Vortex in Maxwell-Chern-Simons models coupled to external backgrounds

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We consider Maxwell-Chern-Simons models involving different nonminimal coupling terms to a non relativistic massive scalar and further coupled to an external uniform background charge. We study how these models can be constrained to support static radially symmetric vortex configurations saturating the lower bound for the energy. Models involving Zeeman-type coupling support such vortices provided the potential has a "symmetry breaking" form and a relation between parameters holds. In models where minimal coupling is supplemented by magnetic and electric field dependent coupling terms, non trivial vortex configurations minimizing the energy occur only when a nonlinear potential is introduced. The corresponding vortices are studied numerically.

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I. INTRODUCTION

Vortex solutions in $(2+1)$-dimensional field theories have received a constant attention (for reviews, see, e.g., [1], [2]) motivated in part by the possible role played by vortices in various phenomena of condensed matter physics, such as Josephson junctions arrays or Quantum Hall Effect. Various field theory models have been considered, starting from the Abelian-Higgs model [3] and then extending to Chern-Simons or Maxwell-Chern-Simons (MCS) [4] theories coupled with relativistic or non relativistic matter systems [5–11] together with $(N=2)$ supersymmetric extensions and/or nonabelian generalizations [12,1]. Basically, these models have been shown to support finite energy vortices which, in most cases, for a suitable choice for the matter potential, saturate the lower bound of the energy for the considered system.

In this paper, we consider two different types of MCS models coupled to a non relativistic massive scalar, hereafter called type I and type II MCS models. Both are further coupled to an external uniform background charge (a situation which may be of interest in condensed matter systems). We study how each type of models can be constrained to support static radially symmetric vortex configurations saturating the lower bound for the energy. In type I models, the usual minimal coupling of the MCS gauge potential to the scalar is supplemented by a magnetic field dependent (Zeeman-type) coupling. Type II models involve the nonminimal coupling introduced and discussed in [13] (see also [14] and first of [11]) in which both magnetic and electric field dependent coupling terms appear. The introduction within MCS theories with matter of such a nonminimal coupling whose strength must be fixed to a specific value has been proposed as a possible alternative way to describe (nonstandard) composite anyonic objects and/or statistical transmutation. Basically, in this approach, the statistical properties of the described anyonic (composite) objects are controlled by a MCS gauge potential with suitable Pauli-type coupling with matter [13]. In this spirit, type II models may be viewed as a modification of the Landau-Ginzburg effective theory proposed in [15] to describe some global physical features of the Quantum Hall Effect. In this latter effective theory [15], the statistical transmutation (between fermions and bosons) is simply controlled by a Chern-Simons statistical gauge potential minimally coupled with matter. This point is presented in the appendix.

We find that type I models support (static) vortices saturating the lower bound for the energy provided the matter potential has a usual "symmetry breaking" form (with minimum linked with the magnitude of the uniform background) and a relation between the masses and the strength of the Zeeman coupling holds. This is shown in Sec. II. In Sec. III, we consider type II models in which the strength of the nonminimal coupling is fixed to the specific value mentioned above. We show that these models do not support vortices saturating the lower bound for the energy unless a nonlinear potential is introduced. The corresponding vortex configurations are studied numerically. The case where type II models are coupled with an external gauge potential is also considered and gives rise to similar conclusions. In Sec. IV, we summarize the results and we conclude.

II. VORTEX SOLUTIONS IN TYPE I MCS MODELS

The gauge-invariant action for the first class of models we consider, hereafter called type I models, is defined by[1]

[1]Our conventions are $\hbar = c = 1$, $g_{\mu\nu} = \mathrm{diag}(+,-,-)$, $x = [x_0; \mathbf{x} = (x_1, x_2)]$. Polar coordinates are $x_1 = r\cos\theta, x_2 = r\sin\theta$. We define $\mathbf{e_\theta} = (\sin\theta, -\cos\theta)$.

$$S_1 = S_{\rm MCS} + S_1^m, \tag{2.1a}$$

$$S_{\rm MCS} = \int d^3x\Big(-\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{\eta}{4}\epsilon_{\mu\nu\rho}A^\mu F^{\nu\rho}\Big), \tag{2.1b}$$

$$S_1^m = \int d^3x\Big[i\phi^\dagger D_0\phi - \frac{1}{2m}|\mathbf{D}\phi|^2 + \kappa F_0\phi^\dagger\phi - V(\phi) - A_0J_0\Big]. \tag{2.1c}$$

In (2.1), $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $F_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho}F^{\nu\rho}$ is the dual field strength whose timelike (spacelike) component F_0 (F_i, $i = 1, 2$) is associated to the magnetic (electric) field, η is the Chern-Simons coefficient, ϕ is a non relativistic scalar with mass m, $D_\mu = \partial_\mu - iA_\mu$ is the covariant derivative. $V(\phi)$ is the potential to be specified in a while, J_0 denotes an external static uniform background charge assumed to be positive and the term involving κ is a Zeeman term. The mass dimensions for the parameters and fields are $[A_\mu] = [\phi] = 1, [J_0] = 2, [e^2] = [m] = 1, [\eta] = 0, [\kappa] = -1$. The equations of motion stemming from (2.1) are

$$-\frac{1}{e^2}\epsilon_{ij}\partial^iF^j + \eta F_0 + j_0 - J_0 = 0, \tag{2.2a}$$

$$\frac{1}{e^2}(\partial_iF_0 - \partial_0F_i) + \eta\epsilon_{ij}F^j + \epsilon_{ij}j^j - \kappa\partial^ij_0 = 0 \qquad (i = 1, 2), \tag{2.2b}$$

$$iD_0\phi + \kappa F_0\phi + \frac{1}{2m}D_iD_i\phi = \frac{\delta V}{\delta\phi^\dagger} \qquad (\text{and h.c.}), \tag{2.2c}$$

