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SELF-DUALITY IN MAXWELL–CHERN–SIMONS TYPE EFFECTIVE THEORIES

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We consider a class of (2+1)-dimensional nonlocal effective models with a Maxwell–Chern–Simons part for which the Maxwell term involves a suitable nonlocality that permits one to take into account some (3+1)-dimensional features of "real" planar systems. We show that this class of models exhibits a hidden duality symmetry stemming from the Maxwell–Chern–Simons part of the action. We discuss and illustrate this result in the framework of a (2+1)-dimensional effective model describing (massive) vortices and charges with realistic interactions.

1. Introduction

Infinite discrete symmetries and duality symmetries have received continuous attention within supersymmetric gauge theories,[1] string theories,[2] statistical systems[3] together with models describing (some of) the physics of condensed matter systems[4] such as Josephson junction arrays[5a,b] and the Quantum Hall Effect.[6a,b,c] The occurrence of a duality symmetry within a model is interesting as it can be used for example to relate the strong coupling to the weak coupling regime, to derive exact non-perturbative results and/or to obtain information on the corresponding phase diagram. For instance, the old Kramers–Wannier duality for the two-dimensional Ising model is a discrete $\mathbf{Z}_2$ map relating high-temperature to low-temperature properties and allows one to determine the critical temperature without having to solve the model explicitly. Basically, a duality symmetry occurs within a model when there exists a set of transformations (the duality transformations) in the corresponding parameter space (the space of the coupling constants) mapping the original action to an action having the same form with coupling constants transformed according to the duality transformations. Generally, partition functions for the original model and its dual counterpart formally differ only by a pre-factor which does not alter the critical properties and disappears necessarily in the computation of correlation functions.

For a wide class of condensed matter planar systems possibly subjected to an external electromagnetic field, the long distance behavior of the fluctuations of the relevant degrees of freedom (e.g. charges and/or vortices) around some ground state can be described with the help of a (2+1)-dimensional effective gauge theory in which generically each type of degree of freedom is associated to a gauge field (see e.g. Ref. 7). One of the basic ingredients underlying this description is that a conserved current in (2+1) dimensions can be represented (locally) as the curl of a (pseudo)-vector field. The resulting actions involve Maxwell and/or Chern–Simons terms for each gauge field while interaction terms, respectively minimal and magnetic Pauli type coupling of a gauge field to a current, can be expressed respectively as "mixed" Chern–Simons terms and "mixed" Maxwell terms. As an illustration and to introduce some of the ingredients involved in the main discussion of this paper, consider the following local action[a] describing the coupling of a Maxwell–Chern–Simons abelian gauge field a_μ ($f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$) to a gauge invariant conserved current J_μ:

$$S = \int_x \left(-\frac{1}{4e^2} f_{\mu\nu} f^{\mu\nu} + \frac{\theta}{2\pi} \epsilon_{\mu\nu\rho} a^\mu \partial^\nu a^\rho + \frac{\kappa}{2\pi} a_\mu J^\mu - \frac{\delta}{2} \epsilon_{\mu\nu\rho} f^{\mu\nu} J^\rho - \frac{1}{2g^2} J_\mu J^\mu \right), \tag{1a}$$

which, owing to the fact that one can write locally $J^\mu = \epsilon_{\mu\nu\rho}\partial^\nu v^\rho \equiv \frac{1}{2}\epsilon_{\mu\nu\rho} w^{\nu\rho}$ ($w_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu$) can be expressed in a more convenient form given by

$$S = \int_x \left(-\frac{1}{4e^2} f_{\mu\nu} f^{\mu\nu} + \frac{\theta}{2\pi} \epsilon_{\mu\nu\rho} a^\mu \partial^\nu a^\rho + \frac{\kappa}{2\pi} \epsilon_{\mu\nu\rho} a^\mu \partial^\nu v^\rho - \frac{\delta}{2} f_{\mu\nu} w^{\mu\nu} - \frac{1}{4g^2} w_{\mu\nu} w^{\mu\nu} \right). \tag{1b}$$

The mass dimensions of the parameters are $[e^2] = [g^2] = 1$, $[\theta] = [\kappa] = 0$, $[\delta] = -1$. In Eqs. (1a) and (1b) one easily recognizes the usual minimal coupling term (involving κ) supplemented by an additional magnetic Pauli-type coupling whose strength is given by δ. Such a non-minimal magnetic coupling has already been considered in the literature from different viewpoints (see e.g. Refs. 8a and 8b and the first and second of Ref. 5b). Physically, in a system where a_μ (resp. J_μ (or v_μ)) is related to vortices (resp. charges) moving in a plane, the κ-term describes the Lorentz force induced by the vortices on the charges or alternatively the Magnus force induced by the charges on the vortices, while the δ-term can be viewed as an intrinsic magnetic moment for the matter and the last term is a current-current interaction. Note that, upon integrating by parts the δ-term in (1a) and further defining $\mathcal{J}_\mu = J_\mu - (2\pi\delta/\kappa)\epsilon_{\mu\nu\rho}\partial^\nu J^\rho$ (still verifying $\partial_\mu \mathcal{J}^\mu = 0$), the terms coupling J_μ to a_μ in (1a) can be recasted into a minimal coupling form, namely $\int_x \kappa a_\mu \mathcal{J}^\mu$ (the current-current part involving now an additional term of the form $\epsilon_{\mu\nu\rho}\mathcal{J}^\mu\partial^\nu\mathcal{J}^\rho$ in addition to the term $\sim \mathcal{J}_\mu\mathcal{J}^\mu$). Thus, the magnetic moment coupling can be

