Non-Abelian generalization of Born-Infeld theory inspired by noncommutative geometry

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We present a new non-Abelian generalization of the Born-Infeld Lagrangian. It is based on the observation that the basic quantity defining it is the generalized volume element, computed as the determinant of a linear combination of metric and Maxwell tensors. We propose to extend the notion of the determinant to the tensor product of space-time and a matrix representation of the gauge group. We compute such a Lagrangian explicitly in the case of the $SU(2)$ gauge group and then explore the properties of static, spherically symmetric solutions in this model. We have found a one-parameter family of finite energy solutions. In the last section, the main properties of these solutions are displayed and discussed.

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I. INTRODUCTION

Recently there has been rising interest in the Born-Infeld nonlinear theory of electromagnetism [1,2] and more general Lagrangians of this type, which appear quite naturally in string theories. Non-Abelian generalizations of a Born-Infeld type Lagrangian were proposed by Hagiwara in 1981 [3], and more recently, including a supersymmetric version, by Schaposnik and co-workers (see [4–6] and the references within). In [7] we analyzed one of the possible non-Abelian generalizations of the Born-Infeld Lagrangian, and showed the existence of sphaleronlike solutions with a qualitative behavior similar to the solutions of the combined Einstein-Yang-Mills field equations found by Bartnik and McKinnon [8]. The non-Abelian generalization proposed in [7] was quite straightforward indeed: it consisted in the replacement of the electromagnetic field invariants

$$F^{\mu\nu}F_{\mu\nu} \quad \text{and} \quad {}^*F^{\lambda\rho}F_{\lambda\rho}=\frac{1}{2}\epsilon^{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho}$$

by similar expressions formed by taking the traces of corresponding Lorentz invariants in the Lie algebra space:

$$g_{ab}F^{a\ \mu\nu}F^{b}_{\mu\nu} \quad \text{and} \quad g^{*}_{ab}F^{a\ \lambda\rho}F^{b}_{\lambda\rho}=\frac{1}{2}g_{ab}\epsilon^{\mu\nu\lambda\rho}F^{a}_{\mu\nu}F^{b}_{\lambda\rho}.$$

However, except for a straightforward analogy, this expression does not seem to come from any more fundamental theory. In addition, this generalization still keeps a particular dependence on second-order invariants of the field tensor, characteristic for a *four-dimensional* manifold only; in higher dimensions the determinant would lead quite naturally to Lagrangians depending on higher-order invariants of the field tensor, too. On the other hand, it is well known that a correct mathematical formulation of gauge theories considers the gauge field tensor associated with a compact and semisimple gauge group G as a connection one-form in a principal fiber bundle over Minkowskian space-time, with values in $\mathcal{A}_G$, the Lie algebra of G. In local coordinates we have

$$A=A^{a}_{\mu}dx^{\mu}L_{a} \tag{1}$$

where L_a, $a=1,2,\ldots,N=\dim(G)$, is the basis of the adjoint representation of $\mathcal{A}_G$. In many cases another representation must be chosen, especially when the gauge fields are supposed to interact with spinors (cf. [9–11]). It is always possible to embed the Lie algebra in an enveloping associative algebra, and to use the tensor product

$$A=A^{a}_{\mu}dx^{\mu}\otimes T_{a}, \tag{2}$$

where T_a is the basis of the matrix representation of $\mathcal{A}_G$, so that now the non-Abelian field tensor will have its values in the enveloping algebra:

$$F=dA+\frac{1}{2}[A,A]=(F^{a}_{\mu\nu}dx^{\mu}\wedge dx^{\nu})\otimes T_{a}. \tag{3}$$

Now, in order to reproduce as closely as possible the classical Born-Infeld Lagrangian, a natural idea is to embed the space-time metric tensor $g_{\mu\nu}$ also into the enveloping algebra, tensoring it simply with the unit element in the appropriate matrix space, i.e., replacing it by $g_{\mu\nu}\otimes\mathbb{1}_N$; then we can add up the metric and the field tensors and take the determinant in the resulting matrix space. This structure of the matrix is similar to the structures found in certain realizations of gauge theory in noncommutative matrix geometries [12], or in Lagrangians found in matrix theories [13,14]. Such a Lagrangian has been proposed by Park [15] and reads as follows:

$$S_{Park}[F,g] = \int_{\mathbb{R}^4} \alpha \left(\left| \det_{\mathcal{M}\otimes R} (g_{\mu\nu}\otimes \mathbb{1}_{d_R} + \beta^{-1} F^a_{\mu\nu} \otimes T_a) \right|^{1/2d_R} - \sqrt{|g|} \right), \tag{4}$$

where α and β are real positive constants. The $2d_R$-order root is introduced to ensure the invariance of the resulting action under the diffeomorphisms. As a matter of fact, with the root of this order we are able to factorize out the usual four-dimensional volume element $\sqrt{|g|}d^4x$ and rewrite the action principle with the subsequent scalar quantity:

$$L_{Park}(F,g) = \alpha \left(\left| \det_{\mathcal{M}\otimes R} (\mathbb{1}_{4\times d_R} + \beta^{-1}\hat{F}) \right|^{1/2d_R} - 1 \right) \tag{5}$$

and

$$S_{Park}[F,g] = \int_{\mathbb{R}^4} L_{Park}(F,g)\sqrt{|g|}d^4x, \tag{6}$$

where

$$\hat{F} = \frac{1}{2} F^a_{\mu\nu} \hat{M}^{\mu\nu} \otimes T_a, \quad (\hat{M}^{\mu\nu})^{\rho}_{\sigma} = g^{\rho\rho'} \delta^{\mu\nu}_{\rho'\sigma}, \tag{7}$$

$\hat{F}$ is an endomorphism of $\mathbb{R}^4 \otimes \mathbb{C}^{d_R}$, and $M_{\mu\nu}$ denote the generators of the Lorentz group (in the defining representation). It is also useful to introduce the notation

$$\hat{F}^a = \frac{1}{2} F^a_{\mu\nu} \hat{M}^{\mu\nu}. \tag{8}$$

The generalization of the Born-Infeld (BI) Lagrangian proposed in this paper results in a variational principle that leads to a highly nonlinear system of field equations, whose general properties can be analyzed using standard techniques [16–18]. Our aim in this article is to check whether stationary regular solutions with finite energy can be found as in [7]. We consider the standard 't Hooft monopole ansatz, which in this particular case leads to one ordinary differential equation for a single function $k(r)$ of radial coordinate r. The structure of this equation is similar to the one found in [7], with a more complicated term corresponding to friction. Nevertheless, the structure of solutions and their energy spectrum are very different, as shown in the last sections of our article. We have not found solutions joining together two different vacuum configurations (called BI *sphalerons*), as in [7]. We find instead a family of solutions labeled by integer winding number n, and a real parameter bounded from below. The energy integral tends with $n\to\infty$ to the energy of the BI magnetic monopole obtained in [7].

