J. Phys. A: Math. Gen. **33** (2000) 39–55. Printed in the UK PII: S0305-4470(00)05337-3

$\Gamma(2)$ modular symmetry, renormalization group flow and the quantum hall effect

Yvon Georgelin†, Thierry Masson‡ and Jean-Christophe Wallet†

† Groupe de Physique Théorique, Institut de Physique Nucléaire, F-91406 Orsay Cedex, France
‡ Laboratoire de Physique Théorique (UMR 8627), Université de Paris-Sud, Bât. 211, F-91405 Orsay Cedex, France

Received 21 June 1999, in final form 14 October 1999

Abstract. We construct a family of holomorphic β-functions whose renormalization group (RG) flow preserves the $\Gamma(2)$ modular symmetry and reproduces the observed stability of the Hall plateaus. The semicircle law relating the longitudinal and Hall conductivities that has been experimentally observed is obtained from the integration of the RG equations for any permitted transition which can be identified from the selection rules encoded in the flow diagram. The generic scale dependence of the conductivities is found to agree qualitatively with the present experimental data. The existence of a crossing point occurring in the crossover of the permitted transitions is discussed.

1. Introduction

The quantum Hall effect (QHE) is a remarkable phenomenon occurring in a two-dimensional electron gas in a strong magnetic field at low temperature [1]. Since the discovery of the quantized integer [2] and fractional [3] Hall conductivity, the QHE has been the subject of an intensive field of theoretical and experimental investigation. The pioneering theoretical contributions [4] analysing the basic features of the hierarchy of the Hall plateaus have triggered numerous works aiming to provide a better understanding of the underlying properties governing the complicated phase diagram associated with the quantum Hall regime, together with the precise nature of the various observed transitions between plateaus and/or focusing on a characterization of a suitable theory.

It has been realized for some time that modular symmetries may well be of interest to aid the deeper understanding of the salient properties of the QHE. For instance, the superuniversality proposed in [5] to explain the apparent similarity of the observed transitions is reminiscent of the modular tranformations. In addition, it has been shown that some properties of the phase diagram may well be explained in terms of modular group transformations in a two-parameter scaling theory. At the present time, a fully satisfactory microscopic or effective theory for the QHE, from which the relevant modular symmetry (if any) would emerge, is still lacking. In some ways, this has motivated studies focused on the derivation of general constraints on the phase diagram (and/or expressions for the conductivities) coming from the full modular group [6] or some of its subgroups [7]. Indeed, it is well known that the existence of a discrete symmetry group acting on the parameter space of a theory induces restrictions on the renormalization group (RG) flow. This has been pointed out [6] in the case of the full modular group which, in that context, can be viewed as a rich extension of the old Kramers–Wannier

Z_2 duality of the two-dimensional Ising model. This interesting aspect has been applied in various areas of physics, such as statistical systems [11], extended sine–Gordon theories [12], as well as in the non-perturbative analysis of $N = 2$ supersymmetric Yang–Mills theory [13].

In this paper, we construct a family of holomorphic β-functions which reproduces the observed stability of the Hall plateaus and whose corresponding RG flow in the conductivity plane (i.e. the parameter space) preserves a $\Gamma(2)$ symmetry acting on it. At this point two comments are in order. The first one concerns the reasons motivating the choice of $\Gamma(2)$. Although the full modular group has a natural action on the longitudinal and Hall conductivities (as originally noticed in the first paper of [6]), the corresponding symmetry is too large (see e.g. [8]), basically because it maps filling factors $p/q \rightarrow p'/q'$ with q odd and q' even which contradicts the experimental observation that only odd denominators are observed in the QHE. Therefore one has to look for other subgroups of the modular group that may be suitable candidates for a relevant symmetry group of the phase structure of the QHE. In addition to $\Gamma(2)$, to date two (level-two) subgroups have been studied in some detail. The first one, known as $\Gamma_V(2)$ in the mathematical literature, has been proposed in [7] as being of possible relevance for the QHE. This subgroup has interesting features. However, a RG flow preserving $\Gamma_V(2)$ would have critical points (corresponding to the fixed points of $\Gamma_V(2)$) that cannot be reconciled with the present experimental situation. Therefore, $\Gamma_V(2)$ appears to be unsuitable, at least in the present context. The second subgroup, called $\Gamma_0(2)$, has been considered extensively in [10] (see, also, [21] and references therein) and used to constrain (under a physically plausible hypothesis) a possible β-function ruling the RG flow of the conductivities for quantum Hall systems. We first note that at the present time no similar analysis has been performed for $\Gamma(2)$. Since there is presently no experimental evidence definitely favouring either of these two subgroups, it is worth examining the $\Gamma(2)$ case, in particular, from the viewpoint of its physical implication on the RG flow and to compare (some of) the corresponding results with those obtained from $\Gamma_0(2)$. Phenomenologically, when this is confronted by further experimental results it may be helpful to operate a selection of the relevant symmetry (or will permit one to rule out these two modular subgroups). Moreover, we notice that the classification of the Hall states stemming from $\Gamma_0(2)$ does not distinguish the even-*numerator* Hall states from the odd ones, whereas the corresponding classification based on $\Gamma(2)$ does [9]. Experimentally, there have been recent indications that even- and odd-*numerator* states may well behave differently, as reported in some studies on magnetization performed in Hall samples [14], which may therefore indicate that a suitable symmetry for the QHE cannot treat even- and odd-numerator states on an equal footing. As a final remark, justifying *a posteriori* our choice of $\Gamma(2)$, in this paper we will show that one can construct a β-function giving rise to a RG flow preserving $\Gamma(2)$ which reproduces the present experimental observations, in particular, the recently observed 'semicircle' law [15] relating the longitudinal and Hall conductivities with the qualitative shape of the crossover for the various transitions.