where summation over space indices i, j is understood ($\epsilon_{12} = +1$) and the components of the gauge-invariant matter current are given by

$$j_0 = \phi^\dagger\phi; \tag{2.3a}$$

$$j_i = \frac{i}{2m}[\phi^\dagger D_i\phi - (D_i\phi)^\dagger\phi]. \tag{2.3b}$$

We now look for static radially symmetric solutions of the equations of motion having finite energy and satisfying

$$\mathbf{A} = \mathbf{e_\theta}\frac{a(r) + n}{r}, \quad n \in \mathbb{Z}; \tag{2.4a}$$

$$A_0(r) = a_0(r); \tag{2.4b}$$

$$\phi = f(r)e^{-in\theta}; \tag{2.4c}$$

$$\lim_{r\to\infty} a(r) = \lim_{r\to\infty} a_0(r) = 0; \tag{2.4d}$$

$$\lim_{r\to 0} a(r) = -n, \tag{2.4e}$$

where $a(r)$ and $a_0(r)$ appearing in the usual vortex Ansatz (2.4a)–(2.4c) are smooth radial functions and (2.4e) corresponds to configurations carrying a quantized magnetic flux Φ, $\Phi = \int d^2xF_0 = -2\pi n$, where $n \in \mathbb{Z}$ is related to the vorticity. The boundary conditions for f will be determined in the course of the discussion.

From now on, we drop the explicit radial dependence on the various functions to simplify the notations. Furthermore we define $X' = \frac{dX}{dr}$ for any radial function X. Then, Eq. (2.2) can be conveniently reexpressed as

$$-\frac{1}{e^2}\Delta a_0 - \eta\frac{a'}{r} + f^2 - J_0 = 0; \tag{2.5a}$$

$$-\frac{1}{e^2}\Big(\frac{a'}{r}\Big)' - \eta a_0' + \frac{f^2}{m}\frac{a}{r} - \kappa(f^2)' = 0, \tag{2.5b}$$

$$\frac{1}{2m}\Delta f + f\Big[a_0 - \kappa\frac{a'}{r} - \frac{1}{2m}\Big(\frac{a}{r}\Big)^2\Big] = \frac{\delta V}{\delta\phi^\dagger}\Big|_{\phi=f}, \tag{2.5c}$$

in which $\Delta = \frac{1}{r}\frac{d}{dr}(r\frac{d}{dr}.)$ The Hamiltonian for the system (2.1) is given by

$$\mathcal{H}_1 = \int d\mathbf{x}\Big[\frac{1}{2e^2}(F_0^2 + \mathbf{F}^2) + \frac{1}{2m}|\mathbf{D}\phi|^2 - \kappa F_0\phi^\dagger\phi + V(\phi)\Big], \tag{2.6}$$

from which one obtains the static energy functional density expressed in terms of the radial variables (2.4) as

$$H_1 = \int d\mathbf{x}\Big\{\frac{1}{2e^2}\Big[\Big(\frac{a'}{r}\Big)^2 + a_0'^2\Big] + \frac{1}{2m}\Big[f'^2 + f^2\Big(\frac{a}{r}\Big)^2\Big] + \kappa f^2\frac{a'}{r} + V(f)\Big\}. \tag{2.7}$$

In the absence of any further requirements, type I models (2.1) involve five free parameters. Furthermore, the potential $V(\phi)$ is still unspecified. This latter will have to be chosen in such a way that the energy is definite positive, as it will be shown in the sequel. We are now in position to select some specific models which admit finite energy vortex solutions for some suitable choice for the potential. Notice that we will focalize on vortex solutions saturating the minimum for the energy.

To do this, one observes that the static energy (2.7) can be conveniently rewritten as $H_1 = H_1' + H_{1V} + H_{10}$ with

$$H_1' = \int d\mathbf{x}\Big[\frac{1}{2e^2}\Big(a_0'^2 + \Big[\frac{a'}{r} + e^2\lambda_\pm(f^2 - J_0)\Big]^2\Big) + \frac{1}{2m}\Big(f' \pm f\frac{a}{r}\Big)^2\Big], \tag{2.8a}$$

$$H_{1V} = \int d\mathbf{x}\Big[V(f) - e^2\frac{\lambda_\pm^2}{2}(f^2 - J_0)^2\Big], \tag{2.8b}$$

$$H_{10} = \lambda_\pm J_0\Phi \mp [f^2a]_0^\infty, \tag{2.8c}$$

where $\lambda_\pm \equiv \kappa \pm \frac{1}{2m}$, $[X]_0^\infty \equiv X(\infty) - X(0)$ and we have explicitly collected the boundary terms in H_{10}. Then, it can be readily observed that (2.8) has a lower bound H_{10} provided the potential is chosen to be

$$V(\phi) = e^2\frac{\lambda_\pm^2}{2}(\phi^\dagger\phi - J_0)^2, \tag{2.9}$$

and that H_{10} is saturated by field configurations satisfying

$$a_0 = 0; \tag{2.10a}$$

$$\frac{a'}{r} + e^2\lambda_\pm(f^2 - J_0) = 0; \tag{2.10b}$$

$$f' \pm f\frac{a}{r} = 0. \tag{2.10c}$$

Consistency with the equations of motion then requires that

$$(1 + e^2\eta\lambda_\pm)(f^2 - J_0) = 0; \tag{2.11a}$$

$$\Delta f - \frac{f'^2}{f} = \pm e^2\lambda_\pm f(f^2 - J_0). \tag{2.11b}$$

Now one observes that (2.11a) is satisfied for any f provided[2]

$$\lambda_\pm = -\frac{1}{\eta e^2}. \tag{2.12}$$

Then one concludes that when (2.12) is satisfied, type I models with $V(\phi)$ given by (2.9) have vortex solutions of the form (2.4) whose matter density obeys

$$\Delta f - \frac{f'^2}{f} = \mp\frac{1}{\eta}f(f^2 - J_0). \tag{2.13}$$

Moreover, these vortex configurations saturate the lower bound for the energy. Notice that since $a_0 = 0$, these configurations have no electric field.

