hg)$ for any $g, h \in G_a$.

It is shown in [29, Theorem 4.2] that this representation is unitarily equivalent to the following one: the Hilbert space is $\ell^2(\mathbb{Z})$ and for any $\xi = (\xi_k)_{k \in \mathbb{Z}}$, one defines $(D_\chi(b)\xi)_k = \chi \circ \alpha^k(b)\xi_k$ and the generator of $\mathbb{Z}$ is $(U\xi)_k = \xi_{k+1}$. As a representation of $\mathcal{U}_a$, one has $(D_\chi(f)\xi)_k = f \circ \widehat{\alpha}^k(\chi)\xi_k$ for any $f \in C^*(B_a)$.

THEOREM 3.7. *Assume $a \neq 1$ is algebraic.*
(i) *There is a natural bijection between the set of orbits of $\widehat{\alpha}$ in S_a and the set of all equivalence classes of induced representations of $\mathcal{U}_a = C^*(G_a)$. This bijection is realized by $\chi \mapsto \pi^{\mathrm{Ind}}_\chi$.*
(ii) *The representation π^{Ind}_χ is irreducible if and only if χ is aperiodic.*
(iii) *The commutant of π^{Ind}_χ for a q-periodic point χ is the commutative algebra $C(\mathbb{T})$.*
(iv) *The right regular representation R of $\mathcal{U}_a = C^*(G_a)$ is unitarily equivalent to the representation*

$$\int^\oplus_{B_a} \pi^{\mathrm{Ind}}_\chi \, d\mu(\chi).$$

Proof: These results are proved in [29] for the corresponding representations of the group G_a: (i) is Theorem 4.13, (ii) and (iii) are Theorem 4.8 and (iv) is Theorem 6.3. $\qquad\square$

For a q-periodic χ, the representation π^{Ind}_χ is reducible. Explicitly one has the following:

PROPOSITION 3.8. *If χ is q-periodic, then π^{Ind}_χ is unitarily equivalent to the representation π_χ on $\mathcal{H}_\chi$, so that its continuous decomposition into irreducible finite-dimensional representations on $\mathbb{C}^q$ is realized by (13) along $\mathbb{T}$.*

Proof: π^{Ind}_χ is unitarily equivalent to the representation D_χ on $\ell^2(\mathbb{Z})$. The unitary map between the two Hilbert spaces $\mathcal{H}_\chi$ and $\ell^2(\mathbb{Z})$ defined by the correspondence $e_n \otimes e_s^{(q)} \mapsto \delta_{nq+r-1}$ with $(\delta_k)_{k \in \mathbb{Z}}$ as natural basis of $\ell^2(\mathbb{Z})$, intertwines the representations π_χ and D_χ. $\qquad\square$

This proposition states that, while the finite-dimensional representations of $\mathcal{U}_a$ are not obtained as induced representations, they are nevertheless reductions of induced representations. The right regular representation R contains the infinite-dimensional irreducible induced representations which are only accessible using aperiodic points. The representations R and π_χ are not quasi-equivalent, this difference will play a crucial role in the construction of different spectral triples, see Remark 5.4.

Despite the fact that the π^{Ind}_χ's yield a von Neumann factor of type I, R gives a type II$_1$ factor because of the integral. So the group G_a is non-type I.

4. The particular case $a = m \in \mathbb{N}^*$

According to Lemma 3.1, the case $a = m \in \mathbb{N}^*$ considered in this section also covers the case $a = 1/m$. We do insist on this special compact version of a κ-deformed space since in this case the algebra is generated by two unitaries related by one relation (see (14) below) in the spirit of the popular noncommutative two-torus: concretely, $G_m = BS(1, m)$ is the Baumslag–Solitar group which is generated by two elements with a one-relator while, when a and a^{-1} are not integers, G_a is not a finitely presented group (even if it has two generators). Thus this assumption simplifies the computations of Section 3.2.

Moreover, the results described in this section rely more on some well-known properties of the Baumslag–Solitar group than on the dynamical system used until now. As a consequence, these results (which for the most are already valid and exposed for generic values of a) are presented and proved independently. These structures appears also naturally in wavelet theory, which could benefit from our analysis.

4.1. The algebra

DEFINITION 4.1. *Let $\mathcal{U}_m$ be the universal C^*-algebra labelled by $m \in \mathbb{Z}^*$ (restricted to $\mathbb{N}^*$ later) and generated by two unitaries U and V such that*

$$UVU^{-1} = V^m. \tag{14}$$

This universal C^*-algebra $\mathcal{U}_m$ is denoted by $\mathcal{O}(E_{1,m})$ in [39], and also $\mathcal{O}_{m,1}(\mathbb{T})$ in [59] (where only $m \in \mathbb{N}^*$ is considered). These algebras are topological graph C^*-algebras which can be seen as transformation group C^*-algebras on solenoid groups as already noticed in [11, 38, 10]. They have been used in wavelets and coding theory [27–29].

Relation (14) also appeared in the Baumslag–Solitar group $BS(1, m)$ introduced in [5] as the group generated by two elements u, v with a one-relator,

$$BS(1, m) := \langle u, \, v \, | \, uvu^{-1} = v^m \rangle.$$

This group plays a role in combinatorial and geometric group theory. It is a finitely generated, meta-abelian, residually finite, Hopfian, torsion-free, amenable (solvable nonnilpotent) group. It has infinite conjugacy classes, a uniformly exponential growth (for $m \neq 1$) but is not Gromov hyperbolic [33]. There exists a monomorphism π from $BS(1, m)$ into the $PSL(2, \mathbb{R})$ group (see also (12)) given by

$$\pi(u) := \begin{pmatrix} \sqrt{m} & 0 \\ 0 & \sqrt{m^{-1}} \end{pmatrix}, \qquad \pi(v) := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Note that $BS(1, 1)$ is the free abelian group on two generators and $BS(1, -1)$ is the Klein bottle group.