II. NON-ABELIAN GENERALIZATION OF BORN-INFELD LAGRANGIAN

A. Basic properties of the Abelian case

Let us recall several basic properties of the Abelian Born-Infeld Lagrangian, which we would like to reproduce in the proposed non-Abelian generalization. In their original paper [1], Born and Infeld considered the now famous least action principle:

$$\begin{aligned} S_{BI}[g,F] &= \int_{\mathbb{R}^4} \mathcal{L}_{BI}(g,F) = \int_{\mathbb{R}^4} L_{BI}(g,F)\sqrt{|g|}d^4x \\ &= \int_{\mathbb{R}^4} \beta^2 \left(\sqrt{|\det(g_{\mu\nu})|} - \sqrt{|\det(g_{\mu\nu} + \beta^{-1}F_{\mu\nu})|} \right) d^4x \\ &= \int_{\mathbb{R}^4} \beta^2 \left(1 - \sqrt{1 + \frac{1}{\beta^2}(F,F) - \frac{1}{4\beta^4}(F,\star F)^2} \right) \sqrt{|g|}d^4x \\ &= \int_{\mathbb{R}^4} \beta^2 \left(1 - \sqrt{1 + \frac{1}{\beta^2}(\vec{B}^2 - \vec{E}^2) - \frac{1}{\beta^4}(\vec{E}.\vec{B})^2} \right) \sqrt{|g|}d^4x, \end{aligned} \tag{9}$$

where $d^4x = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$, $\vec{B}$ is the magnetic field, $\vec{E}$ is the electric field, $(F,F) = \frac{1}{2}F_{\mu\nu}F^{\mu\nu}$, $(F,\star F) = \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$, and $\epsilon^{\mu\nu\rho\sigma} = (1/\sqrt{|g|})\delta^{\mu\nu\rho\sigma}_{0123}$.

This action can be defined not only on the Minkowskian space-time but also on any locally Lorentzian curved manifold, as in the original case. It is useful to recall here three important properties of the Born-Infeld Lagrangian, which we want to maintain in the case of the non-Abelian generalization, also valid in any finite dimension of space-time.

(1) Maxwell's theory (or, respectively, the usual gauge theory with quadratic Lagrangian density) should be found in the limit $\beta\to\infty$:

$$\begin{aligned} S_{BI} &= -\int_{\mathbb{R}^4} \frac{1}{2}(F,F)\sqrt{|g|}d^4x + o\left(\frac{1}{\beta^2}\right) \\ &= -\frac{1}{2}\int_{\mathbb{R}^4} F\wedge\star F + o\left(\frac{1}{\beta^2}\right) \\ &= -\int_{\mathbb{R}^4} \frac{1}{2}(\vec{B}^2 - \vec{E}^2)\sqrt{|g|}d^4x + o\left(\frac{1}{\beta^2}\right). \end{aligned} \tag{10}$$

(2) There exists an upper limit for the electric field intensity, equal to β when there the magnetic component of the field vanishes:

$$L_{BI}|_{B=0} = \beta^2(1 - \sqrt{1 - \beta^{-2}\vec{E}^2}). \tag{11}$$

Due to this fact, the energy of a pointlike charge is finite, and the field remains finite even at the origin. This was the main goal pursued by Mie [2], suggesting the choice of nonlinear generalization of Maxwell's theory. Indeed, one has for a point charge *e*,

$$\vec{E}=\frac{e\hat{r}}{\sqrt{e^2+r^4}},\quad \text{Energy}=\int_0^\infty\left(\frac{\sqrt{e^2+r^4}}{r^2}-1\right)r^2dr<\infty. \tag{12}$$

(3) The Born-Infeld action principle is invariant under the diffeomorphisms of $\mathbb{R}^4$. In this respect, this theory can be viewed as a covariant generalization (in the sense of general relativity) of Mie's theory, as well as an extension of the usual volume element $\sqrt{|g|}d^4x$.

It is also well known that the Born-Infeld electromagnetism has good causality properties (no birefringence and no shock waves) as well as interesting dual symmetries (electric-magnetic duality, Legendre duality, cf. [19–21,18,22]). Here we shall not consider these aspects of the theory, our main interest being focused on static solutions.

B. The new non-Abelian generalization

Our starting point is the gauge field tensor associated with a compact and semisimple gauge group *G*, defined as a connection one-form in the principal fiber bundle over Minkowskian space-time, with its values in $\mathcal{A}_G$, the Lie algebra of *G*. As explained in the Introduction, we chose the representation of the connection in the tensorial product of a matrix representation of the Lie algebra $\mathcal{A}_G$ and the Grassmann algebra of forms over M_4:

$$A=A^a_\mu dx^\mu\otimes T_a, \tag{13}$$

where T_a, $a=1,2,\ldots,N=\dim(G)$, are anti-Hermitian matrices which form a basis of the particular representation *R* of dimension d_R of $\mathcal{A}_G$, specified later on.

By analogy with the Abelian case, we want the Lagrangian to satisfy the following properties:

(1) One should find the usual Yang-Mills theory in the limit $\beta\to\infty$.

(2) The (non-Abelian) analogue of the electric field strength should be bounded from above when the magnetic components vanish. [To satisfy this particular constraint, we must ensure that the polynomial expression under the root should start with terms $1-\beta^{-2}(\vec{E}^a)^2+\cdots$ when $\vec{B}^a=0$.]

(3) The action should be invariant under the diffeomorphisms of $\mathbb{R}^4$.

(4) The action has to be real.