Equation (2.13) has already appeared in various context (see, e.g., [1], [9]). It is known to have solutions whose asymptotic behavior is $f^2 \sim J_0$ for $r \to \infty$ while for $r \to 0$, $f^2 \sim r^p$ with $p \geq 2$. Using these asymptotics together with (2.4d) and (2.4e), one finds that the second term in (2.8c) vanishes and H_{10} reduces to $H_{10} = \frac{J_0}{e^2\eta}\Phi$. The lower bound for the energy is $|H_{10}|$ (within our conventions, positive (negative) magnetic flux corresponds to $\eta > 0$ ($\eta < 0$) and to the lower (upper) sign in the above equations) which is proportional to the magnetic flux. Integration over space of (2.5a) yields $\int d\mathbf{x}(f^2 - J_0) + \eta\int d\mathbf{x}F_0 \equiv q + \eta\Phi = 0$ so that the vortex configurations carry a quantized charge proportional to the flux.

In the limit $\kappa = 0$, (2.12) becomes $\pm\frac{1}{2m} = -\frac{1}{e^2\eta}$. Selecting, for instance, $\eta > 0$, one concludes that in the absence of Zeeman term, the resulting type I model still admits static vortex solutions saturating the lower bound for the energy provided $m = M/2$ where M is the mass for A_μ (i.e., the inverse screening length of the MCS A_μ-mediated interaction). Notice that this latter relation agrees with the one obtained in [9] within MCS theory coupled with a massive non relativistic scalar in the absence of Zeeman interaction term.

When $J_0 = 0$, still assuming that (2.12) holds, (2.13) reduces to a simple Liouville equation while H_{10} vanishes, which is the situation considered in [8]. Again, selecting, for instance, $\eta > 0$, the corresponding solution for f is given by $f(r) = \frac{2(n+1)}{r\eta^{1/2}}[(\frac{r}{r_0})^{n+1} + (\frac{r_0}{r})^{n+1}]^{-1}$, where r_0 is some real constant. The corresponding configuration carries a quantized electric charge Q since one has from (2.5a) $\eta\Phi + \int d\mathbf{x}f^2 \equiv \eta\Phi + Q = 0$ with $\Phi = -2\pi n$. Notice that one has now $f \to 0$ for $r \to \infty$.

[2]Note that (2.11) admits the solution $f^2 = J_0$ which corresponds to the minimum for the potential (2.9).

III. VORTEX IN TYPE II MCS MODELS

The present analysis can be extended to another class of models, hereafter called type II models, in which the coupling of the MCS sector to the non relativistic matter involves terms depending on the electric field, in addition to the magnetic coupling. We will first consider type II models in the presence of an external background charge J_0 and then deal with the coupling to an external gauge potential $\mathcal{A}_\mu$. In the first case, the corresponding action is now defined by

$$S_2 = S_{\rm MCS} + S_2^m; \tag{3.1a}$$

$$S_2^m = \int d^3x\Big[i\phi^\dagger\mathcal{D}_0\phi - \frac{1}{2m}|\mathcal{D}_i\phi|^2 - V(\phi) - A_0J_0\Big], \tag{3.1b}$$

where the operator $\mathcal{D}$ is given by

$$\mathcal{D}_\mu = \partial_\mu - iA_\mu - i\kappa F_\mu, \tag{3.2}$$

and J_0 is assumed to be positive as in Sec. II. Timelike component of the term involving κ gives rise to a magnetic dipole (Zeeman) coupling which is already present in S_1^m (2.1b), while the corresponding spacelike components generate gauge-invariant couplings depending on the electric field, as announced above.

The action for type II models coupled to an external static gauge potential $\mathcal{A}_\mu = [\mathcal{A}_0 = 0; \mathcal{A}(\mathbf{x})]$ can be obtained by replacing $\mathcal{D}_\mu$ in (3.1) (and setting $J_0 = 0$) by

$$\hat{\mathcal{D}}_0 = \mathcal{D}_0 - i\kappa\mathcal{F}_0; \quad \hat{\mathcal{D}}_i = D_i(A + \mathcal{A}) - i\kappa F_i, \tag{3.3}$$

where $\mathcal{F}_0 = \partial_1\mathcal{A}_2 - \partial_2\mathcal{A}_1$ is the external magnetic field.

At this level, some comments concerning the possible physical interpretation of type II models are in order. The nonminimal coupling defined by (3.2) has been proposed and discussed in [13] as providing a possible alternative description of composite anyonic objects. There, the usual minimal coupling to a Chern-Simons statistical gauge potential controlling the attachment of an infinitesimally thin flux tube to the charge carriers, which basically gives rise to standard (Chern-Simons) anyons, is replaced by the nonminimal coupling (3.2) to a MCS statistical gauge potential with κ fixed to a specific value [13], to be given below. This alternative way produces nonstandard anyons. In this respect, type II models can be viewed as modeling in a second quantization framework the planar dynamics of a system of charged nonstandard anyonic composite where now the statistical interaction realizing the flux attachment has a finite range. This is presented in more detail in the appendix for the sake of completeness. Notice that (3.1) cannot be viewed as the naive non relativistic limit of a model involving a relativistic scalar coupled through (3.2) to a MCS action.