As for the $BS(1, m)$ groups, within the algebras $\mathcal{U}_m$, we remark that $\mathcal{U}_1$ and $\mathcal{U}_{-1}$ play a particular role: $\mathcal{U}_1 = C(\mathbb{T}^2)$ (continuous functions on the 2-torus), and $\mathcal{U}_{-1} \supset C(\mathbb{T}^2)$ will not be considered here since we need $a = m > 0$.

For $m \geq 2$, a solenoid appears as in Section 3.2, as well as a crossed product structure, a fact that we recall now in this particular context.

Assume $m \geq 2$ and let the subring of $\mathbb{Q}$ generated over $\mathbb{Z}$ by $\frac{1}{m}$ be

$$B_m = B_{1/m} := \mathbb{Z}[\tfrac{1}{m}] := \bigcup_{l \in \mathbb{N}} m^{-l}\mathbb{Z} \subset \mathbb{Q}.$$

It is the additive subgroup of $\mathbb{Q}$ which is an inductive limit of the rank-one groups $m^{-l}\mathbb{Z}$, for $l = 0, 1, 2, \ldots$, and B_m has a natural automorphism α defined by

$$\alpha(b) := mb.$$

Note that the abelian group B_m is not finitely generated. When $m \to \infty$, $BS(1, m) \to \mathbb{Z} \wr \mathbb{Z}$ (in the space of marked groups on two generators) [54]. This group also appears when $m = e^{-\omega_0/\kappa}$ is replaced by a transcendental number $a \in \mathbb{R}_+^*$ as seen in Section 3.

B_m can be identified with the subgroup of the affine group $\mathrm{Aff}_1(\mathbb{Q})$ generated by the dilatation $u : x \to mx$ and the translation $v : x \to x + 1$. It is the subgroup normally generated in $BS(1, m)$ by v, $\langle v : u^{-1}vu, u^{-2}vu^2, \ldots \rangle$. The Baumslag–Solitar group $BS(1, m)$ is then isomorphic to the crossed product

$$BS(1, m) \simeq B_m \rtimes_\alpha \mathbb{Z},$$

so that one has the group extension

$$1 \to \mathbb{Z}[\tfrac{1}{m}] \to B(1, m) \to \mathbb{Z} \to 1.$$

Using this crossed product decomposition, the group $BS(1, m)$ has the following explicit law: $(b, l)(b', l') = (b + \alpha^l(b'), l + l')$ for $l, l' \in \mathbb{Z}$ and $b, b' \in B_m$. It is of course generated by the elements $u := (0, 1)$ and $f_b := (b, 0)$ with $b \in B_m$. These generators satisfy for $j, l \in \mathbb{Z}, n \in \mathbb{N}$,

$$u f_b u^{-1} = f_{\alpha(b)}, \text{ and if } f_{\alpha^{-n}j} := (\alpha^{-n}j, 0) = u^{-n} f_j u^n,$$

$$\text{then } ((\tfrac{1}{m})^n j, l) = f_{(\frac{1}{m})^n j} u^l. \quad (15)$$

$BS(1, m)$ is a subgroup of the "$ax + b$" group (with law $(b, a)(b', a') := (b + ab', aa')$) and can be viewed as the following subgroup of two-by-two matrices $\{\left(\begin{smallmatrix} m^l & b \\ 0 & 1 \end{smallmatrix}\right) : l \in \mathbb{Z}, b \in B_m\}$. Note that $BS(1, m) \simeq B(1, m')$ is equivalent to $m = m'$ [49].

It is interesting to quote that if $p_1, \cdots, p_n$ are the prime divisors of m, then the diagonal embedding $BS(1, m) \hookrightarrow \mathrm{Aff}_1(\mathbb{Q}_{p_1}) \times \cdots \times \mathrm{Aff}_1(\mathbb{Q}_{p_n}) \times \mathrm{Aff}_1(\mathbb{R})$ has a discrete image.

Let $\widehat{B_m}$ be the Pontryagin dual of B_m endowed with the discrete topology. It is isomorphic to the solenoid

$$S_m = S_{1/m} \simeq \{(z_k)_{k=0}^{\infty} \in \prod_{i=0}^{\infty} \mathbb{T} : z_{k+1}^m = z_k, \ \forall k \in \mathbb{N}_0\} \tag{16}$$

using $z_k := \chi\left((\frac{1}{m})^k\right)$ for any $\chi \in \widehat{B_m}$. The group S_m is compact connected and abelian. Notice that (see Section 3.2) $S_m \simeq \{(z_k)_{k=-\infty}^{\infty} \in \mathbb{T}^{\mathbb{Z}} : z_{k+1}^m = z_k, \ k \in \mathbb{Z}\}$ defining $z_{-k} := z_0^{mk}$ for $k > 0$.

The natural embedding

$$\hat{\iota} : \theta \in \mathbb{R} \mapsto \chi_\theta \in S_m \text{ where } \chi_\theta(b) := e^{i2\pi\theta b} \in \mathbb{T} \text{ for } b \in B_m,$$

identifies S_m as the Bohr compactification $b_{B_m}\mathbb{R}$ of $\mathbb{R}$.