One important hypothesis underlying the present construction is the holomorphy of the β-function to which the second comment is devoted. Obviously, the final determination of the holomorphic or non-holomorphic character of β would require the construction of a complete microscopic (or effective) theory for the QHE which is still lacking. Note that a holomorphic β is inconsistent with the σ-model framework [22] proposed in the context of the QHE (see [1] for a review). This latter framework is appealing but is strictly valid in a weak coupling regime and needs to be extrapolated to the strong coupling regime in order to be really relevant to the QHE, which is a crude extrapolation whose validity is still unclear at the present time. If one does not insist on constructing a β-function whose asymptotic form reproduces some of the gross features of the results derived in [22], then a holomorphic ansatz for β remains an open possibility whose physical consequences must be confronted by experimental results. One may expect that holomorphy would be implied by some specific properties of an underlying theory for the QHE. This may be suggested, for example, in a recent work (see the third paper of [6]) where a class of lattice models attempting to reproduce some features of the conductivities in Hall systems has been considered. In particular, it has been pointed out that holomorphy is related to models where time reversal is broken. In this paper, we will show that holomorphy requirement for the β-function whose RG flow commutes with $\Gamma(2)$ permits one to construct an ansatz which reproduces most of the experimental results for the QHE.

The paper is organized as follows. Section 2 is devoted to the construction of a family of physically acceptable holomorphic β-functions. In section 3, we discuss the corresponding physical consequences. We also compare our results with those obtained in a recent work dealing with the construction of a β-function based on $\Gamma_0(2)$ [10]. In section 4, we collect the main results of this paper and present our conclusions.

2. Construction of the β-functions

2.1. Basic properties of $\Gamma(2)$

The properties of the modular group $\Gamma(1)(\equiv\ PSL(2, Z))$ and its various subgroups can be found in [16]. In this section we collect all the relevant ingredients that we will use in the subsequent analysis. Let $\mathcal{P}$ and z denote, respectively, the open upper-half complex plane and a complex coordinate on $\mathcal{P}$ ($\mathrm{Im}\, z > 0$). One defines $\bar{\mathcal{P}} \equiv \mathcal{P} \cup Q$, where Q is the set of rational numbers. We first recall that the group $\Gamma(2)$ is the set of transformations G acting on $\bar{\mathcal{P}}$ defined by

$$G(z) = \frac{az+b}{cz+d} \qquad a, b, c, d \in \mathbb{Z} \quad (a, d) \text{ odd} \quad \text{and} \quad (b, c) \text{ even} \tag{2.1a}$$