Let us consider type II models in the presence of an uniform background charge as described by (3.1)-(3.2). The equations of motion together with the gauge-invariant matter current are

$$-\frac{1}{e^2}\epsilon_{ij}\partial^i F^j + \eta F_0 + \kappa\epsilon_{ij}\partial^i j^j + j_0 - J_0 = 0, \tag{3.4a}$$

$$\frac{1}{e^2}(\partial_i F_0 - \partial_0 F_i) + \eta\epsilon_{ij}F^j - \kappa(\partial_i j_0 - \partial_0 j_i) + \epsilon_{ij}j^j = 0, \tag{3.4b}$$

$$i\mathcal{D}_0\phi + \frac{1}{2m}\mathcal{D}_i\mathcal{D}_i\phi = \frac{\delta V}{\delta\phi^\dagger} \quad \text{(and h.c.)}, \tag{3.4c}$$

$$j_0 = \phi^\dagger\phi; \tag{3.5a}$$

$$j_i = \frac{i}{2m}[\phi^\dagger\mathcal{D}_i\phi - (\mathcal{D}_i\phi)^\dagger\phi]. \tag{3.5b}$$

Now, when $\kappa = -1/(\eta e^2)$ which we assume from now on and upon setting

$$G_0 = F_0 + \frac{1}{\eta}j_0; \tag{3.6a}$$

$$G_i = F_i + \frac{1}{\eta}j_i, \tag{3.6b}$$

Eqs. (3.4a) and (3.4b) can be rewritten as

$$-\frac{1}{e^2}\epsilon_{ij}\partial^i G^j + \eta G_0 - J_0 = 0; \tag{3.7a}$$

$$\frac{1}{e^2}(\partial_i G_0 - \partial_0 G_i) + \eta\epsilon_{ij}G^j = 0, \tag{3.7b}$$

and are solved by $G_0 = J_0/\eta$ and $G_i = 0$.[3] Therefore, in the static regime, the relevant equations take the form

$$-\frac{a'}{r} + \frac{1}{\eta}(f^2 - J_0) = 0; \tag{3.8a}$$

$$-\Omega a_0' + \frac{f^2}{m\eta}\frac{a}{r} = 0, \tag{3.8b}$$

$$\frac{1}{2m}\Delta f + f\left[a_0 + \frac{1}{\eta e^2}\frac{a'}{r} - \frac{1}{2m}\left(\frac{a}{r} + \frac{1}{\eta e^2}a_0'\right)^2\right] = \frac{\delta V}{\delta\phi^\dagger}\bigg|_{\phi^\dagger = f}, \tag{3.8c}$$

where we have used (2.4) and we have defined $\Omega = (1 - \frac{f^2}{m\eta^2 e^2})$. The conjugate momenta for the fields are

$$\Pi_{\phi^+} = \Pi_{A_0} = 0; \tag{3.9a}$$

$$\Pi_\phi = i\phi^+; \tag{3.9b}$$

$$\Pi_{A_i} = \frac{1}{e^2}F_{0i} - \frac{\eta}{2}\epsilon_{ij}A^j + \frac{1}{\eta e^2}\epsilon_{ij}j^j \quad (i = 1, 2), \tag{3.9c}$$

from which one obtains the Hamiltonian given by

$$\mathcal{H}_2 = \int d\mathbf{x}\left[\frac{1}{2e^2}F_0^2 + \frac{1}{2e^2}\Omega\mathbf{F}^2 + \frac{1}{\eta e^2}F_0\phi^\dagger\phi + \frac{1}{2m}|\mathbf{D}\phi|^2 + V(\phi)\right], \tag{3.10}$$

where $D_i = \partial_i - iA_i$. The positivity of (3.10) has been discussed in [16](see also [11]). Here, we will assume that $\phi^\dagger\phi \le m\eta^2 e^2$ as in [11], a condition which will have to be verified *a posteriori* by any vortex configuration.

The study of the possible existence of non trivial vortex configurations now follows a way similar to the one described for type I models. Again, we restrict ourselves to configurations saturating the lower bound for the energy. We find that some type II models involving a local non polynomial potential do admit vortex type solutions of the form (2.4) which minimize the energy. These configurations are not present in type II models with polynomial potentials (or they do not saturate the lower bound for the corresponding energy).

To see that, one first observes that the static energy functional can be cast into the form

$$H_2 = \tilde{H}_{20} + \int d\mathbf{x}\frac{1}{2m}\left(f' \pm \frac{fa}{r\sqrt{\Omega}}\right)^2 \mp \eta^2 e^2\sqrt{\Omega}\frac{a'}{r} + \left[V(f) - \frac{1}{2e^2\eta^2}(f^4 - J_0^2)\right], \tag{3.11}$$

where $\tilde{H}_{20} = \pm 2\pi\eta^2 e^2[a\sqrt{\Omega}]_0^\infty$ collects boundary contributions and we have used (3.8a) and (3.8b). By further making use of (3.8a), (3.11) can be conveniently reexpressed as

[3] Note that, when $J_0 = 0$, (3.7a) and (3.7b) are solved by $F_i = -\frac{j_i}{\theta}$, $F_0 = -\frac{j_o}{\theta}$. This latter relation is nothing but the usual anyonic relation (Gauss law constraint) enforcing the proportionality between the magnetic field and matter density that would be obtained also within pure Chern-Simons theory with minimal coupling to scalars.

$$H_2 = H_{20} + H_2' + H_{2V}, \tag{3.12a}$$

$$H_{20} = \tilde{H}_{20} + \frac{J_0}{\eta e^2}\Phi; \tag{3.12b}$$

$$H_2' = \int d\mathbf{x} \frac{1}{2m}\left(f' \pm \frac{fa}{r\sqrt{\Omega}}\right)^2, \tag{3.12c}$$

$$H_{2V} = \int d\mathbf{x}\left[V(f) - \frac{1}{2e^2\eta^2}(f^2 - J_0)^2 \mp \eta e^2 \times (f^2 - J_0)\left(1 - \frac{f^2}{m\eta^2 e^2}\right)^{1/2}\right], \tag{3.12d}$$