This enables us to introduce the following generalization of the Born-Infeld Lagrangian density for a non-Abelian gauge field:

$$\begin{aligned}\mathcal{L}&=\sqrt{g}L\\&=\sqrt{|g|}-|\det(\mathbb{1}_2\otimes g_{\mu\nu}\otimes\mathbb{1}_{d_R}+\beta^{-1}J\otimes F^a_{\mu\nu}\otimes T_a)|^{1/4d_R}.\end{aligned} \tag{14}$$

In the expression above, *J* denotes a $SL(2,\mathbb{C})$ matrix satisfying $J^2=-\mathbb{1}_2$, thus introducing a quasicomplex structure. This extra doubling of tensor space is necessary in order to ensure that the resulting Lagrangian is real. We are left with the root of order $4d_R$, so that the invariance of our action under the space-time diffeomorphism is preserved.

Let us recall a few arguments in favor of this construction.

The simplest way to generalize the Born-Infeld action principle to the non-Abelian case seems at first glance the substitution of real numbers by corresponding Hermitian operators, as in quantum mechanics or in noncommutative geometry. Then one would arrive at the following expression:

$$\begin{aligned}U(1)&\quad\leftrightsquigarrow\quad G,\\ iF_{\mu\nu}&\quad\leftrightsquigarrow\quad F^a_{\mu\nu}\otimes T_a,\\ g_{\mu\nu}&\quad\leftrightsquigarrow\quad g_{\mu\nu}\otimes\mathbb{1}_{d_R},\end{aligned} \tag{15}$$

where $\mathbb{1}_{d_R}$ and iT_a are Hermitian matrices. What remains now to make the generalization complete is to extend the notion of the determinant taken over the space-time indices in the usual case. We propose to replace the determinant of a 4×4 matrix (denoted hereafter $\det_{\mathcal{M}}$) by a determinant taken in the tensor product of space-time and matrix indices of the representation *R* (denoted hereafter $\det_{\mathcal{M}\otimes R}$). Notice that this kind of tensor product of algebras appears in the context of the noncommutative geometry of matrices (see [12–14]). Indeed, the general structure of the connection one-form in these noncommutative geometries is very similar to the one in Eq. (13).

In this kind of generalization, one would replace the objects in Eq. (9) following the procedures in Eq. (15). This leads to a complex Lagrangian in the case of a non-Abelian structure group. Indeed, the determinant $\det_{\mathcal{M}\otimes R}(g_{\mu\nu}\otimes\mathbb{1}_{d_R}+\beta^{-1}F^a_{\mu\nu}\otimes iT_a)$ is not real when $\dim(\mathcal{A}_G)>1$. Therefore we must find a different generalization.

Another possibility would consist of taking anti-Hermitian generators tensorized with the field *F*. This was proposed by Hagiwara [3] and studied in more detail by Park [15] for the Euclidean case. This substitution leads to a Lagrangian satisfying the requirements (1), (3), and (4), but not (2) (for details, see the article by Park [15]).

Moreover, Lagrangians obtained with the above choices display invariants of order 3 in the field *F*, destroying the charge conjugation invariance of the theory, $F\mapsto -F$, and possibly leading to indefinite energy densities.

This is why we propose a third choice. We start from an alternative formulation of the Abelian version. As a matter of fact, one can write the Abelian Born-Infeld Lagrangian in the following alternative form:

$$\begin{aligned}S_{BI}&[F,g]\\&=\int_{\mathbb{R}^4}\beta^2\left(\sqrt{|g|}-\left|\det_{\mathbb{C}^2\otimes\mathcal{M}}(\mathbb{1}_2\otimes g_{\mu\nu}+\beta^{-1}J\otimes iF_{\mu\nu})^{1/4}\right)d^4x,\end{aligned} \tag{16}$$

where *J* is a 2×2 complex matrix whose square is equal to $-\mathbb{1}_2$. The Lagrangian is independent of the choice of *J* as can be easily seen. In Eq. (16) [see also Eq. (15)], the imaginary unit *i* can be considered as the anti-Hermitian generator of $\mathfrak{u}(1)$. In our formula (16), we use an obvious notation for the space on which the determinant is defined. With the correspondence displayed in Eq. (15), we end up with the following action principle:

$$S[F,g]=\int_{\mathbb{R}^4}\alpha\left(\sqrt{|g|}-\left|\det_{\mathbb{C}^2\otimes\mathcal{M}\otimes R}(\mathbb{1}_2\otimes g_{\mu\nu}\otimes\mathbb{1}_{d_R}+\beta^{-1}J\otimes F^a_{\mu\nu}\otimes T_a)\right|^{1/4d_R}\right)d^4x, \tag{17}$$

satisfying all the requirements we asked for, (1), (2), (3), and (4), by taking J in $SL(2,\mathbb{C})$. The Lagrangian is again independent of the choice of J. In particular, we find the usual Abelian Lagrangian if we replace T_a by i and set $d_R=1$.

It was supposed in Eq. (17) that α and β are real positive constants. It is clear that only the root of degree $4d_R$ will lead to an expression where $\sqrt{g}$ can be factorized out as an overall factor. This enables one to rewrite the action using a purely scalar quantity as follows:

$$L(g,F)=\alpha\left(1-\left|\det_{\mathbb{C}^2\otimes\mathcal{M}\otimes R}(\mathbb{1}_2\otimes\mathbb{1}_{4\times d_R}+\beta^{-1}J\otimes\hat{F})\right|^{1/4d_R}\right), \tag{18}$$

so that

$$S[g,F]=\int_{\mathbb{R}^4}L(g,F)\sqrt{|g|}d^4x, \tag{19}$$

where $\hat{F}=\frac{1}{2}F^a_{\mu\nu}\hat{M}^{\mu\nu}\otimes T_a$ as defined in the Introduction.