S_m is endowed with a natural group automorphism $\widehat{\alpha}$ given by

$$\widehat{\alpha}(z_0, z_1, z_2, \dots) = (z_0^m, z_0, z_1, \dots) \qquad \text{and} \qquad \widehat{\alpha}^{-1}(z_0, z_1, z_2, \dots) = (z_1, z_2, \dots).$$

There are periodic points in S_m of arbitrary period $q \in \mathbb{N}^*$ and they are all of the following form: if z_0 is a solution of $z^{m^q-1} = 1$, then $(z_0, z_0^{m^{q-1}}, \dots, z_0^m, z_0, \dots) \in S_m$; so there are only finitely many periodic points, namely $c_q(m) = m^q - 1$ such points.

The C^*-algebra $C(S_m) \simeq C^*(B_m)$ is precisely the algebra of almost periodic functions on $\mathbb{R}$, with frequencies in B_m and the isomorphism is the map $f \mapsto f \circ \hat{\iota}$.

It follows that for the C^*-algebra we consider, one has

$$\mathcal{U}_m = C^*(BS(1,m)) \simeq C^*(B_m) \rtimes_\alpha \mathbb{Z} \simeq C(S_m) \rtimes_{\widehat{\alpha}} \mathbb{Z}.$$

The unitary element U of Definition 4.1 is precisely the generator of the action α of $\mathbb{Z}$ on $C^*(B_m)$ while V is one of the generators $\{U^{-\ell}VU^\ell : \ell \in \mathbb{Z}\}$ of the abelian algebra $C^*(B_m)$. As a continuous function on S_m, $U^{-\ell}VU^\ell$ is the function $(z_k)_{k=0}^\infty \mapsto z_\ell$ and in particular, $V : (z_k)_{k=0}^\infty \mapsto z_0$.

Note that the subgroup $\{z := (z_k)_{k=0}^\infty \in S_m : \widehat{\alpha}^q(z) = z \text{ for some } q \in \mathbb{N}^*\}$ of periodic points is dense in S_m and $\widehat{\alpha}$ is ergodic on S_m for $m \geq 2$ as previously seen [11, Proposition 1].

4.2. The representations

The knowledge of *-representations of $\mathcal{U}_m$ is important, and even essential in the context of spectral triples (see Definition 5.1). According to (15), any unitary representation of $BS(1,m)$ is given by a unitary operator U and a family of unitaries T_k, $k \in \mathbb{Z}$, with the constraint $UT_kU^{-1} = T_{mk}$, so there is a bijection between the *-representations of $\mathcal{U}_m$ on some Hilbert space $\mathcal{H}$ and the corresponding unitary representations of $BS(1,m)$. This is rephrased usefully in the following lemma [38].

LEMMA 4.2. *The algebra $\mathcal{U}_m$ is the C^*-algebra generated by $L^\infty(\mathbb{T})$ and a unitary symbol $\tilde{U}$ with commutation relations*

$$\tilde{U} f \tilde{U}^{-1} = f \circ e_m, \ \forall f \in L^\infty(\mathbb{T}), \tag{17}$$

where $e_n(z) := z^n$.

Proof: A *-representation π of $\mathcal{U}_m$ on some arbitrary Hilbert space $\mathcal{H}$ can be obtained from a couple $(\tilde{\pi}, \tilde{U})$ with a *-representation $\tilde{\pi}$ of $L^\infty(\mathbb{T})$ and a unitary operator $\tilde{U}$ on $\mathcal{H}$ related by $\tilde{U}\tilde{\pi}(f(z))\tilde{U}^{-1} = \tilde{\pi}\big(f(z^m)\big)$ and $\tilde{V} := \tilde{\pi}(e_1)$ via $\pi(U) := \tilde{U}$ and $\pi(V) := \tilde{V}$. Conversely, given a representation π of $\mathcal{U}_m$, define $\tilde{\pi}(f) := f\big(\pi(V)\big)$ via the spectral theorem, for any $f \in L^\infty(\mathbb{T})$, and $\tilde{U} := \pi(U)$, so (14) implies $\tilde{U}\tilde{\pi}(f(z))\tilde{U}^{-1} = \tilde{\pi}\big(f(z^m)\big)$. $\qquad\square$

In particular, $\mathcal{U}_m$ contains a family of abelian subalgebras $\mathcal{A}_n := \tilde{U}^{-n} L^\infty(\mathbb{T}) \tilde{U}^n$ for $n \in \mathbb{N}$, which is increasing since $\tilde{U}^{-n} f \tilde{U}^n = \tilde{U}^{-(n+1)} f \circ e_m \tilde{U}^{(n+1)}$.

Remark that $\{\sum_{n \in \mathbb{N}, \text{ finite}} f_n \tilde{U}^n + g_n \tilde{U}^{-n} : f_n, g_n \in L^\infty(\mathbb{T})\}$ is a dense *-subalgebra of $\mathcal{U}_m$.

If we choose the Hilbert space $\mathcal{H} := L^2(\mathbb{R})$, the scaling and shift operators give rise to a representation π of $\mathcal{U}_m$ on $\mathcal{H}$ by $\pi(U) : \psi(x) \mapsto \frac{1}{\sqrt{m}} \psi(\frac{x}{m})$ and $\pi(V) : \psi(x) \mapsto \psi(x - 1)$.

The Haar measure ν on $\widehat{B}_m$ gives rise to a faithful trace on $\mathcal{U}_m$ and, since there are many finite-dimensional representations of $\mathcal{U}_m$ (see proof of Theorem 3.3), there are many traces on it.