so that if the potential is chosen to take the following non polynomial form

$$V(\phi) = \frac{1}{2e^2\eta^2}[(\phi^\dagger\phi) - J_0]^2 \pm \eta e^2(\phi^\dagger\phi - J_0) \times \left(1 - \frac{\phi^\dagger\phi}{m\eta^2 e^2}\right)^{1/2}, \tag{3.13}$$

which will be commented on in a while, H_2 is minimized by configurations verifying

$$f' \pm \frac{fa}{r\sqrt{\Omega}} = 0. \tag{3.14}$$

By combining (3.8a)–(3.8c) with (3.14), one obtains

$$\Delta f - \frac{f'^2}{f\Omega} = \mp \frac{f(f^2 - J_0)}{\eta\sqrt{\Omega}} \tag{3.15}$$

and

$$a_0 = \pm\eta e^2\sqrt{\Omega}, \tag{3.16}$$

indicating that the corresponding configurations have a non vanishing electric field. Note that we have determined the constant term in (3.16) stemming from the integration of (3.8b) by requiring that it cancels all the terms linear in f appearing in (3.8c).

Equation (3.15) has some interesting limits. For small matter density, $f^2 < < me^2\eta^2$ and Ω can be approximated by 1. Then, (3.15) reduces to a nonlinear elliptic equation of the type (2.13) which has physical solutions $f^2 \rightarrow J_0$ for $r \rightarrow \infty (f^2 \rightarrow 0$ for $r \rightarrow 0)$ provided the upper (lower) sign is chosen when $\eta < 0$ $(\eta > 0)$. Note that positivity requires $J_0 < < me^2\eta^2$ (since one must have $\phi^\dagger\phi < < me^2\eta^2$ in this regime). These solutions provide therefore approximate solutions for (3.15) in a small density regime. The corresponding configurations still carry a charge proportional to the flux. When $J_0 = 0$, (3.15) simply reduces to a Liouville equation.

By using (2.4d) and (2.4e), the expression for H_{20} in (3.12) becomes

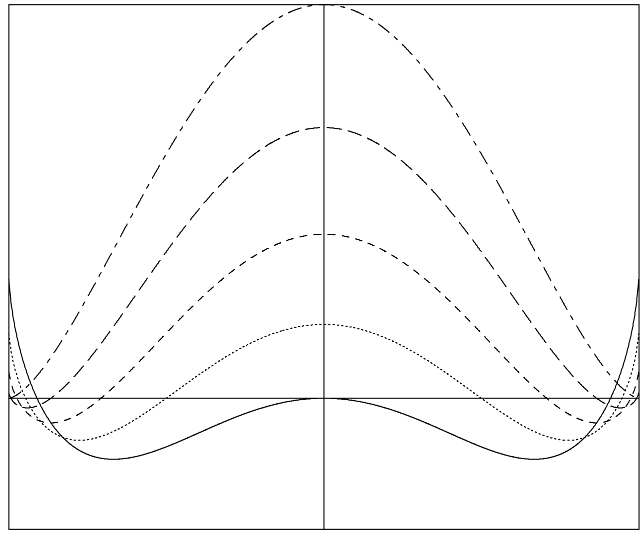

FIG. 1. Qualitative shape of the nonlinear potential, plotted for different values of the ratio $J_0/m\eta^2e^2$, assuming $\eta > 0$ and $m/\eta e^2 = 1$. Choosing $\eta < 0$ and/or other values for $m/|\eta|e^2$ does not change significantly the behavior of the potential. From the lowest curve to the uppermost one, one has $J_0/m\eta^2e^2 = 0, 1/4, 1/2, 3/4, 1$.

$$H_{20} = 2\pi n(\eta e^2)(\pm\eta^3 e^2 - J_0) \tag{3.17}$$

and is proportional to the magnetic flux. It represents the positive lower bound for the static energy provided $n < 0$ $(n > 0)$ for $\eta > 0$ $(\eta < 0)$ corresponding to a positive (negative) magnetic flux.

The nonlinear potential (3.13) is depicted in Fig. 1 for different values of the ratio $J_0/(m\eta^2e^2)$ (and for $\eta > 0$ and $m/\eta^2e^2 = 1$,[4] the lowest (solidline) curve corresponding to $J_0 = 0$. It exhibits a symmetry breaking shape with a minimum obtained at some $|\phi_0| \leq m\eta^2e^2$ which coincides with J_0, $|\phi_0| = J_0$, when $J_0 = m\eta^2e^2$. This potential has a somehow unusual expression in that the usual "symmetry breaking" term (1st term in (3.13)) is supplemented by an additional nonlinear term. Its origin can be traced back by first noticing that nonminimal electric field dependent couplings appearing in (3.2) generate an extra contribution to the (electric) energy (see second term in (3.10)). This gives rise to a (nonlinear) term in the static energy which, if one insists on obtaining non trivial vortex solutions (i.e., solutions with non constant matter density) that saturate the lower bound for the energy, cannot be compensated by a polynomial term of finite degree in $\phi^\dagger\phi$. Correspondingly, in type II models with polynomial potentials, the minimum for the energy is reached only by those configurations having

[4]Other choices for theses parameters do not change significantly the behavior of the potential.

constant matter density (and/or vanishing gauge potential). Vortex solutions with non constant matter density possibly occur but they do not correspond to a minimum for the energy.

We have solved numerically Eq. (3.15). The resulting behavior for the matter density is depicted in Fig. 2 for different values of $J_0/m\eta^2e^2$ (with $\eta > 0$ and $m/\eta e^2 = 1$). We find numerically that physical configurations have the following asymptotic behavior

$$f^2 \sim r^2, \quad r \rightarrow 0; \qquad f^2 \rightarrow J_0, \quad r \rightarrow \infty, \tag{3.18}$$

provided $J_0 \leq m\eta^2e^2$. The corresponding behavior for the magnetic field can be obtained from (3.8a). The magnetic field reaches its maximum $F_0 = J_0/\eta$ at the origin, decreases smoothly as r increases, and vanishes at the infinity. Roughly speaking, matter is repelled away from the area where the magnetic field is concentrated. It can be verified numerically that the closest J_0 stands to the value $m\eta^2e^2$, the fastest f^2 approaches its asymptotic plateau $f^2 \sim J_0$ so that when $J_0 = m\eta^2e^2$, the solution of (3.15) is obtained only for constant matter density $f^2 = J_0$.