III. EXPLICIT COMPUTATION OF THE DETERMINANT

A. General remarks

The determinant defined in Eq. (18) can be written in several equivalent forms:

$$\det_{\mathbb{C}^2\otimes\mathcal{M}\otimes R}(\mathbb{1}_2\otimes\mathbb{1}+\beta^{-1}J\otimes\hat{F}) \tag{20a}$$

$$=\det_{\mathbb{C}^2\otimes\mathcal{M}\otimes R}(s\otimes\mathbb{1}+\beta^{-1}sJ\otimes\hat{F}) \tag{20b}$$

$$=\det_{\mathcal{M}\otimes R}(\mathbb{1}+\beta^{-2}\hat{F}^2), \tag{20c}$$

where s and J are elements of $SL(2,\mathbb{C})$, J satisfying $J^2=-\mathbb{1}$. For example, choosing $s=i\sigma_2$ and $sJ=-i\sigma_3$ in (20b), we get the following determinant:

$$\begin{vmatrix}-i\beta^{-1}\hat{F} & \mathbb{1}\\ -\mathbb{1} & i\beta^{-1}\hat{F}\end{vmatrix} =|g|^{-2d_R}\begin{vmatrix}-i\beta^{-1}F^a_{\mu\nu}\otimes T_a & g_{\mu\nu}\otimes\mathbb{1}\\ -g_{\mu\nu}\otimes\mathbb{1} & i\beta^{-1}F^a_{\mu\nu}\otimes T_a\end{vmatrix}, \tag{21}$$

which is a straightforward generalization of the determinant considered by Schuller [23]. Following Schuller's idea, the matrix (21) in the Abelian case can be interpreted as the matrix defining commutation relations between the coordinates in the phase space of a relativistic point particle minimally coupled to the Born-Infeld field. Similarly, we can extend this interpretation to the case of coordinates taking their values in an appropriate Lie algebra, i.e., by imposing the following relations:

$$[X_\mu,X_\nu]=-\frac{1}{e\beta^2}F^a_{\mu\nu}\otimes T_a,$$

$$[X_\mu,P_\nu]=-ig_{\mu\nu}\otimes\mathbb{1}, \tag{22}$$

$$[P_\mu,P_\nu]=eF^a_{\mu\nu}\otimes T_a,$$

with

$$X_\mu:=X^a_\mu\otimes-iT_a,\quad P_\mu:=P^a_\mu\otimes-iT_a. \tag{23}$$

On the other hand, the particular form (20c) enables us to check that the Lagrangian is indeed real, and at the same time it represents an obvious generalization of the Abelian Born-Infeld action in the form given in [23], and the references therein. It is worthwhile to note that if one chooses $J=-i\sigma_3$ in (20a) the determinant can be written as an absolute value of a complex number. Indeed, one has

$$\begin{vmatrix}\mathbb{1}-i\beta^{-1}\hat{F} & 0\\ 0 & \mathbb{1}+i\beta^{-1}\hat{F}\end{vmatrix}=\left|\det_{\mathcal{M}\otimes R}(\mathbb{1}-i\beta^{-1}\hat{F})\right|^2. \tag{24}$$

We shall use this particular form of the determinant in the subsequent computations.

B. Comparison with the symmetric trace prescription

Let us recall a useful formula relating the determinant of a linear operator M to traces:

$$[\det(1+M)]^\beta =\exp\{\beta\,\mathrm{tr}[\log(1+M)]\} =\sum_{n=0}^{\infty}\sum_{\substack{\underline{\alpha}=(\alpha_1,\ldots,\alpha_n)\\ \in[S_n]}}(-1)^n\prod_{p=1}^{n}\frac{1}{\alpha_p!}\left(-\frac{\beta\,\mathrm{tr}(M^p)}{p}\right)^{\alpha_p}, \tag{25}$$

where $\underline{\alpha}\in[S_n]$ and $[S_n]$ is the set of equivalence classes of the permutation group of order n. The multi-index $\underline{\alpha}$ is given by a Ferrer-Young diagram or equivalently satisfies the relation

$$\sum_{p=1}^{n} p\alpha_p = n, \quad \alpha_p \geqslant 0. \tag{26}$$

Using this trace formula, we can develop our Lagrangian up to any order in F. In order to avoid ambiguities, we shall-denote by $\mathrm{tr}_{\mathcal{M}}$ the trace taken over the space-time indices, by tr_R the trace over the representation indices, and by $\mathrm{tr}_\otimes$ the trace over the tensor product of these two spaces. For the sake of simplicity, we have absorbed the scale factor β^{-1} in the definition of the field tensor F. When needed, the appropriate powers of β^{-1} can easily be recovered. Following (20c), we have,

$$\left(\det_{\mathcal{M}\otimes R}(1+\hat{F}^2)\right)^{1/4d_R} = \sum_{n=0}^{\infty} \sum_{\underline{\alpha}=(\alpha_1,\ldots,\alpha_n)} (-1)^n \prod_{k=1}^{n} \frac{1}{\alpha_k!}\left(\frac{-\mathrm{tr}_\otimes(\hat{F}^{2k})}{4d_R\times k}\right)^{\alpha_k}$$
$$= \sum_{n=0}^{\infty} \sum_{\underline{\alpha}=(\alpha_1,\ldots,\alpha_n)} (-1)^n \prod_{k=1}^{n} \frac{1}{\alpha_k!} \prod_{\substack{m=1\\ \alpha_k\neq 0}}^{\alpha_k} \left(-\frac{\mathrm{tr}_{\mathcal{M}}(\hat{F}^{a_1^m}\cdots\hat{F}^{a_{2k}^m})}{4k}\,\frac{\mathrm{tr}_R(T_{a_1^m}\cdots T_{a_{2k}^m})}{d_R}\right), \tag{27}$$

where $\underline{\alpha}\in[S_n]$ satisfies $\Sigma_{k=1}^{n} k\alpha_k = n$.