[a]We choose $\hbar = c = 1$, $g_{\mu\nu} = \mathrm{diag}(+1,-1,-1)$, $\epsilon_{012} = +1$, $dp \equiv \frac{d^3p}{(2\pi)^3}$ and p is the momentum; $\int_{x,y,p} \equiv \int dxdydp$.

reabsorbed through the above redefinition of the current (unless one is interested in keeping explicit all the intrinsic physical properties of the J_μ (v_μ-charge) sector). Notice that keeping a magnetic moment coupling explicit may prove useful when considering some lattice version of (1a) and (1b).[b]

While effective models pertaining to the general class considered above are in general able to capture some physical properties of planar systems nicely, they do not quite describe a truly real situation. Indeed, the plan, in which for instance, charges and vortices are confined is actually embedded into a 3-dimensional space and possibly subjected to a (real) (3+1)-dimensional electromagnetic field which in particular gives rise to a $\sim 1/r$ interaction between, say, static charges and for which the coupling constant involved in the corresponding Maxwell term is dimensionless. Obviously, the class of actions considered above gives rise to a logarithmic potential between (static) charges and the Maxwell couplings are dimension-full. These actions however can be modified to take into account the (3+1)-dimensional origin of the (electromagnetic) field and/or the $1/r$ character of more realistic potentials. A way to achieve this goal is provided by the introduction of a nonlocality affecting (at least) one of the Maxwell terms in the action[c]; namely one performs the substitution:

$$-\frac{1}{4e^2} f_{\mu\nu} f^{\mu\nu} \to -\frac{1}{4e^2\sqrt{\partial^2}} f_{\mu\nu} f^{\mu\nu}\,, \tag{1c}$$

where now the coupling appearing in the RHS of (1c) is dimensionless. This procedure has already been described in the literature.[9] In the static case, the derivation leading to (1c) can be summarized as follows. One has to find essentially the solution of the three-dimensional time independant Maxwell equations for the two in-plane components of the electric field $\mathbf{E}_{\rm in}$ and the magnetic field component perpendicular to the plane B_T, assuming that charge and current densities ρ and $\mathbf{J}$ are non-vanishing only in the plane (indexed by the coordinates x and y). From this, it can be realized that the two (space)-dimensional fields $\mathbf{E}_{\rm in} = (E^x_{\rm in}, E^y_{\rm in})$ and B_T (the components of a (2+1)-dimensional field strength $f_{\mu\nu}$) obey the following equations of motion:

$$\begin{gathered}\left(\frac{1}{(\partial^2/\partial x^2+\partial^2/\partial y^2)^{1/2}}\right)\frac{\partial B_T}{\partial x} = J_y, \qquad \left(\frac{1}{(\partial^2/\partial x^2+\partial^2/\partial y^2)^{1/2}}\right)\frac{\partial B_T}{\partial y} = -J_x,\\ \frac{1}{(\partial^2/\partial x^2+\partial^2/\partial y^2)^{1/2}}\left(\frac{\partial E^x_{\rm in}}{\partial x}+\frac{\partial E^y_{\rm in}}{\partial y}\right) = \rho\end{gathered} \tag{1d}$$

(where now $J_{x(y)}$ and ρ represent the densities restricted to the plane), therefore stemming (in the static case) from the RHS of (1c). This procedure, which can

[b] The presence of this term permits one to simplify in some cases the obtaining of a Coulomb gas representation for topological excitations of the corresponding models on a cubic lattice (see e.g. first and second of Ref. 5b).

[c] A similar substitution possibly affecting the "mixed" Maxwell terms corresponding to Pauli-type interactions.

be roughly viewed as a kind of dimensional reduction, can be generalized to the non-static case and gives rise again to (1c).[9]

In Sec. 2 of this paper, we show the existence of a duality symmetry occurring within a class of (2+1)-dimensional effective models that can be obtained from the procedure described above. In Sec. 3, we summarize the main features of the duality symmetry among which some are illustrated in the context of models relevant to physical systems.

2. Duality Symmetry in a Class of Effective Actions

Owing to the discussion presented in the introduction, we consider a class of (2+1)-dimensional effective models whose corresponding action can be generically written as $S(e^2,\theta;\pi_4,\pi_6)=\int_{x,y,p} e^{ip(x-y)}\mathcal{L}$ with

$$\mathcal{L}=-\frac{1}{4e^2\sqrt{p^2}}f_{\mu\nu(x)}f^{\mu\nu}_{(y)}+\frac{\theta}{2\pi}\epsilon_{\mu\nu\rho}a^{\mu}_{(x)}\partial^{\nu}a^{\rho}_{(y)}+\pi_4(p)\epsilon_{\mu\nu\rho}a^{\mu}_{(x)}\partial^{\nu}v^{\rho}_{(y)}+\pi_6(p)w_{\mu\nu(x)}w^{\mu\nu}_{(y)}\,, \tag{2}$$

where the structure functions $\pi_i(p),\ i=4,6$ parametrize the possible modifications (including therefore a possible momentum dependence[d]) occurring in the coupling of a_μ and v_μ that are induced by the introduction of the modified Maxwell term. For the moment, we do not consider an explicit magnetic moment coupling ($\sim \pi_5(p)f_{\mu\nu(x)}w^{\mu\nu}_{(y)}$) in (2) which could be introduced in a straighforward way without altering the conclusions obtained in the following discussion. Such a coupling will be reinstalled for convenience in Sec. 3. In the following, the Maxwell–Chern–Simons (MCS) action in (2) (first two terms) will be denoted by $S_{\mathrm{MCS}}(e^2,\theta)$. We now define the complex coupling constant

$$z=-\frac{1}{e^2}+i\frac{\theta}{\pi}\,. \tag{3a}$$

$\bar{z}$ (resp. $|z|$) will denote the complex conjugate (resp. modulus) of z. We will show that the class of effective models defined by (2) exhibits a hidden duality symmetry with duality transformations acting in the parameter space defined by

$$z\to\frac{1}{\bar{z}}\,, \tag{3b}$$

$$\pi_4\to\frac{\pi_4}{|z|}\,, \tag{3c}$$

$$\pi_6\to\pi_6\,. \tag{3d}$$

We point out that this duality symmetry stems essentially from the nonlocal MCS part of the action and seems to be a general feature of the class of models defined in (2). In particular, the $1/\sqrt{p^2}$ momentum dependence in the Maxwell term of

[d]From now on, we will not explicitly write the momentum dependence of the various structure functions involved in the subsequent calculations.