The above analysis can be extended to the case where type II models are coupled to an external static gauge potential $\mathcal{A}_\mu[\mathcal{A}_0 = 0; \mathcal{A}(\mathbf{x})]$ instead of a uniform background charge density. The corresponding action together with the equations of motion are obtained by substituting (3.2) by (3.3) in (3.1) and (3.4), (3.5) and further setting $J_0 = 0$ and $\kappa = -1/\eta e^2$, this latter constraint ensuring that (3.7) and (3.6) still hold (with (3.2) replaced by (3.3)). Notice that the action coincides with (A9) (in which $\mathcal{A}_0 = 0$ and $\gamma = -1/(2\eta)$). We consider here the case where the vector potential $\mathcal{A}$ gives rise to an external magnetic field localized at the origin, $\mathcal{A}_i = \frac{\alpha}{2\pi}\epsilon_{ij}\frac{x^j}{|\mathbf{x}|^2}$ ($\mathbf{x} = (x_1, x_2)$) with $\alpha > 0$. The relevant static equations of motion are then deduced from (3.8) by replacing a by $(a + \alpha)$ in (3.8b), (3.8c) while (3.8a) (where now $J_0 = 0$) is unchanged. The Hamiltonian for the system is

$$\mathcal{H}(\mathcal{A}) = \int d\mathbf{x}\bigg[\frac{1}{2e^2}F_0^2 + \frac{1}{2e^2}\Omega\mathbf{F}^2 + \frac{1}{\eta e^2}(F_0 + \mathcal{F}_0)\phi^\dagger\phi + \frac{1}{2m}|\mathbf{D}(A + \mathcal{A})\phi|^2 + V(\phi)\bigg], \tag{3.19}$$

from which we obtain after some algebraic calculation the static energy functional

$$H(\mathcal{A}) = \hat{H}_0 + \hat{H}' + \hat{H}_V; \tag{3.20a}$$

$$\hat{H}_0 = \pm 2\pi\eta^2e^2[(a + \alpha)\sqrt{\Omega}]_0^\infty + \frac{\alpha}{\eta e^2}f^2(0), \tag{3.20b}$$

$$\hat{H}' = \int d\mathbf{x}\frac{1}{2m}\bigg(f' \pm \frac{f(a + \alpha)}{r\sqrt{\Omega}}\bigg)^2; \tag{3.20c}$$

$$\hat{H}_V = \int d\mathbf{x}\bigg[V(f) - \frac{1}{2e^2\eta^2}f^4 \mp \eta e^2 f^2 \times\bigg(1 - \frac{f^2}{m\eta^2e^2}\bigg)^{1/2}\bigg]. \tag{3.20d}$$

Therefore, if one chooses again the nonlinear potential (3.13) (in which $J_0 = 0$), the static energy is minimized by configurations such that $f' \pm \frac{f(a+\alpha)}{r\sqrt{\Omega}} = 0$ from which one easily realizes that the matter density still verifies the differential Eq. (3.15) (for $J_0 = 0$) while the electric potential is still given by (3.16).

The latter differential equation reduces to a Liouville equation when $f^2 << m\eta^2e^2$ (small density regime). In this limit, physically admissible solutions are obtained when $\eta > 0$ ($\eta < 0$) and the upper (lower) sign in (3.20) is chosen. One then obtains $\hat{H}_0 = \pm\eta^2e^2(2\pi n)$ which represents the lower bound for the energy provided $n > 0$ ($n < 0$) for $\eta > 0$ ($\eta < 0$). As for the first situation studied in the first part of this section, the present type II models with polynomial potentials do not support non trivial vortex solutions minimizing the energy. In particular, for $V = \lambda(\phi^\dagger\phi - v)^2$, a simple calculation shows that the minimum for the energy is obtained for configurations such that $f^2 = v$, $A_0 = 0$, and $A_i + \mathcal{A}_i = 0$.

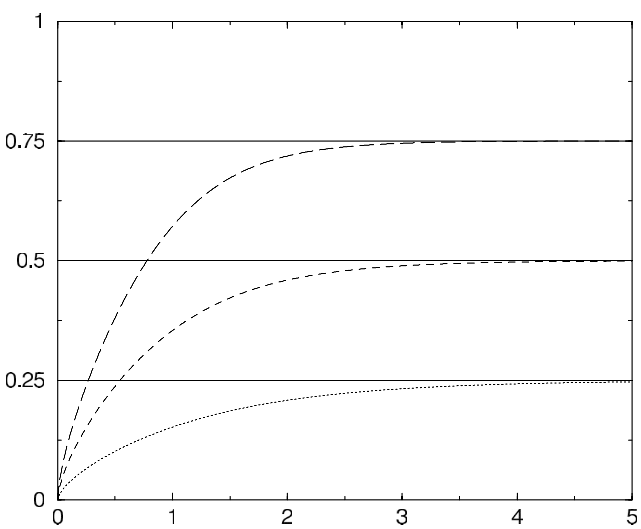

FIG. 2. Radial behavior for the matter density. The quantity $\frac{f^2}{m\eta^2e^2}$ (vertical axis) is plotted versus $\rho \equiv \frac{r}{\eta e^2}$ (horizontal axis). As in Fig. 1, $\eta > 0$ and $m/\eta e^2 = 1$. The three curves from bottom to top correspond, respectively, to $\frac{J_0}{m\eta^2e^2} = 1/4, 1/2, 3/4$, indicated by the three horizontal asymptotic lines.