$$ad - bc = 1 \quad \text{(unimodularity condition)}. \tag{2.1b}$$

$\Gamma(2)(\subset \Gamma(1))$ is the free group generated by

$$T^2(z) = z + 2 \qquad \Sigma(z) = ST^{-2}S(z) = \frac{z}{2z+1} \tag{2.2}$$

where $T(z) = z+1$ and $S(z) = -\frac{1}{z}$ are the two generators of the modular group $\Gamma(1)$, which is known in the mathematical literature as the principal congruence unimodular group at level 2. The corresponding principal fundamental domain $\mathcal{D}_{\Gamma(2)}$, depicted in figure 1, has three cusps denoted by [0], [1] $\simeq$ [-1] and [i∞] that are respectively identified with the three points 0, 1 and i∞ of $\bar{\mathcal{P}}$ which are the only fixed points of $\Gamma(2)$ on $\mathcal{D}_{\Gamma(2)}$. The whole set of fixed points of $\Gamma(2)$ on $\bar{\mathcal{P}}$ is obtained in the usual manner by successive $\Gamma(2)$ transformations of these three points†. Notice the identification of the frontiers on $\mathcal{D}_{\Gamma(2)}$ as indicated in figure 1.

It has been pointed out recently in [8, 17] that $\Gamma(2)$ can be used to derive a model for a classification of fractional (as well as integer) Hall states. This classification, which refines the Jain one [18] and involves a kind of generalization of the 'law of the corresponding states' derived in [5], seems to successfully reproduce the observed hierarchical structure of the Hall states. The salient feature of the proposed construction is that each family of quantum fluid states indexed by fractions with odd denominators (plus the insulator state(s)) is generated from a metallic state labelled by an even-denominator fraction through specific $\Gamma(2)$ transformations. To be more precise, we first rewrite the transformations (2.1a), (2.1b) as

$$G(z) = \frac{(2s+1)z + 2n}{2rz + (2k+1)} \tag{2.3a}$$

$$(2s+1)(2k+1) - 4rn = 1 \tag{2.3b}$$

† Observe that $\Gamma(2)$ has only real fixed points.

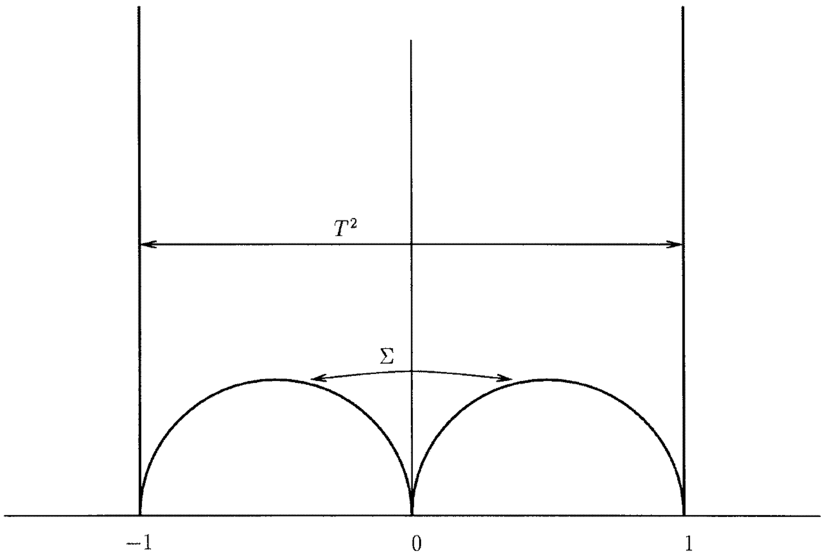

Figure 1. The principal fundamental domain of $\Gamma(2)$. The arrows indicate the frontiers which have to be identified.

where $k, n, r, s \in \mathbb{Z}$. We identify for the moment z with a filling factor $\nu = p/q$ and select a given Hall metallic state labelled by $\lambda = \frac{(2s+1)}{2r}$ $(r \geqslant 0, s \geqslant 0)$. Then, as shown in [17], one obtains a hierarchy of Hall (liquid) states surrounding the metallic state λ from the images $G^{\lambda}_{n,k}(0)$ and $G^{\lambda}_{n,k}(1)$ of 0 and 1, by the family of transformations $G^{\lambda}_{n,k} \in \Gamma(2)$ (n and k satisfying (2.3*b*)) which sends $z = \mathrm{i}\infty$ onto λ. As an example, the double-Jain family of states surrounding the metallic state $\lambda = \frac{1}{2}$

$$\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \ldots, \frac{N}{2N+1} \tag{2.4a}$$

$$\frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \ldots, \frac{N}{2N-1} \tag{2.4b}$$

can be easily recovered in this scheme from the images $G^{1/2}_{n,k}(0)$ and $G^{1/2}_{n,k}(1)$ with, respectively, $n \geqslant 0$ for (2.4*a*) and $n < 0$ for (2.4*b*). We recall that this construction separates the even-numerator Hall fractions from the odd numerator ones so that it may be possible to take into account a possible particle–hole symmetry within the present scheme. Other families surrounding any state indexed by an even-denominator (metallic state) can be constructed in the same way so that all the experimentally observed Hall states can be taken into account in the present construction. It has been further shown in [9] that the corresponding predicted global organization of the various Hall conductivity states stemming from the action of $\Gamma(2)$ fits quite well with (some of) the present experimental data.