If we choose $\mathcal{H} := L^2(S_m, \nu) \simeq \ell^2(B_m)$, then $C(S_m)$ acts on $\mathcal{H}$ by left pointwise multiplication and we define $U : \psi \in \mathcal{H} \mapsto \psi \circ \widehat{\alpha} \in \mathcal{H}$ and get a covariant representation of (S_m, U) of the dynamical system $(C(S_m), \widehat{\alpha}, \mathbb{Z})$, so a representation of $\mathcal{U}_m$ on $\mathcal{H}$. Since $\widehat{\alpha}$ is ergodic, this representation is irreducible and faithful [11, Theorem 1].

If we choose $\mathcal{H} := \ell^2(\mathbb{Z})$, then for each $\theta \in \mathbb{R}$, we get an induced representation of $\mathcal{U}_m$ by

$$\pi_\theta(U)\psi(k) := \psi(k - 1) \text{ and } \pi_\theta(V)\psi(k) := \chi_\theta(m^{-k})\,\psi(k), \text{ for } k \in \mathbb{Z}.$$

We summarize a few results on the algebras $\mathcal{U}_m$ (independently of Theorem 3.3).

PROPOSITION 4.3. *Let $m \in \mathbb{N}$, $m \geq 2$.*

(i) *$\mathcal{U}_m$ is a NGCR, AF-embeddable, nonsimple, residually finite-dimensional C^*-algebra and its generated von Neumann algebra for the left representation is a type II_1-factor.*

(ii) $\big(K_0(\mathcal{U}_m), [1_{\mathcal{U}_m}], K_1(\mathcal{U}_m)\big) \simeq (\mathbb{Z}, 1, \mathbb{Z} \oplus \mathbb{Z}_{m-1})$.

(iii) *$\mathcal{U}_m \simeq \mathcal{U}_{m'}$ if and only if $m = m'$.*

Proof: (i): See [10, 11] and [59, Theorem 2.2], [35].

(ii): See [9] or [39, Example A6].

(iii): Consequence of (ii) (or Corollary 3.5). $\qquad\square$

5. On the existence of spectral triples

Since we want to construct spectral triples on $\mathcal{U}_a$, it is worthwhile to know the heat decay of $G_a = B_a \rtimes_\alpha \mathbb{Z}$ via a random walk on the Cayley graph of G_a with

generators $S = \{x, x^{-1}, y, y^{-1}\}$ where $x = (0, 1)$ and $y = (1, 0)$, and with a constant weight and standard Laplacian.

The decay of the heat kernel p_t, with $t \in \mathbb{N}$, has been computed on the diagonal in [51, Theorem 1.1], [22, Theorem 5.2]: when $t \to \infty$, $p_{2t} \sim e^{-t^{1/3} (\log t)^{2/3}}$ if a is transcendental while $p_{2t} \sim e^{-t^{1/3}}$ if a is algebraic. This is related to the fact that G_a has exponential volume growth.

However, for a finite-dimensional connected noncompact Lie group, the behaviour of the heat kernel p_t depends on $t \in \mathbb{R}_+^*$ and can diverge for the short time behaviour when $t \to 0$. Let us explain how this point is related to the dimension.

In noncommutative geometry, a (regular simple) spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ has a (spectral) dimension which is given by $\max\{n \in \mathbb{N} : n$ is a pole of $\zeta_\mathcal{D} : s \in \mathbb{C} \to \mathrm{Tr}(|\mathcal{D}|^{-s})\}$ (here $\mathcal{D}$ is assumed invertible). In particular, when M is an n-dimensional compact Riemannian spin manifold, and $\mathcal{A} = C^\infty(M)$, $\mathcal{H} = L^2(S)$ where S is the spinor bundle and $\mathcal{D}$ is the canonical Dirac operator, the spectral dimension coincides with n. Via the Wodzicki residue, an integral $\int X := \mathrm{Res}_{s=0} \mathrm{Tr}(X|\mathcal{D}|^{-s})$ is defined on (classical) pseudodifferential operators X acting on the smooth sections of S. For instance, $\int |\mathcal{D}|^{-n}$ coincides (up to a universal constant) with the Dixmier trace $\mathrm{Tr}_{\mathrm{Dix}}(|\mathcal{D}|^{-n}) = \lim_{N\to\infty} \log(N)^{-1} \sum_{k=1}^N |\lambda_k|^{-n}$ where the λ_k are the singular values of $\mathcal{D}$. The dimension of M appears in

$$\mathrm{Tr}(e^{-t\mathcal{D}^2}) \sim \sum_{N \geq 0} \frac{1}{t^{(n-N)/2}} \, a_N(\mathcal{D})$$

when $t \to 0$. In particular, when $M = \mathbb{R}^n$ with Lebesgue measure and $\mathcal{D}^2 = -\triangle$ is the standard Laplacian (noncompactness is not a problem), the heat kernel is

$$p_t(x, x) = \frac{1}{(4\pi t)^{n/2}}$$

for all $x \in M$ (see [16, 20, 32]). As a consequence, $\mathrm{Tr}(|\mathcal{D}|^{-(n+\epsilon)}) < \infty$ for all $\epsilon > 0$.

We will see in this section that, depending on the chosen representation of $\mathcal{U}_a$, such an n does not always exist, meaning that the "dimension is infinite".