(2), which reflects physically some of the (3+1)-dimensional features of the planar system, is essential to obtain the duality transformation for the complex coupling constant z defined in (3b). Furthermore, this duality symmetry has nothing to do with some "exchange duality" which may occur within (2) whenever some (*ad hoc*) suitable choices and/or relations among the structure functions are assumed for which, after performing suitable redefinitions of the fields, the resulting action could be cast into a form left unchanged upon the simultaneous exchange of redefined fields and corresponding (redefined) coupling constants. This will be illustrated in a specific example in Sec. 3.

Let us first show that (2) has the duality symmetry defined by (3a)–(3d) and that the transformation (3b) reflects rigidly the momentum dependence $\sim 1/\sqrt{p^2}$ of the Maxwell term in (2). To do this, it is convenient to start from a more general form for (2), hereafter denoted by $\tilde{S}$, which is obtained by replacing $S_{\rm MCS}(e^2,\theta)$ by $\int_{x,y,p} e^{ip(x-y)}((\pi_1/4)f_{\mu\nu(x)}f^{\mu\nu}_{(y)} + (\pi_3/2)\epsilon_{\mu\nu\rho}a^{\mu}_{(x)}\partial^{\nu}a^{\rho}_{(y)})$ where we will set $\pi_1 = -1/(e^2\sqrt{p^2})$ and $\pi_3 = \theta/\pi$ shortly. Consider now the partition function built from $\tilde{S}$ which can be written as

$$\mathcal{Z}(\pi_1,\pi_3;\pi_4,\pi_6) = \int [\mathcal{D}v][\mathcal{D}a]e^{i(\tilde{S}(\pi_i)+S_{gf}(a)+S_{gf}(v))}\,, \tag{4}$$

where $S_{gf}(a)+S_{gf}(v) = \int_x \lambda/2(\partial_\mu a^\mu)^2 + \sigma(\partial_\mu v^\mu)^2$ represents a gauge-fixing term which permits one to deal properly with each one of the two abelian gauge invariances of $\tilde{S}$ (one for a_μ and the other for v_μ). Then, we perform the Gaussian integration over the gauge field a_μ in (4). To do this, we have to invert the operator appearing in the part of the action in (4) quadratic in a_μ. This can be represented in momentum space as

$$S_{\rm MCS}(\pi_1,\pi_3) + S_{gf}(a) = \int_p \frac{1}{2}a_\mu(-p)K^{\mu\nu}(\pi_1,\pi_3;p)a_\nu(p)\,, \tag{5a}$$

with

$$K_{\mu\nu}(\pi_1,\pi_3;p) = \pi_1 p^2 T_{\mu\nu}(p) + i\pi_3\sqrt{p^2}C_{\mu\nu}(p) + \lambda p_\mu p_\nu\,, \tag{5b}$$

in which the parity conserving $T_{\mu\nu}(p)$ and parity violating $C_{\mu\nu}(p)$ are respectively defined by $T_{\mu\nu}(p) = g_{\mu\nu} - (p_\mu p_\nu/p^2)$ and $C_{\mu\nu}(p) = \epsilon_{\mu\rho\nu}(p^\rho/\sqrt{p^2})$. The inverse of the operator (5b) is easily found to be

$$K^{-1}_{\mu\nu}(\pi_1,\pi_3;p) = \frac{\pi_1}{\pi_1^2p^2-\pi_3^2}T_{\mu\nu}(p) - i\frac{\pi_3}{\sqrt{p^2}(\pi_1^2p^2-\pi_3^2)}C_{\mu\nu} + \frac{1}{\lambda}\frac{p_\mu p_\nu}{p^4}\,. \tag{6}$$

Notice that for $\pi_1 = -1/(e^2\sqrt{p^2})$ and $\pi_3 = \theta/\pi$, which is the case of interest here, the operator (6) is not defined when $1/e^4 - \theta^2/\pi^2 = 0$ since both denominators in the first two terms of (6) vanish for these values. The counterpart of this for constant $\pi_1(=1/e^2)$ and π_3 would be the appearance of a pole in the propagator for a_μ at $p^2 = \pi_3^2/\pi_1^2$ corresponding to the mass for a_μ as can be expected in local MCS theory. The singularity occurring at $1/e^4 - \theta^2/\pi^2 = 0$ will be examined at the end of this section. For the moment, we assume that $1/e^4 - \theta^2/\pi^2 \neq 0$ $(\pi_1^2p^2 - \pi_3^2 \neq 0)$.