The possible importance of the role played by modular symmetries in the QHE has been considered for some time. Some of the related works have emphasized that (most of) the main features of the (up to now) experimentally observed phase structure of QHE seem to be recovered from the action of a suitable subgroup of the modular group on the complex conductivity plane: hereafter denoted by $\bar{\mathcal{P}}$ and parametrized by $z = \frac{h}{e^2}(\sigma_{xy} + \mathrm{i}\sigma_{xx})$† (with $\mathrm{Im}\, z \equiv \sigma_{xx} \geqslant 0$). In particular, the group $\Gamma_0(2)$ has been the one considered most by some

† In the following, $e^2 = h = 1$.

authors, and was used recently in [10] to constrain the β-function governing the RG flow of the conductivities for quantum Hall systems. The corresponding studies have been performed under various physically acceptable sets of hypotheses. In the next section, we will consider somewhat similar hypotheses to study the restrictions which can be obtained from $\Gamma(2)$ on the β-function of the RG flow for the conductivities. We therefore assume that the action of $\Gamma(2)$ on real filling factors, $\nu = p/q$, which has been described above can be extended to an action on $\bar{\mathcal{P}}$.

2.2. Holomorphic β-functions from $\Gamma(2)$ symmetry

Let t be a scale parameter. We first recall that scale transformations on the complex conductivity plane $\bar{\mathcal{P}}$ generate a RG flow $z \to R(t; z, \bar{z}) \equiv z(t)$, from which the β-function is defined to be the (contravariant) vector-field tangent to this flow, namely

$$\beta(z, \bar{z}) = \frac{\mathrm{d}R(t; z, \bar{z})}{\mathrm{d}t} = \frac{\mathrm{d}z(t)}{\mathrm{d}t}. \tag{2.5}$$

It is well known that the existence of a discrete symmetry group acting on the parameter space of a theory may induce restrictions on the RG flow and, in turn, provides some non-perturbative information on the RG flow (basically stemming from the reasonable ansätze for the corresponding β-functions). As mentioned in the introduction, this aspect has already been investigated in various areas of physics. Most of the considerations involved in these investigations can be adapted to the present situation for which we now outline the main steps of the analysis.

First of all, the crucial mathematical hypothesis is that the action of $\Gamma(2)$ commutes with the RG flow, which basically means that if the $\Gamma(2)$ symmetry of the parameter space (which in the present case is the conductivity plane) holds at a given scale, it will be preserved by the RG downwards to lower scales. This hypothesis, in particular, determines the $\Gamma(2)$ transformations of the β-function, given by

$$\beta(G(z), \overline{G(z)}) = (cz + d)^{-2}\beta(z, \bar{z}) \tag{2.6}$$

for any $G \in \Gamma(2)$, and may account for the apparent observed superuniversality in the quantum Hall transitions [5]†.

Equation (2.6) indicates that β transforms as a modular form of $\Gamma(2)$ with weight -2, whenever β is *holomorphic in* z on $\mathcal{P}$. In the latter case, the application of general results stemming from complex analysis already permits one to strongly constraint the possible expression for an admissible β-function. In the rest of this section, we will therefore assume that β is holomorphic in z. This hypothesis will be commented upon again at the beginning of section 3.

Now, a general theorem on modular forms [16] states that any modular form $\omega(z)$ of any subgroup Γ (of finite index) of the modular group with even weight k can be represented on $\mathcal{D}_\Gamma$, the fundamental domain of Γ, as

$$\omega(z) = (\lambda'(z))^{k/2} R(\lambda) \tag{2.7}$$

where $\lambda(z)$ is a modular function of Γ‡ defined on $\mathcal{D}_\Gamma$, $\lambda'(z) = \frac{\mathrm{d}\lambda(z)}{\mathrm{d}z}$ and $R(\lambda)$ is a rational function in λ. In the present case, $k = -2$ and λ, the modular function of $\Gamma(2)$, can be chosen as

$$\lambda = \frac{\theta_2^4}{\theta_3^4} \tag{2.8}$$

† It is easy to show that distinct critical points of the RG flow related by a $\Gamma(2)$ transformation will have the same scaling exponents.