DEFINITION 5.1. A *spectral triple* (or unbounded Fredholm module) $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is given by a unital C^*-algebra A with a faithful representation π on a Hilbert space $\mathcal{H}$ and an unbounded self-adjoint operator $\mathcal{D}$ such that
    — the set $\mathcal{A} = \{a \in \mathcal{A} : [\mathcal{D}, \pi(a)]$ is bounded $\}$ is norm dense in A,
    — $(1 + \mathcal{D}^2)^{-1} \in J$, where J is a symmetrically-normed ideal of the compact operators $\mathcal{K}(\mathcal{H})$ on $\mathcal{H}$.
    (Note that $\mathcal{A}$ is always a *-subalgebra of A.)
    The triple is *p-summable* if $J = \mathcal{L}^p(\mathcal{H})$ for $1 \leq p < \infty$ (i.e. $\mathrm{Tr}\big((1 + \mathcal{D}^2)^{-p/2}\big) < \infty$.
    It is *p^+-summable* if $J = \mathcal{L}^{p+}(\mathcal{H})$.
    It is *finitely summable* if it is *p*-summable for some *p*.
    It is *θ-summable* if there exists $t_0 \geq 0$ such that $\mathrm{Tr}\big(e^{-t\mathcal{D}^2}\big) < \infty$ for all $t > t_0$ (thus $J = \mathcal{K}(H)$).

Note that p-summability implies θ-summability.

Connes proved in [15] that, for an infinite, discrete, nonamenable group G, there exist no finitely summable spectral triples on $A = C^*_{\text{red}}(G)$. However, in this case, there always exist θ-summable spectral triples on A (even with $\mathcal{D} > 0$). Using a computable obstruction to the existence of quasicentral approximate units relative to J for A, Voiculescu was able to derive, for solvable groups with exponential growth, the nonexistence result for unbounded (generalized) Fredholm modules using the Macaev ideal $J = \mathcal{L}^{\infty,1}(\mathcal{H})$ [57]. We use these results in the following.

THEOREM 5.2. *Nonexistence of finite-summable spectral triples.*

Let $A = \mathcal{U}_a$, $G_a = B_a \rtimes_\alpha \mathbb{Z}$ and $\mathcal{A} = \mathbb{C}[G_a]$.

(i) There is no finitely summable spectral triple $\big(\pi(\mathcal{A}), \mathcal{H}_\pi, \mathcal{D}\big)$ when the representation π is quasi-equivalent to the left regular one.

(ii) There exist θ-summable spectral triples $\big(\pi(\mathcal{A}), \mathcal{H}_\pi, \mathcal{D}\big)$ with $t_0 = 0$ where the representation π is quasi-equivalent to the left regular one.

Proof: (i) Since $\mathcal{U}_a \simeq C^*_{\text{red}}(G_a)$ and G_a is a solvable group with exponential growth, this follows from [57, 4.12. Prop].

(ii) Such a triple is given in [15, Prop. 24] where the representation π is the left regular one on $\mathcal{H}_\pi = \ell^2(G_a)$ and $\mathcal{D} := \mathcal{D}_\ell$ is the multiplication by the length function ℓ on the group G_a associated to the generating set $\{u, v, u^{-1}, v^{-1}\}$ where $u = (0,1)$ and $v = (1,0)$ (see Section 3). In particular $\mathcal{D}_\ell > 0$. $\square$

Despite the previous result, we add a few explicit examples of spectral triples using the fact that the algebra $\mathcal{U}_a$ is residually finite. Clearly, these triples deal with a restrictive part of the geometry of the κ-deformation based on $\mathcal{U}_a$, namely the dynamical system which is behind. The residually finite property is seen via the periodic points of this dynamics.

We gather in the following theorem all these results.

THEOREM 5.3. *Existence of finite-summable spectral triples.*

Let $A = \mathcal{U}_a$ and $\mathcal{A} = \mathbb{C}[G_a]$.

(i) There exist spectral triples $\big(\pi(\mathcal{A}), \mathcal{H}_\pi, \mathcal{D}\big)$ which are compact, i.e. $[\mathcal{D}, \pi(x)]$ is compact for all $x \in \mathcal{A}$.

(ii) There exist spectral triples $\big(\pi(\mathcal{A}), \mathcal{H}_\pi, \mathcal{D}\big)$ such that $[\mathcal{D}, \pi(x)] = 0$, $\forall x \in \mathcal{U}_a$, and with arbitrary summability $p \geq 2$.

(iii) When a is algebraic, there exist spectral triples $\big(\pi(\mathcal{A}), \mathcal{H}_\pi, \mathcal{D}\big)$ such that $[\mathcal{D}, \pi(v)] = 0$, $[\mathcal{D}, \pi(u)] \neq 0$ and with an arbitrary summability.

In case (i), $[\mathcal{D}, \pi(x)]$ is not necessarily zero but the summability is not controlled while for case (ii), the condition $[\mathcal{D}, \pi(x)] = 0$ enables us to control summability. In a sense, case (iii) is a mixed situation requiring that a be algebraic. In that situation, we have a concrete description of the representation π so that an explicit formula for the Dirac operator can be proposed.