Using (6), the integration over a_μ in (4) gives rise to

$$\mathcal{Z}(\pi_1, \pi_3; \pi_4, \pi_6) = \frac{1}{N(\pi_1, \pi_3)} \int [\mathcal{D}v] e^{i(S_{\text{eff}}(\eta_1, \eta_3; v) + S_{gf}(v))} , \tag{7}$$

where the pre-factor $N(\pi_1, \pi_3) = \text{Det}^{1/2}(K(\pi_1, \pi_3))$ and

$$S_{\text{eff}}(\eta_1, \eta_3; v) = \int_{x,y,p} e^{ip(x-y)} \left(\frac{1}{4} \eta_1 w_{\mu\nu(x)} w^{\mu\nu}_{(y)} + \frac{1}{4} \eta_3 \epsilon_{\mu\nu\rho} v^\mu_{(x)} w^{\nu\rho}_{(y)} \right) , \tag{8a}$$

and

$$\eta_1 = \frac{-4\pi_1 \pi_4^2}{\pi_1^2 p^2 - \pi_3^2} + 4\pi_6 , \tag{8b}$$

$$\eta_3 = \frac{4\pi_3 \pi_4^2}{\pi_1^2 p^2 - \pi_3^2} . \tag{8c}$$

Note that the gauge-dependant term in (6) does not contribute to $S_{\text{eff}}(\eta_i; v)$ in (7) as expected, since it is annihilated when combined with terms proportional to $\epsilon_{\mu\rho\nu} \partial^\rho v^\nu$, which reflects the fact that v_μ is associated to a conserved current ($J_\mu \sim \epsilon_{\mu\nu\rho} \partial^\nu v^\rho$). The determinant pre-factor in front of the RHS of (7) depending only on π_1 and π_3 comes out from the Gaussian integration and can be ignored as usual in the calculation of correlation functions. When $\pi_1 = -1/(e^2 \sqrt{p^2})$ and $\pi_3 = \theta/\pi$, the effective action for v_μ (8) is singular for $1/e^4 - \theta^2/\pi^2 = 0$, which reflects the corresponding singularity occurring in (6) that was mention above. A possible physical interpretation of this will be discussed at the end of this section.

Now, define the following transformations:

$$\pi_1^D = \frac{\pi_1}{\pi_1^2 p^2 + \pi_3^2} , \tag{9a}$$

$$\pi_3^D = \frac{\pi_3}{\pi_1^2 p^2 + \pi_3^2} , \tag{9b}$$

$$\pi_4^D = \frac{\pi_4}{(\pi_1^2 p^2 + \pi_3^2)^{1/2}} , \tag{9c}$$

$$\pi_6^D = \pi_6 . \tag{9d}$$

These transformations map the action involved in (4) to an action having the same form. The corresponding partition function is readily obtained by performing in (4) the substitution $\pi_1 \to \pi_i^D$, $i = 1, 3, 4, 6$ in all. Then, all the steps leading to (7) can be thoroughly reproduced simply by replacing the π_i's by their transformed counterparts. This gives rise to

$$\mathcal{Z}(\pi_1^D, \pi_3^D; \pi_4^D, \pi_6^D) = \frac{1}{N(\pi_1^D, \pi_3^D)} \int [\mathcal{D}v] e^{i(S_{\text{eff}}(\eta_1^D, \eta_3^D; v) + S_{gf}(v))} , \tag{10}$$

where the pre-factor $N(\pi_1^D, \pi_3^D) = \text{Det}^{1/2}(K(\pi_1^D, \pi_3^D))$ and $S_{\text{eff}}(\eta_1^D, \eta_3^D; v)$ has an expression similar to (8a) with

$$\eta_i^D \equiv \eta_i(\pi_1^D, \pi_3^D, \pi_4^D), \qquad i = 1, 3 , \tag{11}$$

and the η_i's are still given by (8b) and (8c). Again, the contribution of the gauge-dependant term of $K_{\mu\nu}(\pi_1^D, \pi_3^D)$ disappears from $S_{\rm eff}(\eta_i^D; v)$ as expected and the determinant pre-factor stemming from the Gaussian integration over a_μ can be safely ignored in the computation of correlation functions.

Then, combining (11) with (9a)–(9d) and (8b) and (8c), it can be easily observed that $\eta_i^D = \eta_i(\pi_1, \pi_3, \pi_4)$ $i = 1, 3$, that is, the structure functions appearing in the effective actions for v_μ are invariant under (9a)–(9d) so that both actions appearing in the exponential of (7) and (10) coincide. Assume now $\pi_1 = -1/(e^2\sqrt{p^2})$ and $\pi_3 = \theta/\pi$, $\tilde{S}$ reduces to S defined in (2) while (9a)–(9d) reduce to (3b)–(3d) upon introducing the complex coupling constant defined in (3a). This, combined with (7) and (10), keeping in mind that $\eta_i^D = \eta_i$ yields

$$\mathcal{Z}(z, \bar{z}; \pi_4, \pi_6) = \mathcal{N}\mathcal{Z}\left(\frac{1}{\bar{z}}, \frac{1}{z}; \frac{\pi_4}{|z|}, \pi_6\right), \tag{12}$$

where the pre-factor in the RHS of (12) $\mathcal{N} = \left(\frac{N(\frac{1}{\bar{z}}, \frac{1}{z})}{N(z,\bar{z})}\right)$. Note that this pre-factor is equal to unity on the "self-dual line" $z\bar{z} = 1$ since from (5b) and (3a) one has $K_{\mu\nu}(\frac{1}{\bar{z}}, \frac{1}{z}) = \frac{1}{z\bar{z}}K_{\mu\nu}(z, \bar{z})$. In short, the relation (12) shows that correlation functions calculated from $\mathcal{Z}(z, \bar{z}; \pi_4, \pi_6)$ and corresponding correlations functions for the dual objects calculated from $\mathcal{Z}(\frac{1}{\bar{z}}, \frac{1}{z}; \frac{\pi_4}{|z|}, \pi_6)$ with the duality transformations defined by (3) are the same.