‡ That is, a function invariant under the action of Γ.

and satisfies on $\mathcal{D}_{\Gamma(2)}$

$$\lambda(\mathrm{i}\infty) = 0 \qquad \lambda(0) = 1 \qquad \lambda(1) = \infty. \tag{2.9}$$

In (2.8), the Jacobi θ functions θ_2 and θ_3 (together with θ_4 given here for further convenience) are defined by

$$\theta_2 = 2\sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2} = 2q^{\frac{1}{4}}\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n})^2 \tag{2.10a}$$

$$\theta_3 = \sum_{n=-\infty}^{\infty} q^{n^2} = \prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1})^2 \tag{2.10b}$$

$$\theta_4 = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \prod_{n=1}^{\infty}(1-q^{2n})(1-q^{2n-1})^2 \tag{2.10c}$$

where $q = \exp(\mathrm{i}\pi z)$.

Therefore, consistency with the previous two hypotheses requires the general expression for the β-function on $\mathcal{D}_{\Gamma(2)}$ to be $\beta(z) = \lambda'(z)^{-1}R(\lambda)$ with λ given in (2.8), which can then be straightforwardly extended to $\mathcal{P}$ by the action of $\Gamma(2)$. We now have to expose this general expression to experiment.

The strongest experimental constraint comes from the observed stability of the Hall plateaus labelled by integer as well as odd-denominator fractional filling factors, which must presumably correspond to attractive stable fixed points of the β-function. To apply this constraint, we proceed as follows. First, observe that in the present framework, the fixed points of $\Gamma(2)$ must be derived by construction of the critical points of the β-function. On $\mathcal{D}_{\Gamma(2)}$, the only fixed points of $\Gamma(2)$ are 0, 1 and $\mathrm{i}\infty$ which can be respectively identified with the (Hall) insulator state, the first Landau level and some (unobserved) superconducting state. Next, observe that in the classification of the Hall states based on the $\Gamma(2)$ symmetry [8, 17], the Hall plateaus correspond to the images of 0 and 1 by $\Gamma(2)$ as stated in section 2.1 (the images of $\mathrm{i}\infty$ correspond to even-denominator (metallic) Hall states). From these observations and under the further assumption that $\beta(z)$ has no other critical points than those given by the fixed points of $\Gamma(2)$, it is easy to realize that $\beta(z)$ can be conveniently parametrized on $\mathcal{D}_{\Gamma(2)}$ as

$$\beta(z) = \frac{\alpha}{\lambda'(z)}\lambda^p(z)(\lambda(z)-1)^q \tag{2.11}$$

where α is a complex constant, λ is still given by (2.8), $p, q \in \mathbb{Z}$ and use has been made of (2.9).

Now, according to the previous discussion, equation (2.11) must have zeros at $z = 0$ and $z = 1$. This is realized provided that

$$q \geqslant 1 \tag{2.12a}$$

for $\beta(z = 0) = 0$ and

$$p + q - 1 \leqslant 0 \tag{2.12b}$$

for $\beta(z = 1) = 0$. These constraints can be easily derived by combining (2.11) and (2.9) with the explicit expression for $\lambda'(z)$ given by $\lambda'(z) = \mathrm{i}\pi\lambda(z)\theta_4^4(z)$, obtained from (2.8) and the functional relation

$$\frac{\theta_2'}{\theta_2} - \frac{\theta_3'}{\theta_3} = \frac{\mathrm{i}\pi}{4}\theta_4^4 \tag{2.13}$$

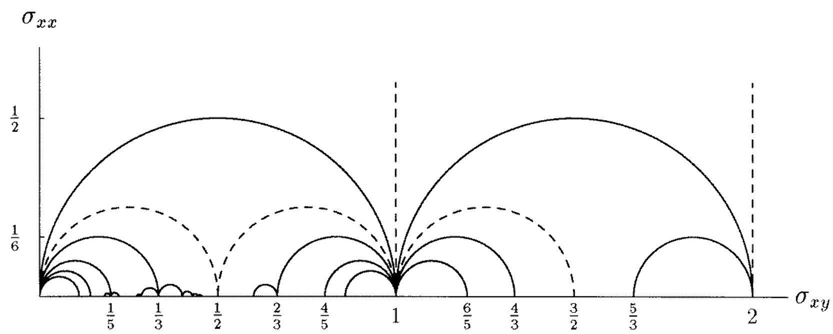

Figure 2. Some semicircle flow lines for the β-function corresponding to $\alpha = -1$, $p = 0$ and $q = 1$. Solid curves correspond to plateau–plateau or plateau–insulator transitions. Dashed curves indicate unstable transitions.