Proof: (i) Since $\mathcal{U}_a$ is a residually finite C^*-algebra (i.e. faithfully embeddable into the algebra $\prod_{n \in N} M_{k_n}(\mathbb{C})$), it is quasi-diagonal [12, Prop. 7.1.15]. Thus the result follows from [53, Theorem 2.1]: let π be a faithful quasi-diagonal representation which means that there exists an increasing family of finite rank projections $(P_n)_{n \in \mathbb{N}}$ such that P_n strongly converges to the identity and $[P_n, \pi(x)] \to 0$ for all $x \in \mathcal{U}_a$. Define $Q_n := P_n - P_{n-1}$ $(P_0 = 0)$ and choose a sequence $d_n \in \mathbb{R}$, $|d_n| \nearrow \infty$. Select a subsequence $(P_{n_k})_k$ such that $\sum_k |d_{n_k}| \, \|[Q_{n_k}, x]\| < \infty$ for $x = u, v$. Then for $\mathcal{D} := \sum_k d_{n_k} Q_{n_k}$, $[\mathcal{D}, \pi(x)]$ is compact for $x \in \mathcal{A}$ and $(\pi(\mathcal{U}_a), \mathcal{H}_\pi, \mathcal{D})$ is a spectral triple. Note that the family $(Q_{n_k})_k$ depends on the sequence $(d_n)_n$, so this latter cannot be adjusted to control summability.

(ii) [53]: since $\mathcal{U}_a$ is *-isomorphic to a subalgebra of $\prod_{n \in \mathbb{N}} M_{k_n}(\mathbb{C})$, there exists a faithful representation π of $\mathcal{U}_a$ acting on $\mathcal{H}_\pi = \bigoplus_{n \in \mathbb{N}} \mathbb{C}^{k_n}$. Let P_n be the projection on $\mathbb{C}^{k_n}$ and as above, $\mathcal{D} := \sum_{n=1}^\infty d_n Q_n$. Then $[\mathcal{D}, \pi(x)] = 0$ for $x \in A$ (we can even choose $\mathcal{A} = A$) and the summability is arbitrary by choosing $(d_n)_n$.

(iii) We first construct a faithful representation of $\mathcal{U}_a$ inspired by [59]. For any nonzero positive integer q, we already defined

$$\mathrm{Per}_q(S_a) = \{z = (z_k)_{k=0}^\infty \in S_a : \widehat{\alpha}^k(z) \neq z, \ \forall k < q, \ \widehat{\alpha}^q(z) = z\},$$

the space of q-periodic points in S_a for the action $\widehat{\alpha}$ of $\mathbb{Z}$.

Since $c_q := \#\mathrm{Per}_q(S_a) = |q_d|^q \prod_{k=1}^d |1 - r_k^q| < \infty$ when a is algebraic, we can enumerate these periodic points as $\{z^{(i)}\}_{i=1,\ldots,c_q} = \mathrm{Per}_q(S_a)$.

Consider the Hilbert space

$$\mathcal{H} := \bigoplus_{q \geq 1} L^2(\mathbb{T}) \otimes \mathbb{C}^q \otimes \mathbb{C}^{c_q}.$$

We denote by $E_i^{(q)} \in M_q(\mathbb{C})$, for $i = 1, \ldots, q$, the matrix whose only nonvanishing entry is 1 on the diagonal at the ith entry from the top. We denote by $\iota : \mathbb{T} \to \mathbb{T}$ the identity map, which we identify also with $\iota : \mathbb{R} \to \mathbb{T}$, $\iota(\theta) = e^{i\theta}$. Let us introduce a convenient basis in $\mathcal{H}$: denote by $\{e_s^{(q)}\}_{s=1,\ldots,q}$, the canonical basis of $\mathbb{C}^q$, $\{e_i^{(c_q)}\}_{i=1,\ldots,c_q}$, the canonical basis of $\mathbb{C}^{c_q}$, and by $e_n : \theta \mapsto e^{in\theta}$, for $n \in \mathbb{Z}$, the natural basis of $L^2(\mathbb{T}) \simeq \ell^2(\mathbb{Z})$. Then, for $n \in \mathbb{Z}$, $q \geq 1$, $r = 1, \ldots, q$ and $i = 1, \ldots, c_q$, the vectors $e_n \otimes e_r^{(q)} \otimes e_i^{(c_q)}$ define an orthonormal basis of $\mathcal{H}$: any $\psi \in \mathcal{H}$ can then be decomposed as

$$\psi = \bigoplus_{n \in \mathbb{Z}} \bigoplus_{q \geq 1} \sum_{r=1}^q \sum_{i=1}^{c_q} \psi_{n,q,r,i} \ e_n \otimes e_r^{(q)} \otimes e_i^{(c_q)}$$

$$= \bigoplus_{n \in \mathbb{Z}} \bigoplus_{q \geq 1} \sum_{i=1}^{c_q} e_n \otimes \begin{pmatrix} \psi_{n,q,1,i} \\ \vdots \\ \psi_{n,q,q,i} \end{pmatrix} \otimes e_i^{(c_q)}$$

for $\psi_{n,q,r,i} \in \mathbb{C}$.