We can compare the resulting expansion with the symmetrized trace prescription given by Tseytlin in [24]. With the notation adopted above, we have

$$\frac{1}{d_R}\mathcal{S}\mathrm{tr}_R\left(\det_{\mathcal{M}}(1+i\hat{F}^a T_a)\right)^{1/2} = \frac{1}{d_R}\mathcal{S}\mathrm{tr}_R\left(\det_{\mathcal{M}}(1+\hat{F}^a\hat{F}^b T_a T_b)\right)^{1/4}$$
$$= \frac{1}{d_R}\mathcal{S}\mathrm{tr}_R \sum_{n=0}^{\infty} \sum_{\underline{\alpha}=(\alpha_1,\ldots,\alpha_n)} (-1)^n \prod_{k=1}^{n}\frac{1}{\alpha_k!}\left(-\frac{\mathrm{tr}_{\mathcal{M}}(\hat{F}^{a_1}\cdots\hat{F}^{a_{2k}})}{4k} T_{a_1}\cdots T_{a_{2k}}\right)^{\alpha_k}$$
$$= \sum_{n=0}^{\infty} \sum_{\underline{\alpha}=(\alpha_1,\ldots,\alpha_n)} (-1)^n \left[\prod_{k=1}^{n}\frac{1}{\alpha_k!}\prod_{\substack{m=1\\ \alpha_k\neq 0}}^{\alpha_k}\left(-\frac{\mathrm{tr}_{\mathcal{M}}(\hat{F}^{a_1^m}\cdots\hat{F}^{a_{2k}^m})}{4k}\right)\frac{1}{d_R}\mathcal{S}\mathrm{tr}_R\left(\prod_{k=1}^{n}\prod_{m=1}^{\alpha_k} T_{a_1^m}\cdots T_{a_{2k}^m}\right)\right]. \tag{28}$$

Now we can easily compare the series resulting from similar expansions of two different Lagrangians: the symmetrized trace prescription, and the generalized determinant prescription, i.e., comparing Eqs. (27) and (28) with the corresponding expansion of the Abelian version of Born-Infeld electrodynamics. In both cases, the third-order and higher odd-order invariants that are possible in a non-Abelian case do not appear (as they are absent in the Abelian version, of course).

Let us compare, up to the fourth order, the expansion in powers of F of the two Lagrangians. Our Lagrangian (17) yields the following series:

$$L[F,g] \simeq -\frac{1}{4d_R}\mathrm{tr}_\otimes\hat{F}^2 + \frac{1}{8d_R}\mathrm{tr}_\otimes\hat{F}^4 - \frac{1}{32d_R^2}(\mathrm{tr}_\otimes\hat{F}^2)^2$$
$$\simeq -\frac{1}{2}(F^a,F^b)K_{ab} + \frac{1}{8}(F^a,F^b)(F^c,F^d)$$
$$\times(-K_{ab}K_{cd}+K_{abcd}+K_{acbd})$$
$$+\frac{1}{8}(F^a,\star F^b)(F^c,\star F^d)K_{acbd}, \tag{29}$$

whereas the symmetrized trace prescription of [24] gives

$$L_{sym}[F,g] = \frac{1}{d_R}\mathcal{S}\mathrm{tr}_R\left(\mathbb{1} - \sqrt{\det_{\mathcal{M}}(\mathbb{1}+i\hat{F})}\right)$$
$$\simeq \frac{1}{d_R}\mathcal{S}\mathrm{tr}_R\left(-\frac{1}{4}\mathrm{tr}_{\mathcal{M}}\hat{F}^2 + \frac{1}{8}\mathrm{tr}_{\mathcal{M}}F^4 - \frac{1}{32}(\mathrm{tr}_{\mathcal{M}}\hat{F}^2)^2\right)$$
$$\simeq -\frac{1}{2}(F^a,F^b)K_{ab} + \frac{1}{8}[(F^a,F^b)(F^c,F^d)$$
$$+(F^a,\star F^b)(F^c,\star F^d)]K_{\{abcd\}}, \tag{30}$$

with $K_{\{abcd\}} = \frac{1}{3}(K_{ab}K_{cd}+K_{ac}K_{bd}+K_{ad}K_{bc}+\frac{1}{4}S_{ab}^{e}S_{cde} + \frac{1}{4}S_{ac}^{e}S_{bde}+\frac{1}{4}S_{ad}^{e}S_{bce})$. As usual we note that

$$T_a T_b = -g_{ab}\mathbb{1} + \frac{1}{2}C_{ab}^{c}T_c + \frac{i}{2}S_{ab}^{c}T_c, \tag{31}$$

where $g_{ab}=(c_R/d_R)\delta_{ab}$, $S_{cab}=g_{cd}S_{ab}^{d}$ is completely symmetric and real, $C_{cab}=g_{cd}C_{ab}^{d}$ is completely antisymmetric and real, and

$$K_{a_1\cdots a_n}=\frac{(-1)^{[n/2]}}{d_R}\mathrm{tr}_R(T_{a_1}\cdots T_{a_n}). \tag{32}$$

C. Explicit calculus for $G=SU(2)$

We use the fundamental representation of $G=SU(2)$, with generators defined by $T_a=-(i/2)\sigma_a$. In order to simplify the calculus, we have rescaled the formula (24) replacing β by 1/2, so that it compensates the factor 1/2 in the definition of T_a. It is useful to note that in the formula (24), the expression $\det_{\mathcal{M}\otimes R}(\mathbb{1}-i\beta^{-1}\hat{F})$ is a perfect square (as noticed already in [15]). As a matter of fact, one can multiply this determinant by $1=\det_{\mathcal{M}\otimes R}(\mathbb{1}\otimes -i\sigma_2)$, to obtain

$$\det_{\mathcal{M}\otimes R}(\mathbb{1}-2i\hat{F})=\det_{\mathcal{M}\otimes R}[\mathbb{1}\otimes(-i\sigma_2)+\hat{F}^a\otimes(i\sigma_2\sigma_a)] \tag{33}$$

$$=|g|^{-2}\det_{\mathcal{M}\otimes R}[g_{\mu\nu}\otimes(-i\sigma_2)+F^a_{\mu\nu}\otimes(i\sigma_2\sigma_a)]. \tag{34}$$

It is easily seen that the matrix in the last expression is antisymmetric, so its determinant is a perfect square. This implies that the highest power of F in the expansion of $\exp[\frac{1}{2}\mathrm{tr}\log(1+2i\hat{F})]$ is 4; therefore

$$\det_{\mathcal{M}\otimes R}(\mathbb{1}+2i\hat{F})=\left[\exp\left(\frac{1}{2}\mathrm{tr}\log(1+i\hat{F})\right)\right]^2 \tag{35}$$

$$=\left\{1+\frac{1}{2}\mathrm{tr}_\otimes\left(\frac{\hat{F}^2}{2}\right)-i\frac{1}{2}\mathrm{tr}_\otimes\left(\frac{\hat{F}^3}{3}\right)-\frac{1}{2}\mathrm{tr}_\otimes\left(\frac{\hat{F}^4}{4}\right)+\frac{1}{2!}\left[\frac{1}{2}\mathrm{tr}_\otimes\left(\frac{\hat{F}^2}{2}\right)\right]^2\right\}^2 \tag{36}$$

$$=\left(1+\frac{t_2}{4}-i\frac{t_3}{6}-\frac{t_4}{8}+\frac{t_2^2}{32}\right)^2, \tag{37}$$

where $t_i=\mathrm{tr}_\otimes[(\hat{F})^i]$.