and making use of the following asymptotic expansions for the Jacobi θ functions:

$$\theta_2(z) \sim \sqrt{\frac{\mathrm{i}}{z}} \qquad \theta_3(z) \sim \sqrt{\frac{\mathrm{i}}{z}} \qquad \theta_4(z) \sim \sqrt{\frac{\mathrm{i}}{z}} \exp\left(-\frac{\mathrm{i}\pi}{4z}\right) \qquad \text{for} \quad z \sim 0 \tag{2.14a}$$

$$\theta_2(z) \sim \sqrt{\frac{\mathrm{i}}{z-1}} \qquad \theta_3(z) \sim \sqrt{\frac{\mathrm{i}}{z-1}} \exp\left(\frac{-\mathrm{i}\pi}{4(z-1)}\right) \qquad \theta_4(z) \sim \sqrt{\frac{\mathrm{i}}{z-1}} \qquad \text{for} \quad z \sim 1. \tag{2.14b}$$

From the combination of (2.11) with (2.12a) and (2.12b), one can easily deduce that $\beta(z)$ must be singular for $z = \mathrm{i}\infty$, using, for instance,

$$\theta_2(z) \sim \exp\left(\frac{\mathrm{i}\pi z}{4}\right) \qquad \theta_3(z) \sim 1 \qquad \theta_4(z) \sim 1 \qquad \text{for} \quad z \to \mathrm{i}\infty \tag{2.15}$$

a result that can be expected from general results on holomorphic functions. Finally, we find that the complex constant α appearing in (2.11) must be chosen to have a non-zero real part in order to obtain a flow from the β-function with the desired experimental properties. Moreover, further requiring that the flow reproduces the experimentally observed semicircle law [15] forces α to be real whereas any non-zero imaginary part for α would produce some overall deformation of the flow lines (the flow diagrams in both cases have the same topology). From now on, we will assume that α is real. This is illustrated in figures 2 and 3 where the flow of (2.11) with $q = 1$, $p = 0$ and $\alpha = -1$ is represented.

3. Discussion

Let us first summarize what has been derived up to this point. Putting (2.8), (2.11), (2.12) together with $\lambda'(z) = \mathrm{i}\pi\lambda(z)\theta_4^4(z)$, we finally obtain

$$\beta = \mathrm{i}^{-1}\frac{\alpha}{\pi}\frac{\theta_2^{4(p-1)}\theta_4^{4(q-1)}}{\theta_3^{4(p+q-1)}} \qquad p, q \in Z \quad q \geqslant 1 \quad p+q-1 \leqslant 0 \quad \alpha < 0 \tag{3.1}$$

which is defined on $\mathcal{D}_{\Gamma(2)}$ and can be straightforwardly extended to $\bar{\mathcal{P}}$. This is the main result of the previous section. Equation (3.1) represents a physically admissible family of *holomorphic* β-functions reproducing, in particular, the experimentally observed stability of the Hall plateaus and whose corresponding RG flow in the complex conductivity plane (i.e. the parameter space) preserves a $\Gamma(2)$ symmetry acting on it.