The above discussion indicates that the class of effective models given in (2) has a duality symmetry defined by the transformation (3a)–(3d). We note that the duality symmetry that we have exhibited in (2) stems essentially from the nonlocal MCS action $S_{\rm MCS}$ involved in (2), as should be clear now. In particular, the momentum dependence $\sim 1/\sqrt{p^2}$ of the corresponding Maxwell term is essential to obtain (3b). This can be easily realized from a combination of (9a) and (9b) and the definition of z, (3a).

An alternative way to show that the duality symmetry is hidden in $S_{\rm MCS}$ can be obtained as follows. Consider the functional integral given by $\mathcal{Z}_0(z, \bar{z}) = \int[\mathcal{D}a]e^{i(S_{\rm MCS}(z,\bar{z})+S_{gf}(a))}$ in which the action is given by (5a) and (5b) where $\pi_1 = -1/(e^2\sqrt{p^2})$ and $\pi_3 = \theta/\pi$. Now, this action can be conveniently written in the form:

$$S_{\rm MCS} = \int dp\big(zA_+^\mu(-p)A_{+\mu}(p) + \bar{z}A_-^\mu(-p)A_{-\mu}(p)\big), \tag{13}$$

where $A_\pm^\mu(p) = (u_1 T^{\mu\nu}(p) \pm u_2 C^{\mu\nu}(p) + \gamma p^{-2}p^\mu p^\nu)a_\nu(p)$ in which the real numbers u_1 and u_2 verify $u_1 u_2 = (\sqrt{p^2})/8$, $u_1^2 - u_2^2 = (\sqrt{p^2})/4$ and $\gamma^2 = \lambda p^2/4$. Notice that the gauge-fixing term is now involved in the expressions for the $A_\pm$'s and insures that these expressions are invertible. Let us now apply a Hubbard–Stratonovitch (HS) transformation to $\mathcal{Z}_0$. The procedure is standard and is achieved by making use of the following functional relation:

$$e^{i\int dpzA^{\mu}_{+}(-p)A_{+\mu}(p)}$$
$$= \frac{1}{N_+}\int[\mathcal{D}\Lambda_+]e^{-i\int dp\frac{1}{z}\Lambda^{\mu}_{+}(-p)\Lambda_{+\mu}(p)-\Lambda^{\mu}_{+}(-p)A_{+\mu}(p)-A^{\mu}_{+}(-p)\Lambda_{+\mu}(p)}\,, \quad (14)$$

where Λ_+ is the HS field for A_+ together with a similar expression relating A_- to its HS partner deduced from (9) through the following substitution: $\Lambda_+ \to \Lambda_-$, $A_+ \to A_-$, $z \to \bar{z}$ and $N_+ \to N_-$. The $N_\pm$'s are pre-factors which disappear in the evaluation of the correlation functions.

Now, combining (13) and (14) with $\mathcal{Z}_0$ and further integrating over the gauge field a_μ, after some algebra we obtain a constraint given by

$$-\left(u_1T^{\mu\nu}+u_2C^{\mu\nu}+\gamma\frac{p^\mu p^\nu}{p^2}\right)\Lambda_{+\nu} = \left(u_1T^{\mu\nu}-u_2C^{\mu\nu}+\gamma\frac{p^\mu p^\nu}{p^2}\right)\Lambda_{-\nu} \quad (15)$$

stemming from the terms linear in a_μ appearing in the action, which gives rise to a functional δ-function upon integration over a_μ. This constraint is then found to be solved by setting

$$\Lambda^\mu_\pm = -\frac{1}{2}\left(u_2T^{\mu\nu} \pm u_1C^{\mu\nu} \pm \alpha\frac{p^\mu p^\nu}{p^2}\right)\tilde{a}_\nu\,, \quad (16)$$

where $\tilde{a}_\mu$ is some vector field and the last term, insuring that (16) is invertible provided the real parameter α is nonzero, will give rise to a gauge-fixing contribution in the action expressed in terms of $\tilde{a}_\mu$. Since (15) is solved by (16) for any non-vanishing α, it is convenient to set $\alpha^2 = \lambda p^2/4$ in the following. Then, using (16), the functional integral $\mathcal{Z}_0$ can be re-expressed as[e]

$$\mathcal{Z}_0(z,\bar{z}) = \frac{1}{\mathcal{N}}\int[\mathcal{D}\tilde{a}]e^{i(S^D_{\mathrm{MCS}}(z,\bar{z})+S_{gf}(\tilde{a}))}\,, \quad (17\text{a})$$

with

$$S^D_{\mathrm{MCS}}(z,\bar{z})+S_{gf}(\tilde{a}) = \frac{1}{2}\int_p\sqrt{p^2}\tilde{a}_\mu(-p)\left(-\frac{1}{e^2|z|^2}T^{\mu\nu}+i\frac{\theta}{\pi|z|^2}C^{\mu\nu}+\frac{\lambda}{\sqrt{p^2}}p^\mu p^\nu\right)\tilde{a}_\nu(p), \quad (17\text{b})$$

from which, by inspection of the first two terms in (17b), it is easy to realize that $S^D_{\mathrm{MCS}}(z,\bar{z}) = S_{\mathrm{MCS}}(\frac{1}{\bar{z}},\frac{1}{z})$ while the last term in (17b) is a gauge-fixing term similar to the one involved in the action (5a) and (5b). This finally leads to $\mathcal{Z}_0(z,\bar{z}) = \frac{1}{\mathcal{N}}\mathcal{Z}_0(\frac{1}{\bar{z}},\frac{1}{z})$ showing that the duality symmetry defined by (3a)–(3d) is hidden in S_{MCS}.