Using formula (24), we get

$$L=1-\sqrt[4]{(1+2P-Q^2)^2+(2K_3)^2}, \tag{38}$$

where

$$2P=\frac{1}{4}t_2=(F^a,F_a),$$

$$Q^2=\frac{1}{8}t_4-\frac{1}{32}t_2^2=\frac{1}{4}(F^a,\star F^b)(F^c,\star F^d)K_{acbd}, \tag{39}$$

$$K_3=-\frac{1}{12}t_3=\frac{1}{6}\epsilon_{abc}\mathrm{tr}_{\mathcal{M}}(\hat{F}^a\hat{F}^b\hat{F}^c).$$

It is also interesting to note that our Lagrangian depends exclusively on three invariants of F (the third-order invariant entering via its square), although the determinant can lead to expressions up to the eighth order in F. In this particular case, there exist many relations between the traces, so that the complicated expressions can finally be simplified and expressed as functions of three invariants only, even though there are eight for a general $SU(2)$ Lagrangian (cf. [25,26]).

IV. SPHERICALLY SYMMETRIC STATIC CONFIGURATIONS

A. The magnetic ansatz and equations of motion

Our aim now is to study static, spherically symmetric solutions of purely "magnetic" type. They are given by the so-called 't Hooft–Polyakov ansatz [27]:

$$\begin{aligned}A&=\frac{1-k(r)}{2}UdU^{-1}\quad\text{with}\quad U=e^{i\pi T_r}\\&=[1-k(r)][T_r,dT_r]\\&=[1-k(r)](T_\theta\sin\theta d\varphi-T_\varphi d\theta)\\&=\frac{1-k(r)}{r^2}(\vec{r}\wedge\vec{T})\cdot\vec{dx},\end{aligned} \tag{40}$$

where the usual notation is used. When expressed in components, the same formula becomes

$$A^a_k=\frac{[1-k(r)]}{r^2}\epsilon^a{}_{km}x^m, \tag{41}$$

where

$$a,b,c,\ldots=1,2,3;\quad i,j,k,\ldots,=1,2,3;\quad \epsilon^a{}_{km}=\epsilon^{aij}g_{ik}g_{jm}.$$

The notion of spherical symmetry for gauge potentials in Yang-Mills theory has been analyzed by Forgacs and Manton in [28]; see also [29]. The most general form for a spherically symmetric $SU(2)$ gauge potential is often called "the Witten ansatz" (cf. [30]); an exhaustive discussion of its properties can be found in [31]. When this form of potential is chosen, there remains a residual $U(1)$ symmetry preserving the field, and the situation can be interpreted as an Abelian gauge theory on two-dimensional de Sitter space, coupled to a complex scalar field w with a Higgs-like potential. Then the problem is parametrized by four real functions a_0, a_1, Re(w), and Im(w) (we use the notation introduced in [31]). Fixing the gauge enables one to set $a_1=0$. Next, one can eliminate a_0 if one restrain the solutions to the "magnetic" type only. In the static case, the remaining equations of motion possesses a first integral [due to the residual global $U(1)$ symmetry]. The condition that the energy must be finite at infinity forces it to vanish in this case. This means that we can choose the phase of the function w at will, thus reducing the form of the potential to the one proposed by 't Hooft in 1974 [27].

Therefore, the only nonvanishing components of the curvature F can be identified as the "magnetic" components of the Yang-Mills field:

$$B_i^a = \frac{1}{er^2}[\hat{r}_i \hat{r}^a (1-k^2) - rk' P_i^a], \tag{42}$$

where $\hat{r}_i = x_i / r$ and $P_i^a = \delta_i^a - \hat{r}^a \hat{r}_i$ is the projection operator onto the subspace perpendicular to the radial direction.

The only nonvanishing invariants of the field appearing in the Lagrangian density can now be expressed by means of the spherical variable r, one unknown real function $k(r)$, and its first derivative $k'(r)$:

$$2P = \frac{1}{r^4}[(1-k^2)^2 + 2(rk')^2],$$

$$K_3 = \frac{1}{r^6}[(1-k^2)(rk')^2], \tag{43}$$

$$Q^2 = 0.$$

Then the action takes on the following form:

$$S = \int \left[1 - \left\{ \left(1 + \frac{(1-k^2)^2 + 2(rk')^2}{r^4} \right)^2 + \frac{4}{r^{12}}(1-k^2)^2 (rk')^4 \right\}^{1/4} \right] r^2 dr. \tag{44}$$

For the subsequent analysis, it is very useful to change the independent variable by introducing its logarithm $\tau = \log(r)$. Then the action can be expressed as follows:

$$S = \int (1 - \sqrt[4]{A}) e^{3\tau} d\tau, \tag{45}$$

where

$$A = (1 + a^2 + 2b)^2 + 4a^2 b^2 = (1+a^2)[(1+2b)^2 + a^2],$$

$$a = (1-k^2)/r^2,$$

$$b = \dot{k}^2 / r^4.$$

Now the equation of motion can be written as

$$A_k + A_{\dot{k}} \left(\frac{3}{4} \frac{\dot{A}}{A} - 3 \right) - \frac{d}{d\tau} A_{\dot{k}} = 0, \tag{46}$$

or equivalently, in a more standard form,

$$\dot{k} = u,$$

$$\dot{u} = \gamma(k,u,\tau) u + k(k^2 - 1), \tag{47}$$

with

$$\gamma(k,u,\tau) = 1 - 2\frac{u^2 + 2uk(1-k^2) + (1-k^2)^2}{r^4 + (1-k^2)^2} + \frac{6u(1-k^2)[ku^2 + 2u(1-k^2) + k(1-k^2)^2][r^4 + 2u^2 + (1-k^2)^2]}{[r^4 + (1-k^2)^2][(r^4 + 2u^2)^2 + (1-k^2)^2(r^4 + 6u^2)]}. \tag{48}$$

The coefficient γ, which plays the role of dynamic friction, is quite similar to the one found in [7] (except for a missing factor 2, due to a printing error). In the usual Yang-Mills theory with the same ansatz, the corresponding factor is just $\gamma_{YM} = 1$.