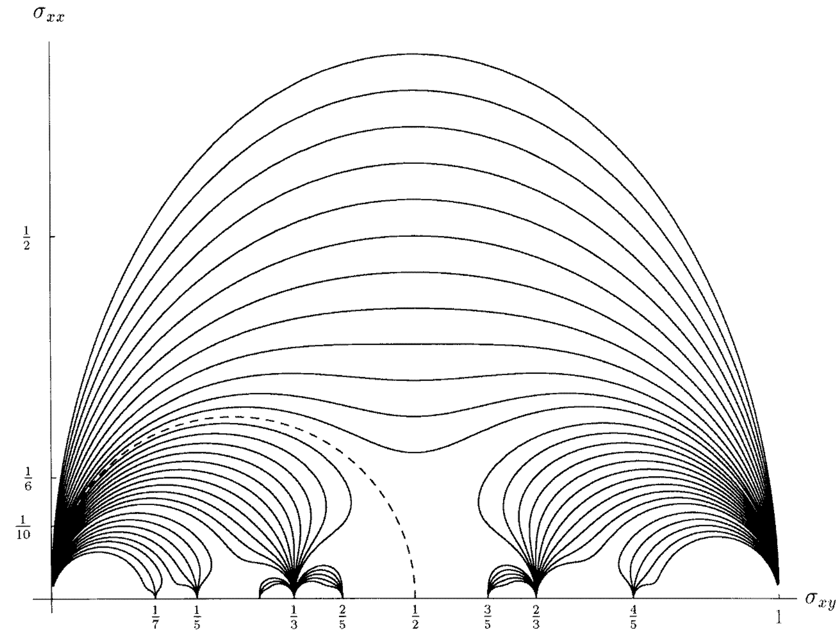

Figure 3. Some general flow lines for the β-function corresponding to $\alpha = -1$, $p = 0$ and $q = 1$. Solid curves correspond to plateau–plateau or plateau–insulator transitions. Dashed curves indicate unstable transitions.

In this section we will show that the physical predictions that can be extracted from (3.1) are in good agreement with the present experimental observations. Namely, we will show that the 'semicircle' law which has been recently observed [15] can be recovered from the behaviour of σ_{xy} and σ_{xx} obtained from the integration of (3.1). We will also show that the crossover in the plateau–plateau and plateau–insulator (observed) transitions described by (3.1) agrees qualitatively with the present experimental observations.

Before starting the discussion, some important remarks concerning the holomorphy hypothesis as well as the already proposed candidates for β-functions are in order. As is well known, the holomorphy constraint is a very strong one which severely restricts the possible expression for β. Relaxing this constraint permits one to have much more freedom for the construction of physically admissible β-functions. For a recent comprehensive analysis of the non-holomorphic case, see, for example, [21] and references therein. Here, we notice that the non-holomorphic β which were proposed in [21] can obviously be related through their asymptotic (large σ_{xx}) behaviour to the β-functions stemming from the dilute-instanton gas calculations performed in the framework of nonlinear sigma models [22], whereas the family of holomorphic β-functions given in (3.1) cannot be. In particular, the corresponding asymptotic (large σ_{xx}) behaviour is different and, indeed, cannot be finite as it is for those non-holomorphic β. Imposing finitude at large σ_{xx} for (3.1) would produce necessarily unstable Hall plateaus, in clear contradiction with the experimental observations. Therefore, one of the characteristic features of (3.1) is the existence of a singularity as $\sigma_{xx} \to \infty$ and, consequently, at any even-denominator rational point on the real axis in the conductivity plane. Anticipating the discussion, the physical consequence on the corresponding flow diagram is the existence of unstable paths connecting even-denominator (metallic) Hall states to odd-denominator (or insulator) states which might be associated with 'unfavoured' (but nevertheless observable) transitions.

Let us now turn to already proposed candidates for the β-function and RG flows. Recall that the early developments on this subject were essentially based on a field theory of the type found in the nonlinear sigma model [22] (see also [23]) mentioned previously, with two (dimensionless) coupling constants identified with the longitudinal (σ_{xx}) and Hall (σ_{xy}) conductivities of the disordered electron gas. Starting from this framework, a RG flow diagram for the conductivities has been conjectured [23] whose characteristic feature is the existence of fixed points occurring at some σ_{xx} and σ_{xy} equal to half-integer values. Although this proposal is appealing and seems to capture some experimental features of the (mainly integer) QHE, it is plagued by a problem. Indeed, the postulated fixed points (if they exist as fixed points of the nonlinear sigma model [22], a fact which is not clear at the present time) correspond to a small σ_{xx} (strong coupling) regime whereas the dilute-instanton gas calculation that gives rise to the non-holomorphic β-function underlying the conjectured RG flow is only valid in the large σ_{xx} (weak coupling) regime. Whether this crude extrapolation from the weak to the strong coupling regime is finally correct or not is unclear at the present time. Keeping in mind the above remarks and that, as far as we know, there are no experimental facts favouring either holomorphic or non-holomorphic β, one can reasonably regard the family of holomorphic β (3.1) based on $\Gamma(2)$ as possible candidates for a description of aspects of the physics of the QHE effect.