We conclude this section with some remarks. First, the above analysis can be extended in a straightforward way to the case where (2) is augmented by an explicit magnetic moment coupling ($= \pi_5(p)f_{\mu\nu(x)}w^{\mu\nu}_{(y)}$). Such a term will be re-inserted for convenience in Sec. 3. Then, the duality symmetry still holds with duality transformations (3) augmented with $\pi^D_5 = \pi^D_5/|z|$ while the structure functions involved in

[e]Pre-factors coming from (14) are absorbed into the overall factor $1/\mathcal{N}$.

the effective action for v_μ become

$$\eta_1' = \eta_1(8b) + \frac{4\pi_5}{\pi_1^2 p^2 - \pi_3^2}(2\pi_3\pi_4 - p^2\pi_1\pi_5)\,, \tag{18b}$$

$$\eta_3' = \eta_3(8c) + \frac{4p^2\pi_5}{\pi_1^2 p^2 - \pi_3^2}(\pi_3\pi_5 - 2\pi_1\pi_4)\,. \tag{18c}$$

Next, consider any effective model with action given by $\tilde{S}$. If, in some limit (e.g. long wavelength limit), the structure functions for at least one Maxwell and one Chern–Simons part behave respectively as $\sim 1/\sqrt{p^2}$ and $\sim$ constant, then one can expect that a duality symmetry of the type discussed in this paper becomes valid in that limit.

Finally, let us consider more closely the singularity occurring in the operator $K^{-1}_{\mu\nu}$ given in (6) when $\pi_1 = -1/(e^2\sqrt{p^2})$ and $\pi_3 = \theta/\pi$. For these values of the coupling constants, the denominator of (6) given by $\pi_1^2 p^2 - \pi_3^2$ reduces to $1/e^4 - \theta^2/\pi^2$ and therefore vanishes for any z_0 (3a) satisfying

$$\frac{1}{e^4} - \frac{\theta^2}{\pi^2} = 0 \tag{19}$$

giving rise to an effective action for v_μ (8) becoming singular at that z_0. For a *local* MCS action (π_1 and π_3 both constant), one would have obtained instead a pole in the propagator for a_μ (6) at $p^2 = \pi_3^2/\pi_1^2$ corrsponding to the mass for a_μ. In the present case, the appearance of singular couplings satisfying (19) merely reflects the momentum dependence for $\pi_1(\sim 1/\sqrt{p^2})$ (and $\pi_3 \sim$ constant). But this does not correspond to singularity in the duality tranformation (3) which remains well defined even when (19) is satisfied. A tentative physical interpretation of (19) may be obtained by computing the electromagnetic response for the a_μ obtained by coupling $S_{\mathrm{MCS}}(e^2,\theta)$ to an external field A_μ, namely $S_{\mathrm{MCS}}(e^2,\theta) \to S_{\mathrm{MCS}}(e^2,\theta) + a_\mu\epsilon^{\mu\nu\rho}\partial_\nu A_\rho$, then integrating over a_μ and making use of (6). From the response function in the long wavelength limit, the conductivities σ_{xy} and σ_{xx} are easily found to be $\sigma_{xy} \sim 1/[e^2(\frac{1}{e^4} - \frac{\theta^2}{\pi^2})]$ and $\sigma_{xx} \sim \theta/(\pi(\frac{1}{e^4} - \frac{\theta^2}{\pi^2}))$, therefore describing a resistive state. When (19) is satisfied, both conductivities become infinite so that (19) would be interpretable within this framework as corresponding to a superconducting state.

3. Discussion and Conclusion

Let us summarize the analysis presented above. We have shown the existence of a hidden duality symmetry defined by (3) which occurs within a class of (2+1)-dimensional effective models given by (2) aiming to incorporate some (3+1)-dimensional features of "real" planar systems. The appearance of this duality symmetry stems from the nonlocal MCS action in (2) which is obtained through (1c). The duality transformation can be put into a simple form through the introduction of a complex coupling constant z built from the Maxwell and Chern–Simons coupling constants. The duality transformation appears to be a $\mathbf{Z}_2$ duality of Kramers–Wannier type acting on a complex parameter. As should be clear from (9a), (9b) and (3a), the z transformation reflects the momentum dependence of the Maxwell term involved in the action.

As far as other (2+1)-dimensional models are concerned, we note that a somehow similar $\mathbf{Z}_2$ duality has been exhibited within the $\mathbf{Z_N}$ abelian Higgs model with Chern–Simons term and bare mass term (on the lattice).[10] There, the relevant complex parameter is found to be $\zeta = \theta/2\pi + i(2\pi m/Ng)$ ($\zeta \to -iz$ and $\theta \to 2\theta$ in our conventions) where θ still denotes the Chern–Simons coupling, g is the (dimension-full) Maxwell coupling and m is the bare mass. The corresponding duality transformation is again found to be[10] $\zeta \to 1/\bar{\zeta}$ and leaves the partition function for the model invariant, up to an inessential (field independant) pre-factor as in (12). It has been pointed out in Ref. 10 that an additional approximate (periodicity) symmetry given by $\zeta \to \zeta + 1$ appears in the limit of weak coupling and large bare mass. In this limit however, this additional transformation combined with $\zeta \to 1/\bar{\zeta}$ (which still holds) does not generate, as the full (discrete) symmetry group, the modular group $SL(2, Z)$ which is indeed generated by supplementing these two transformations with $\zeta \to -\bar{\zeta}$, that is, a time reversal symmetry, as indicated in Ref. 10. Within the class of effective models we have considered, the duality symmetry is a $\mathbf{Z}_2$ symmetry in the absence of some additional periodicity symmetry. The appearence of such an additional symmetry (if for instance the physics of the system would remain unchanged upon $\theta \to \theta + k2\pi, k \in Z$) would again not be sufficient to give rise to $SL(2, Z)$ as the full discrete symmetry group relevant for the model. This would further require additional time reversal invariance as is the case for the abelian Higgs model mentioned above.