The system (47) is not autonomous (i.e., some of the coefficients depend explicitly on the variable τ), so that the qualitative analysis of solutions should be performed in an extended three-dimensional phase space (τ, k, u) (see, for example, [32]). Of course, one cannot expect to find true singular points, because the ''time'' variable τ never stands still. Instead, one can find asymptotic behaviors of the function k whose dominant terms for $\tau \to -\infty$ $(r \to 0)$ or for $\tau \to \infty$ $(r \to \infty)$ satisfy the equations of motion up to a required order, neglecting infinitely small terms. However, for $r \to \infty$ there are two genuine fixed points $(k=1, u=0)$ and $(k=-1, u=0)$. Having found these asymptotic expansions, we then try to extend them from both sides so that they can meet and produce a regular solution valid for all values of τ.

Although our equations display asymptotic expansions analogous to those found in [33,34,7], careful analysis shows that solutions of the Bartnik-McKinnon type [8] are excluded here.

B. Asymptotic expansions

We have found two expansions in positive powers of r which satisfy the equations of motion up to a certain finite order in r near $r=0$. The first one depends on two free parameters k_0 and a, and starts with the following expressions:

$$k = k_0 + ar - k_0 \left(\frac{5a^2}{6g} + \frac{g}{12a^2} \right) r^2 + \frac{a^8(52-70g) - 9a^4 g^3 + (g-1)g^4}{108 a^5 g^2} r^3 + O(r^4), \tag{49}$$

where $g = 1 - k_0^2$, $a \neq 0$, and $g \neq 0$. This expansion displays a certain similarity with the expressions found in [33,34], which depend on the same parameter k_0.

The second one depends on only one free parameter b, and starts as follows:

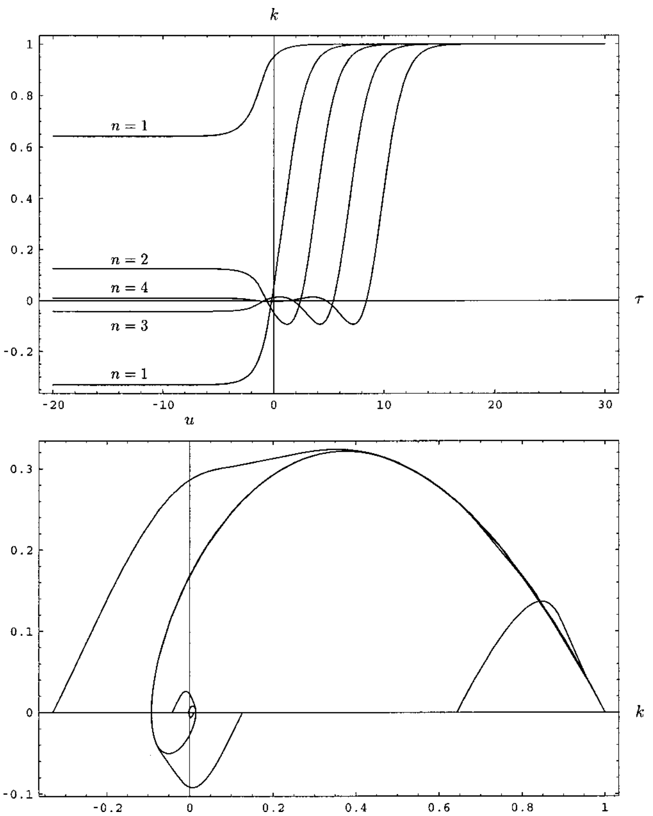

FIG. 1. Plots of solutions for the parameters $\tau_c = -3,1.2,4,7,10$.

$$k = \pm\left(1 - br^2 + \frac{3b^2 + 92b^4 + 608b^6}{10 + 200b^2 + 1600b^4} r^4 + O(r^6)\right). \tag{50}$$

Near $r = \infty$, the Taylor expansion can be made with respect to r^{-1}. It depends on one free parameter, denoted by c:

$$k = \pm\left[1 - \frac{c}{r} + \frac{3c^2}{4r^2} + O\left(\frac{1}{r^3}\right)\right]. \tag{51}$$

It is remarkable that the asymptotic behavior at $r = \infty$ is the same here [up to the order $O(r^{-7})$] as the corresponding behavior of the spherically symmetric static ansatz in the usual Yang-Mills theory, which makes it easier to interpret the characteristic integrals as magnetic charge, energy, etc.

Taking these expansions as the first approximation either at $r = 0$ or at $r = \infty$, we then use standard techniques in order to generate solutions valid everywhere. It is interesting to note that, when we started from infinity, no fine-tuning was necessary, and an arbitrarily fixed constant c would lead to a solution which, when extrapolated to $r = 0$, would define a particular pair of values of constants k_0 and a. On the contrary, starting from $r = 0$, arbitrarily chosen values of k_0 and a would not necessarily lead to good extrapolation at $r = \infty$. We shall discuss the properties of numerical solutions so obtained in the following subsection.

C. Numerical solutions

The search for numerical solutions was based on the same method as in [33,34,7]. With the expansions (49) and (51), we evaluate the initial conditions used as starting point for the numerical integration of Eq. (47).

The three parameters occurring in the asymptotic expansions (two at $r = 0$ and one at $r = \infty$) are interrelated by two constraint equalities, therefore the solutions can be labeled by only one real parameter. We chose to index the solutions with the parameter c of Eq. (51), with $c > 0$, or its logarithm $\tau_c = \log(c)$.