We define on $\mathcal{H}$ a faithful representation π of $\mathcal{U}_a$ on any $f \in C(S_a)$ and the unitary generator u of $\mathbb{Z}$ in $\mathcal{U}_a = C(S_a) \rtimes_{\widehat{\alpha}} \mathbb{Z}$ by

$$
\pi(f) := \bigoplus_{q \geq 1} 1_{L^2(\mathbb{T})} \otimes \sum_{i=1}^{c_q} \left[\begin{pmatrix} f(z^{(i)}) & & 0 \\ & \ddots & \\ 0 & & f \circ \widehat{\alpha}^{q-1}(z^{(i)}) \end{pmatrix} \otimes E_i^{(c_q)} \right],
$$

$$
\pi(u) := \bigoplus_{q \geq 1} M_\iota \otimes \begin{pmatrix} 0 & 1 \\ 0_{q-1} & 0 \end{pmatrix} \otimes 1_{c_q} + 1_{L^2(\mathbb{T})} \otimes \begin{pmatrix} 0 & 0 \\ 1_{q-1} & 0 \end{pmatrix} \otimes 1_{c_q}
$$

where M_ι is the multiplication by the function ι. In particular,

$$
\pi(v) := \bigoplus_{q \geq 1} 1_{L^2(\mathbb{T})} \otimes \sum_{i=1}^{c_q} \left[\begin{pmatrix} z^{(i)}{}_0 & & 0 \\ & \ddots & \\ 0 & & (\widehat{\alpha}^{q-1} z^{(i)})_0 \end{pmatrix} \otimes E_i^{(c_q)} \right].
$$

We define now a self-adjoint unbounded operator on $\mathcal{H}$. It is taken to be diagonal, with eigenvalues given by a sequence $\{\mu_{n,q,i}\}_{n \in \mathbb{Z}, q \geq 1, i=1,\ldots c_q}$ of real parameters and it is given by

$$
\mathcal{D} \left(e_n \otimes e_r^{(q)} \otimes e_i^{(c_q)} \right) := \mu_{n,q,i} \left(e_n \otimes e_r^{(q)} \otimes e_i^{(c_q)} \right).
$$

Then, one has the following relations:

$$
[\mathcal{D}, \pi(f)] = 0, \text{ (so } [\mathcal{D}, \pi(v)] = 0),
$$

$$
[\mathcal{D}, \pi(u)] \, e_n \otimes e_r^{(q)} \otimes e_i^{(c_q)} = \delta_{r,q} \left(\mu_{n+1,q,i} - \mu_{n,q,i} \right) e_n \otimes e_1^{(q)} \otimes e_i^{(c_q)}.
$$

By construction, $\mathcal{D}$ has well-defined eigenvectors and eigenvalues,

$$
\mathcal{H}_{n,i}^{(q)} := \mathrm{Span}\{e_{n,r,i}^{(q)} := e_n \otimes e_r^{(q)} \otimes e_i^{(c_q)} \ : \ r = 1, \ldots, q\}
$$

is a q-dimensional eigenspace of $\mathcal{D}$ for the eigenvalue $\mu_{n,q,i}$.

Thus, for any $0 < p < \infty$,

$$
\mathrm{Tr} \left((1 + \mathcal{D}^2)^{-p/2} \right) = \sum_{n \in \mathbb{Z}, q \geq 1, i=1\ldots c_q} \frac{q}{\left(1 + (\mu_{n,q,i})^2 \right)^{p/2}}.
$$

Let us take $\mu_{n,q,i} = \lambda_q(|n| + b_q)$ where b_q does not depend on n and i and $\lambda_q = \pm 1$. Then $\mathcal{D}$ is invertible.

Since $|\mu_{n+1,q,i} - \mu_{n,q,i}| = ||n+1| - |n|| = 1$, one proves that $[\mathcal{D}, \pi(u)]$ is a bounded operator on $\mathcal{H}$.

For $p > 2$, take $b_q = q(qc_q)^{2/p}$. Then, one has

$$\sum_{n\in\mathbb{Z}, q\geq 1, i=1...c_q} \frac{q}{\left(1+(\mu_{n,q,i})^2\right)^{p/2}} = \sum_{n\in\mathbb{Z}, q\geq 1} \frac{qc_q}{\left(1+(|n|+b_q)^2\right)^{p/2}}$$

$$\leq \sum_{n\in\mathbb{Z}, q\geq 1} \frac{qc_q}{(2|n|b_q)^{p/2}}$$

$$= \frac{1}{2^{p/2}} \sum_{n\in\mathbb{Z}, q\geq 1} \frac{1}{|n|^{p/2}} \frac{1}{q^{p/2}}$$

$$< \infty \text{ as a product of two finite sums.}$$

For $p = 2$, the previous inequalities are inefficient but one has

$$\sum_{n\in\mathbb{Z}, q\geq 1} \frac{qc_q}{1+(|n|+b_q)^2} \leq \sum_{n\in\mathbb{Z}, q\geq 1} \frac{qc_q}{(|n|+b_q)^2}.$$

Moreover, $\sum_{n\geq 0} \frac{qc_q}{(|n|+b_q)^2} = qc_q\zeta(2, b_p)$, where $\zeta(s, z)$ is the generalized Riemann zeta function. For $x \to \infty$, $\zeta(2, x) \sim \frac{1}{x}$. Thus, with the choice $b_q = q^3 c_q$, the sum $\sum_{q\geq 1} qc_q\zeta(2, b_p)$ is finite, so that $\mathrm{Tr}\left((1+\mathcal{D}^2)^{-1}\right) < \infty$.   □

REMARK 5.4. There is no contradiction between Theorems 5.2 and 5.3 since the faithful quasi-diagonal representation (or residually finite one) π of $\mathcal{U}_a$ used above to construct $\mathcal{D}$ is not quasi-equivalent to the left regular one: actually, as already mentioned, the von Neumann algebra generated by $\pi(\mathcal{U}_a)$ is a II_1 factor when π is the left regular representation, while it is of type I when π is the quasi-diagonal or residually finite one [26, 5.4.3.].