IV. DISCUSSION AND CONCLUSION

We have studied the possible occurrence of radially symmetric static vortex configurations saturating the lower bound for the energy in two types of MCS models which differ from each other by their gauge-invariant coupling to a non relativistic massive scalar field. Both models are further coupled to an external uniform charge background. In type I models, where the scalar has a minimal and magnetic dipole coupling to the MCS gauge potential A_μ, non trivial vortex configurations satisfying the above requirement do occur when a relation between the strength of the magnetic coupling, the scalar and the A_μ masses is satisfied. The relevant scalar potential must be $V(\phi) = \frac{1}{2\eta^2 e^2}(\phi^\dagger\phi - J_0)^2$. The corresponding vortex configurations have a zero electric field, carry a (quantized) magnetic flux proportional to the charge and saturate the lower bound for the energy which is proportional to the flux.

Type II models[5] involve both magnetic dipole and electric field dependent couplings to the non relativistic scalar as described by (3.2) in addition to the usual minimal coupling. In the present work, we have assumed that the strength for the magnetic coupling reaches the special value already considered in [13]. These models are related to the planar dynamics of nonstandard anyonic composite objects, as indicated in the appendix. Type II models with polynomial potential (of finite degree) cannot support non trivial vortex minimizing the energy. This is due to the contributions coming from the electric field dependent coupling terms generating a (nonlinear) term in the static energy (see (3.11)) which cannot be compensated by a finite number of polynomial potential terms. When the potential is polynomial, the minimum for the energy is obtained for configurations with zero or constant matter density and/or vanishing gauge potential. However, non trivial vortex configurations appear within type II models involving a non polynomial potential whose expression is given by (3.13). These vortex solutions have a non zero electric field and still carry a charge proportional to the magnetic flux. The differential Eq. (3.15) constraining the matter density f^2 reduces in the small matter density regime ($f^2 << m\eta^2 e^2$) to a nonlinear elliptic equation somehow similar to the one constraining the matter density for the vortex configurations obtained within type I models. We have solved (3.15) numerically and found that f^2 vanished at the origin and increases smoothly until it reaches some asymptotic plateau whose value is fixed by the magnitude of the uniform background charge. Finally, we have considered the case of type II models coupled to an external gauge potential corresponding to a magnetic field localized at the origin. Again, those type II models with the nonlinear potential (3.13) support non trivial vortex solutions minimizing the energy.

[5] In these models, non trivial vortex configurations must have a non identically zero electric field, a fact which is already apparent in the equations of motion which, in particular, imply that either the matter density and/or the gauge potential function $a(r)$ is zero whenever the electric field is zero.

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APPENDIX

Consider the usual Hamiltonian for a system of N particles or quasiparticles moving in a plane and submitted, for instance, to an external static gauge potential $\mathcal{A}_\mu$

$$H_{qp} = \frac{1}{2m}\sum_{I=0}^{N}[i\boldsymbol{\partial}_i^{(I)} + \mathcal{A}_i(\mathbf{x}_I)]^2 + \sum_{I=0}^{N}\mathcal{A}_0(\mathbf{x}) + \ldots, \quad (A1)$$

where m is the mass for the particles or quasiparticles, capital Latin indices $I, J, \ldots$ refer to the particles $I, J, \ldots$, $\mathbf{x}_{IJ} = \mathbf{x}_I - \mathbf{x}_J$, the upper indice appearing in the derivative operator means that it acts on the I-th particle, and the ellipsis corresponds to possible potential terms whose explicit form will not influence significantly the present discussion. The second quantized Lagrangian counterpart of (A1) is readily found to be

$$\mathcal{L} = i\phi^\dagger[\partial_0 - i\mathcal{A}_0(\mathbf{x})]\phi + \frac{1}{2m}\phi^\dagger[\partial_i - i\mathcal{A}_i(\mathbf{x})]^2\phi + \ldots, \quad (A2)$$

where $\phi = \phi(t, \mathbf{x})$ is a complex scalar field. Following the usual way to introduce a statistical gauge potential [2], [1], we first define the singular gauge transformation $\Psi'(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \exp(i\frac{\gamma}{\pi}\sum_{I<J}\alpha_{IJ})\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_N)$ acting on the N-particles wave function where γ is a real constant and α_{IJ} denotes the angle between $\mathbf{x}_{IJ} = \mathbf{x}_I - \mathbf{x}_J$ and, says the x axis.[6] The Hamiltonian relevant to Ψ' can then be expressed as

$$H'_{qp} = \frac{1}{2m}\sum_{I=0}^{N}[i\boldsymbol{\partial}_i^{(I)} + \mathcal{A}_i(\mathbf{x}_I) + A_i(\mathbf{x}_I)]^2 + \sum_{I=0}^{N}\mathcal{A}_0(\mathbf{x}) + \ldots, \quad (A3)$$

where the statistical gauge potential carrying the Aharonov-Bohm type singularities is given by

[6] Interchange of any two particles gives $\alpha_{IJ} \to \alpha_{IJ} \pm \pi$ so that it changes the phase of the wave function by $\exp(i\gamma)$ whose statistics is unchanged (changed) for $\gamma = k2\pi$ ($\gamma = (2k+1)\pi$), $k \in \mathbb{Z}$.

$$A_i(\mathbf{x}_I) = \frac{\gamma}{\pi}\sum_{I\neq J}\partial_i^{(I)}\alpha_{IJ} = -\frac{\gamma}{\pi}\sum_{I\neq J}\epsilon_{ij}\frac{(x_I - x_J)^i}{|\mathbf{x}_I - \mathbf{x}_J|^2}. \quad \text{(A4)}$$

This, translated into a second quantized formalism, yields

$$\mathcal{H}_{qp} = \int d\mathbf{x}\bigg[-\frac{1}{2m}\phi^\dagger(\mathbf{x})[\partial_i - i\mathcal{A}_i(\mathbf{x}) - iA_i(\mathbf{x})]^2\phi(\mathbf{x})\bigg] + \ldots \quad \text{(A5a)}$$

and

$$A_i(\mathbf{x}) = -\frac{\gamma}{\pi}\int d\mathbf{y}\epsilon_{ij}\frac{(x^i - y^i)}{|\mathbf{x} - \mathbf{y}|^2}\rho(\mathbf{y}), \quad \text{(A5b)}$$

with $\rho(\mathbf{x}) = \phi^\dagger\phi(\mathbf{x})$. The final step amounts to treat the statistical gauge potential as a dynamical variable of some action which must be suitably chosen and coupled to the matter part such that (3.5b) is solution of the corresponding equations of motion. Two different inequivalent ways do exist to achieve this goal.