As in the Bartnik-McKinnon case, we can assign to each solution an integer n, with $n-1$ denoting the number of zeros of the function u or the winding number of the corresponding trajectory in the phase plane (k,u), as seen in Fig. 1, where a few solutions are plotted. When the parameter τ_c goes from $-\infty$ to $+\infty$, we observe that this integer n grows from 1 to ∞. At certain special values of the parameter τ_c, this integer increases by 1. Here are the first critical values of τ_c:

τ_c	1.658	4.781	7.510	10.092	13.218	16.530	19.813

The two graphs in Fig. 1 should be combined in order to give a correct representation of the trajectories as they appear in the extended three-dimensional phase space including the variable $\tau = \log r$. The graph on the left represents the cut k,τ, and the graph on the right represent the cut k,u, i.e., the usual phase space of the function $k(r)$ and its first derivative $u = \dot{k}$. One can see some trajectories on the plane k,u with various winding numbers.

Our solutions do not interpolate between the two singular points at $k = 1$ and $k = -1$, but between the singular point at $k = 1$ for $r = \infty$ and a certain value k_0 (related to τ_c) which is always lower than 1 and bigger than -1 (as a matter of fact $k_0 = 0$ is a solution). This is radically different from the sphaleronlike solutions or solutions of Bartnik-McKinnon type found in [8,7].

The two parameters k_0 and a of Eq. (49) are functions of the parameter τ_c. We have evaluated the energy E of the solutions and the values of the parameter k_0 for τ_c varying from -10 to 20. The energy E is represented as a function of the parameter τ_c in Fig. 2. This figure represents two enlargements of the upper graph with the precision of 10^{-2} in order to show local minima of the energy curve. The energy minima of each class of solutions are found near the critical

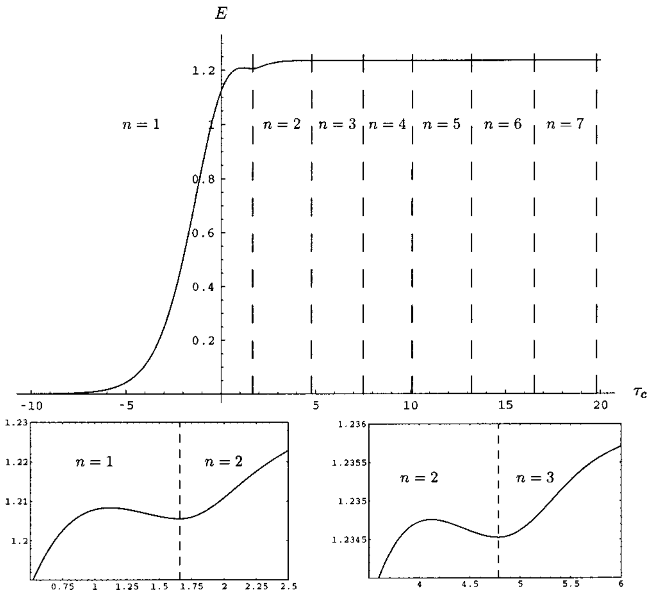

FIG. 2. Energy as function of the parameter τ_c, with local minima visible (the magnification is 100 times higher for the second minimum display).

values of the parameter τ_c, and as far as we can judge, given the precision of numerical calculus employed, coincide with their positions on the τ_c axis. Supposing that the solutions attaining local minima of energy are stable, we conjecture that these most stable solutions can be grouped in couples, with winding numbers n and $n+1$, starting with the couple $n=1, n=2$. The energies converge to the limit $E_{\tau_c=\infty} = E_{n=\infty} = 1.23605\ldots$, which coincides with the energy of the pointlike magnetic Born-Infeld monopole computed in [7].

The last two graphs in Fig. 3 show the specific features of the dependence of the parameter k_0 (the initial value of function k at $r=0$) with respect to τ_c. The dependence is smooth only between the critical values of parameter τ_c, at which the change of winding number n occurs, as can be viewed in the second graph where the second derivative of k_0 with respect to τ_c is plotted.

It is important to notice that our version of generalized non-Abelian Born-Infeld theory is quite different from the symmetrized trace prescription. Nevertheless, the nonpolynomial character of the Lagrangian, common to all generalizations, still ensures a very rich spectrum of solutions, although very different and specific to the choice of the Lagrangian. All our solutions tend to the genuine vacuum

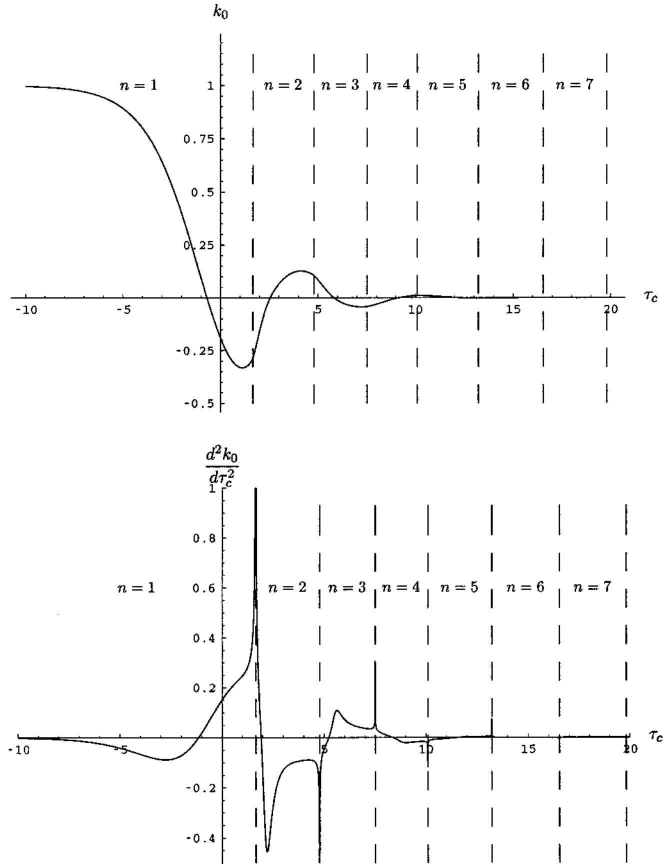

FIG. 3. k_0 as a function of τ_c, and its second derivative. The singularities of the second derivative $d^2k_0/d\tau_c^2$ occur at values of τ_c that coincide with the change of winding number n.

configuration at $r\rightarrow\infty$, but their behavior near the origin $r=0$ is very different from the sphaleronlike solutions. At the origin, our solutions look like monopole configurations whose magnetic charge has been renormalized, as suggested in [33], where the constant $1-k_0^2$ is also integrated in this manner.

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