A recent interesting paper[11] has examined the role and the properties of some duality symmetry which may occur in effective models, possibly involving MCS parts, which are expected to be relevant for the description of planar systems such as for example thin films (possibly exhibiting conductor-insulator transitions) and Quantum Hall systems. The analysis has focussed in particular on particle-vortex duality[11] which may occur in these effective models. It has been shown that particle-vortex duality supplemented with periodicity symmetry takes a simple form when expressed in terms of the low energy electromagnetic response functions. The assumptions underlying this analysis is that response functions are dominated by the motion of quasi-particles or vortices and that the dynamics of quasi-particles and vortices are similar (which is expected to be valid in any system clean enough so that particle-vortex duality becomes relevant). For conductors in particular, the non-commutativity of particle-vortex duality and periodicity symmetry gives rise basically to an infinite discrete group of duality relations amongst the response functions. For femionic charge carriers, this group (acting on the complex conductivity) appears to be a particular subgroup of the modular group whose potential interest for the Quantum Hall Effect has been extensively analyzed (see fourth, fifth and sixth of Ref. 6c and references therein). For bosonic charge carriers, another subgroup of the modular group is obtained, whose physical implications have been analyzed in Ref. 11. A complete comparison of the particle-vortex duality considered in Ref. 11 with the one we discussed here is beyond the scope of the present paper but here, we point out that there is an essential difference between these two dualities. This can be realized by considering the class of effective actions obtained in Ref. 11 after integrating out the quasi-particles and vortices coordinates. The particle-vortex duality stems from the exchange of the two sectors (quasi-particle and vortices) and is valid basically provided the effective action is symmetric under exchange of quasi-particle and vortex degrees of freedom and parameters (apart from possibly a Chern–Simons term responsible for flux attachment). In other words, using the notations from Ref. 11, if P (resp. V) denotes the complex response function stemming from the integration of quasi-particle (resp. vortex) coordinates, one must have $P \leftrightarrow V$ for particle-vortex duality to hold. The occurrence of the duality symmetry we have considered does not require the effective action to be symmetric under exchange, but instead that the effective action involves at least one nonlocal MCS part for one sector with momentum dependence of the corresponding structure functions similar to the one for $S_{\rm MCS}(e^2, \theta)$. For instance, using again the notations of Ref. 11, such a duality is expected to occur whenever P or V corresponds to an MCS part with the suitable momentum dependence, even if the effective action is not exchange symmetric.

Up to now, the discussion has been kept general. In the rest of this section, we present, as a first illustration, a specific model pertaining to the general class (2), which can be obtained by a suitable application of (1c) to (1b). It describes a system of charges and massive vortices in which charge-charge interaction behaves realisticaly (as $\sim 1/r$). The model involves the above duality symmetry. Furthermore, it gives rise on the "self-dual line" $|z| = 1$ to a relation between the resistivities which mimics the one derived in Ref. 13 in the framework of Cooper pairs and vortex dynamics in Josephson junction arrays or superconductor-insulator transition in thin films. First, let us start from (1b), assuming furthermore that a_μ (resp. v_μ) is related to vortices (resp. charges) and $\delta = \theta/(\kappa g^2)$. The resulting action describes a planar system of interacting charges and vortices with logarithmic type charge-charge and vortex-vortex potentials. The effective action for the vortices, obtained after integrating over v_μ, describes massive excitations with mass M ($M^2 = \kappa^2 g^2 e'^2/4\pi^2$, and $1/e'^2 = 1/e^2 - \theta^2/\kappa^2 g^2 \geq 0$). In the charge sector, the electromagnetic response tells us that the transverse conductivity $\sigma_{xy} = \theta(4\pi/\kappa^2)$ while the longitudinal conductivity vanishes, as it can be realized by adapting the general expressions for the response functions (14a) and (14b) to the present situation and taking their long wavelength limit. It can be further shown[12] that the cubic lattice Euclidean version of this model (with $f_{\mu\nu}$, $w_{\mu\nu} \to 0$ at the infinity) supplemented with integer-valued new link variables ensuring the periodicity of the "mixed" Chern–Simons term coincides with a model describing the zero temperature physics of a Josephson junction array in the limit of vanishing capacitance to the ground, provided the following identifications hold: $e'^2 = 2\kappa E_c$, $g^2 = 4\pi^2 E_J/\kappa$, $\kappa = 2$ (for Cooper pairs), where E_c (resp. E_J) is the capacitance (resp. Josephson junction) energy.

We now modify (1b), using as a guideline the arguments underlying (18c), to obtain an action for which (static) charges (e.g. Cooper pairs) interact through a $\sim 1/r$ potential *and* which still describes massive excitations for the vortices. This latter requirement can be obtained by introducing an additional dimensionfull coupling constant g' ($[g'^2] = 1$). The former requirement is motivated by the fact that "real" charges (Cooper pairs) actually interact as $1/r$. A possible relevant candidate is given by the following action:

$$S = \int_{x,y,p} e^{ip(x-y)} \left(-\frac{1}{4}\left(\frac{1}{e^2\sqrt{p^2}} + \frac{1}{g'^2}\right) f_{\mu\nu(x)} f^{\mu\nu}_{(y)} + \frac{\theta}{2\pi}\epsilon_{\mu\nu\rho} a^{\mu}_{(x)} \partial^{\nu} a^{\rho}_{(y)} \right. \tag{20}$$