The first most currently used possibility which gives rise to the standard anyons [2] is obtained when the statistical gauge potential is involved in a Chern-Simons action minimally coupled to the matter. This can be summarized as follows: One notices[7] that (A5b) yields $\epsilon^{ij}\partial_i A_j(\mathbf{x}) = 2\gamma\rho(\mathbf{x})$(i). Then, by further allowing a time dependence in A and ρ, differentiating (i) with respect to time and restoring Lorentz covariance through the introduction of a scalar potential A_0, one obtains $\epsilon^{ij}\partial_i(\partial_0 A_j - \partial_j A_0) = 2\gamma\partial_0\rho = -2\gamma\partial_i J^i$, where $J_\mu = (\rho; J_i)$ is the gauge-invariant matter current and current conservation has been used. This produces $\epsilon^{ij}(\partial_0 A_j - \partial_j A_0) = -2\gamma J^i$(ii). One then easily realizes that (i) and (ii) can be obtained as the equations of motion for $S_{CS} = \int d^3x(-\frac{1}{4\gamma}\epsilon_{\mu\nu\rho}A^\mu\partial^\nu A^\rho + A_\mu J^\mu)$. Then, allowing ϕ to depend on time and restoring again the Lorentz covariance through the introduction of A_0, one easily obtains the second quantized Lagrangian version describing the planar dynamics of particles or quasiparticles with Chern-Simons statistical interaction which is related to the standard anyons [2], [1]

$$\mathcal{L} = i\phi^\dagger\bigg[\partial_0 - i(\mathcal{A}_0 + A_0)\bigg]\phi + \frac{1}{2m}\phi^\dagger[\partial_i - i(\mathcal{A}_i + A_i)]^2\phi - \frac{1}{4\gamma}\epsilon_{\mu\nu\rho}A^\mu\partial^\nu A^\rho + \ldots, \quad \text{(A6)}$$

where still $\mathcal{A}_\mu = \mathcal{A}_\mu(\mathbf{x})$ and corresponds therefore to (A2) with minimal coupling to a Chern-Simons action for A_μ.

[7] Owing to $\partial^2\ln|\mathbf{x} - \mathbf{y}| = 2\pi\delta(\mathbf{x} - \mathbf{y})$.

The alternative possibility has been proposed and further discussed in [13]. It is obtained by noticing that (A5b) is also the solution of the equations of motion stemming from a MCS action for the statistical field with minimal and nonminimal coupling to matter as given in (3.2), provided the strength κ of the nonminimal coupling is fixed to a special value. To see that, consider the following action for the statistical gauge potential

$$S = \int d^3x - \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{8\gamma}\epsilon_{\mu\nu\rho}A^\mu F^{\nu\rho} + A_\mu J^\mu + \kappa F_\mu J^\mu, \quad \text{(A7)}$$

where F_μ is the dual field strength. When $\kappa = \frac{\gamma}{e^2}$, the corresponding equation of motion can be expressed as

$$-\frac{1}{e^2}\epsilon_{\alpha\nu\rho}\partial^\mu(F^\rho - 2\gamma J^\rho) - \frac{1}{2\gamma}(F_\alpha - 2\gamma J_\alpha) = 0, \quad \text{(A8)}$$

which is formally similar to the equations of motion for a free MCS theory and are solved by $F_\mu = 2\gamma J_\mu$ whose time and space components are nothing but equations (i) and (ii) given above. Accordingly, the resulting second quantized Lagrangian version obtained from (A5) can be found to be given by

$$\mathcal{L}_{II} = i\phi^\dagger[\partial_0 - i(\mathcal{A}_0 + A_0)]\phi + \frac{2\gamma}{e^2}F_0\phi^\dagger\phi + \frac{1}{2m}\phi^\dagger\bigg[\partial_i - i(\mathcal{A}_i + A_i) - i\frac{2\gamma}{e^2}F_i\bigg]^2\phi - \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{8\gamma}\epsilon_{\mu\nu\rho}A^\mu F^{\nu\rho} + \ldots, \quad \text{(A9)}$$

where $F_0 = \epsilon_{ij}\partial^i A^j$ and $F_i = \partial_i A_0 - \partial_0 A_i$ which is similar to the Lagrangian defining type II MCS models considered in Sec. III (with $\gamma = -\frac{1}{2\eta}$).

We note that the Lagrangian (A6) is an important piece appearing in the Landau-Ginzburg type effective model proposed in [15] to describe some of the global properties of the Quantum Hall Effect. In this latter description, one of the building ingredients was the reformulation of the problem of interacting fermions in an external magnetic field as a problem of interacting bosons with (minimal) coupling to a Chern-Simons gauge field, the statistical field, controlling the statistical transmutation of bosons to fermions. Now, in the alternative description of anyons proposed in [13], the statistical transmutation is obtained through the minimal and suitable nonminimal coupling of a MCS statistical gauge field to matter. In this spirit, (A9) can be viewed as a modification of the above effective theory in which statistical transmutation is controlled now by the coupling of a MCS statistical field with a suitable coupling to the scalar.

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