A more direct way to confirm that the representation π used in the proof of point (iii) of Theorems 5.3 is not quasiequivalent to the left regular representation (or to the right regular representation which is unitarily equivalent to the left one) is to notice that π is the direct integral

$$\pi = \int_{\mathrm{Per}}^{\oplus} \pi_\chi \, d\mu(\chi)$$

of the *finite-dimensional* representations π_χ defined in (13). As such, this representation is strictly contained in the right regular representation R as can be checked using (iv) of Theorem 3.7. The part of R which is not in π is given by the induced *infinite-dimensional* irreducible representations constructed on aperiodic χ's.

As noticed in [53], if $(\mathcal{A}, \mathcal{H}_\pi, \mathcal{D})$ is a spectral triple such that $[\mathcal{D}, \pi(x)] = 0$, $\forall x \in A$, then A is a residually finite C^*-algebra.

REMARK 5.5. Theorem 5.2 essentially says that the 2-dimensional κ-deformed space reflected by the algebra $\mathcal{U}_a$ with $\kappa = -\omega_0 \log^{-1}(a)$ is in fact "infinite-dimensional" as a metric noncommutative space. Theorem 5.3 is a tentative to restore a metric. For instance, the distances on the state space $\mathcal{S}(\mathcal{U}_a)$ generated by

Connes' formula

$$d(\omega, \omega') := \sup\{|\omega(a) - \omega'(a)| \,:\, a \in \mathcal{A}, \, ||[\mathcal{D}, a]|| \leq 1\}, \qquad \omega, \omega' \in \mathcal{S}(\mathcal{U}_a),$$

are infinite in case (ii) of Theorem 5.3, while in case (iii) some states can be at finite distances.

REMARK 5.6. In the noncommutative geometry given by a spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, the differentiability is defined by the operator $\mathcal{D}$ in a subtle way.

In case $ii)$ of Theorem 5.2, let $\partial_\ell := i[\mathcal{D}_\ell, \cdot]$ be a closed unbounded $*$-derivation from $\mathcal{B}(\ell^2(G))$ onto itself with $G = G_a$. Then $\bigcap_{k \in \mathbb{N}} \mathrm{Dom}\,(\partial_\ell^k)$ is a Fréchet sub-$*$-algebra $\mathcal{B}_\ell$ of $\mathcal{B}(\ell^2(G))$ which is stable under smooth functional calculus and which contains $\mathbb{C}[G]$; thus $\mathcal{C}_\ell = \mathcal{B}_\ell \bigcap C^*(G)$ is a Fréchet sub-$*$-algebra of $C^*(G) = C^*_{\mathrm{red}}(G)$ which is stable under smooth functional calculus and which contains $\mathbb{C}[G]$ [21]. Since G_a (being amenable with exponential growth) does not enjoy the Rapid Decay property ([37, Corollary 3.1.8] and [36, (1.3) Theorem]), we can conclude that if $\mathcal{H}_\ell^\infty := \bigcap_{k \in \mathbb{N}} \mathrm{Dom}\,\mathcal{D}_l^k$, then $\rho\left(\mathcal{C}_\ell\right) \neq \mathcal{H}_\ell^\infty$ where ρ is the map: $x \in C^*_{\mathrm{red}}(G) \mapsto x(\delta_e) \in \ell^2(G)$.

REMARK 5.7. The operator $\mathcal{D}$ given in Theorem 5.3 (iii) is not directly related to the group structure of G_a but rather connected to the underlying dynamical system associated to the algebraic nature of a: it depends explicitly of the isomorphism-invariant $\{c_q(a) \,:\, q \in \mathbb{N}^*\}$.

6. Conclusion

We have shown that κ-Minkowski space defined by (1) can be reduced to a compact or discrete version. Depending on κ, or on a defined in (7), this involves discrete amenable groups G_a, in particular the well-known Baumslag–Solitar ones. The associated C^*-algebras $\mathcal{U}_a$ can be viewed as a deformation of the two-torus. They are different when a varies within the rational numbers (of zero Lebesgue measure) because of the structure of the underlying dynamical system. We also have shown that for quadratic algebraic a, the K-theory does not give the full classification of the algebras $\mathcal{U}_a$. For transcendental values of a, which are dense in $\mathbb{R}_+$ and of full Lebesgue measure, all these algebras are isomorphic to each other.

Due to the exponential growth of G_a, we have proved that the algebras $\mathcal{U}_a$ are not only quasi-diagonal but also residually finite-dimensional. They admit different spectral triples: the ones which are quasi-equivalent to the left regular representation and are never p-summable but only θ-summable, i.e. they are of "infinite metric dimension". This situation reminds us of the passage from nonrelativistic to relativistic quantum mechanics: in quantum field theory, the θ-summability (and not the p-summability) naturally occurs due to the behaviour of $\mathrm{Tr}(e^{-tH})$ (when $t \to 0$) where H is the Hamiltonian (or $\mathcal{D}^2$), see for instance [14].

The other faithful representations can generate fancy spectral triples which can have arbitrary summability $p \geq 2$ (or "dimension") depending on the algebraic properties of the real parameter a, but are in fact degenerate to some extent.

Unfortunately, the topological content of these unbounded Fredholm modules may be trivial, in the sense that they correspond to trivial elements of K-homology (for instance, when $\lambda_q = 1$, the associated bounded Fredholm module is degenerate).

The nonexistence theorem, though powerful, does not preclude the possible existence of a genuine, nondegenerate, nontrivial spectral geometry on the κ-deformation spaces presented here, they only restrict the possible algebra representations that could be used in the construction.

This shows how delicate the notion of spectral or metric dimensions of κ-Minkowski space is, and how subtle its analysis through noncommutative geometry.

Acknowledgements

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