$$\left. + \frac{\kappa}{2\pi}\epsilon_{\mu\nu\rho} a^{\mu}_{(x)} \partial^{\nu} v^{\rho}_{(y)} - \frac{\theta}{2\sqrt{p^2} g^2 \kappa} f_{\mu\nu(x)} w^{\mu\nu}_{(y)} - \frac{1}{4g^2\sqrt{p^2}} w_{\mu\nu(x)} w^{\mu\nu}_{(y)} \right), \tag{21}$$

where now all the parameters (except g') are dimensionless. From the general discussion presented in Sec. 2, it is easy to see that (20) has a duality symmetry stemming from the nonlocal MCS terms in (20). It is defined by $z \to 1/\bar{z}$, $\kappa \to \kappa|z|^{-1}$, $g'^2 \to |z|^2 g'^2$ where $z = -1/e^2 + i(\theta/\pi)$ still holds. No other duality symmetry is present in this action. When $g'^2 \to \infty$, the action (20) involves an additional duality symmetry which can be exhibited upon redefining $c_\mu = v_\mu + (\theta/\kappa)a_\mu$ in (20). It is defined by $e'^2 \to g^2$ (where $1/e'^2 = 1/e^2 - \theta^2/\kappa^2 g^2$) and the exchange of fields $c_\mu \to a_\mu$. This provides an example of an "exchange duality symmetry" which was already mentioned in the introduction of this paper. It has obviously nothing to do with the former duality.

The effective action for the vortices obtained from (20) is given by

$$S^V_{\text{eff}} = \int_{x,y,p} e^{ip(x-y)} \left(\frac{\gamma(z)}{4\sqrt{p^2}} - \frac{1}{4g'^2} \right) f_{\mu\nu}(x) f^{\mu\nu}(y)\,, \tag{21}$$

with $\gamma(z) = -1/e'^2 + \kappa^2 g^2/4\pi^2$ and may be identified with a (longitudinal) conductivity for the vortices sector. The computation of the electromagnetic response for the charge system shows that the corresponding longitudinal and transverse conductivity σ_{xx} and σ_{xy} are both non-vanishing so that (20) describes a resistive state. Using (3a), one finds that the corresponding resistivities ρ_{xx} and ρ_{xy} satisfy the following relation:

$$\rho^2_{xx} + \rho^2_{xy} = \frac{1}{|z|^2}\left(\frac{16}{g^4}\right)\gamma^2(z)\,, \tag{22a}$$

which reduces *at the self-dual points* $|z| = 1$ to

$$\rho^2_{xx} + \rho^2_{xy}|_{|z|=1} = \left(\frac{16}{g^4}\right)\gamma^2(1)\,. \tag{22b}$$

This latter relation bears some similarity to the one derived in Ref. 13 in the framework of Cooper pairs and vortices dynamics in Josephson junction arrays or of superconductor-insulator transition in thin films. While this observation suggests that the set of self-dual points $|z| = 1$ may actually separate two different regimes for the present system of charges and vortices, a full determination of the status of this self-dual set would require an investigation of the occurrence of phase transitions within the model. In this respect, a lattice description may provide a more convenient representation than a continuum one. To conclude, we notice that a euclidean lattice version of (20) which still has the duality symmetry exhibited in this paper can be constructed. For the sake of simplicity, we assume $g'^2 \to \infty$. The lattice action can be written as

$$S_L = \sum_{\{x\}} \left(\frac{L^3}{2e'^2\sqrt{-\nabla^2}} f_\mu^2 - i\frac{L^3\kappa}{2\pi} a_\mu K_{\mu\nu} c_\nu \right.$$
$$\left. + \frac{L^3}{2g^2\sqrt{-\nabla^2}} \left(w_\mu + \frac{\theta}{\kappa} f_\mu \right)^2 + iL\sqrt{\theta} a_\mu W_\mu + iL\frac{\kappa}{\sqrt{\theta}} c_\mu M_\mu \right), \tag{23}$$

defined on a cubic lattice with spacing L. In (23), $\{x\}$ denotes lattice summation, the gauge fields are associated with links (x, μ) between the sites x and $x + L\hat{\mu}$ ($\hat{\mu}$ is a unit vector in the μ direction), $c_\mu = v_\mu + \frac{\theta}{\kappa} a_\mu$ as defined above, $K_{\mu\nu}$ is the lattice Chern–Simons operator,[14] f_μ and w_μ are the lattice dual field strength for a_μ and v_μ, W_μ and M_μ are integer-valued link variables (the excitations) ensuring the periodicity of the mixed Chern–Simons term.[5b] The corresponding partition function $\mathcal{Z} = \sum_{M,W\in Z} \int_{-\infty}^{+\infty} [\mathcal{D}a][\mathcal{D}c] e^{-S_L}$ can then be cast into the factorized form $\mathcal{Z} = \mathcal{Z}_0 \mathcal{Z}_{ex}$ with

$$\mathcal{Z}_0 = \int_{-\infty}^{+\infty} [\mathcal{D}a][\mathcal{D}c] e^{-S_L(W=M=0)}\,, \tag{24a}$$

$$\mathcal{Z}_{ex} = \sum_{W,M\in Z} e^{-S_{ex}} \tag{24b}$$

and

$$S_{ex} = \sum_{\{x\}} -\frac{g^2\kappa^2}{2\theta e'^2\gamma L} M_\mu \frac{\delta_{\mu\nu}}{\sqrt{-\nabla^2}} M_\nu - \frac{\theta}{2\gamma L} W_\mu \frac{\delta_{\mu\nu}}{\sqrt{-\nabla^2}} W_\nu + i\frac{g^2\kappa^2}{2\pi\gamma L} M_\mu \frac{K_{\mu\nu}}{\nabla^2} W_\nu \tag{24c}$$

and can be easily found to be invariant under the duality symmetry defined by (3).

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