We conclude these remarks by noting that similar constructions of holomorphic β-functions based on larger subgroups of the full modular group have been performed recently, most of them focusing on the subgroup $\Gamma_0(2)$. Recall that $\Gamma_0(2)$ is generated by $T(z) = z + 1$ and $\Sigma(z) = z/(2z + 1)$, and that one has $\Gamma(2) \subset \Gamma_0(2)$, i.e. $\Gamma(2)$ is a subgroup of $\Gamma_0(2)$. It has been shown [10] that a $\Gamma_0(2)$-based construction gives rise to a qualitative behaviour of the crossover in the various transitions that is similar to the one obtained from our $\Gamma(2)$-based construction (in both cases, it is assumed that the only critical points of the β-function are fixed points of the corresponding subgroup of the modular group). There are, however, specific differences appearing in the corresponding flows. In particular, one of the fixed point of $\Gamma_0(2)$ in its fundamental domain has to be identified naturally with the crossing point appearing in the plateau–insulator transitions, whose position is therefore entirely ('rigidly') determined in the conductivity plane. Recall that it corresponds to $\sigma_{xy} = \frac{1}{2}$ and $\sigma_{xx} = \frac{1}{2}$ ($z = \frac{1+\mathrm{i}}{2}$) for the $0 \to 1$ transition. Moreover, the proposed β-function has a pole at this point. In this case, the unique path in the conductivity plane connecting $z = 0$ to $z = 1$, which must obviously go through the pole $z = \frac{1+i}{2}$, appears to be unstable (and, indeed, can evolve from either 0 toward ∞ or to 0 toward $\frac{1}{2}$), a fact that can be easily verified numerically by plotting the flow generated by the corresponding β. Further investigations are needed to clarify the experimental and theoretical status of this predicted crossing point. In the $\Gamma(2)$ case, the situation is quite different. Indeed, the point $z = \frac{1+\mathrm{i}}{2}$ does not play a distinguished role simply because it is not a fixed point of $\Gamma(2)$. Therefore, according to the assumption we made that the only poles or zeros of β are fixed points of $\Gamma(2)$, $z = \frac{1+\mathrm{i}}{2}$ cannot be either a zero or a pole of the β-function given by (2.11) (or (3.1)). If this assumption is relaxed, one could, of course, construct a β-function which can have poles or zeros anywhere in the fundamental domain in addition to the fixed points of $\Gamma(2)$. This is the situation we will consider at the end of this section. In particular, we will show that this permits one to incorporate, as a zero of a slightly modified β-function, a crossing point deviating from the suggested value $z = \frac{1+\mathrm{i}}{2}$, a deviation which seems to have been observed recently [20]. For the moment, we will critically examine (3.1) and show that it encodes most of the observed features of the QHE.

First, notice that β-functions defined by (3.1) and such that $p + 2q - 2 = 0$ holds, generate a flow approaching $z = 0$ and $z = 1$ in the same manner. This can be easily seen by combining (3.1) with (2.14*a*), (2.14*b*) and studying the behaviour of β in the vicinity of $z = 0$ and $z = 1$. In particular, it can be straightforwardly realized that the fastest approach of 0 and 1 is obtained when $q = 1$ and $p = 0$ which corresponds to asymptotic expressions for β that do not involve exponential factors. From now on we will focus on this latter situation.

Now, we point out that the RG equation can be formally integrated along the single trajectories in the parameter space, leading to an algebraic relation between σ_{xy} and σ_{xx} which reproduces the 'semicircle' law that has been experimentally observed, at least at low temperature, in the study of the plateau–insulator and plateau–plateau transitions [15]. Indeed, combining (2.5) with (2.11) (in which now $p = 0$, $q = 1$, and we take $\alpha = -1$†), one obtains

$$\mathrm{d}t = -\frac{\mathrm{d}\lambda}{\lambda - 1} \tag{3.2}$$

where λ is still given by (2.8). The integration of (3.2) gives

$$t = -\log(\lambda - 1) + \chi \tag{3.3}$$

where χ is a complex constant and log denotes a determination of the complex logarithm. It follows, for any determination of the logarithm, that

$$\lambda = 1 + \exp(\chi - t) \tag{3.4}$$

and one has $\lambda \to 1$(resp. $+\infty$) for $t \to +\infty$ (resp. $-\infty$).

Let us assume χ is real so that λ is real for any t. Then, it is a general result that the inversion of the map $z(t) \mapsto \lambda(z(t)) = 1 + \exp(\chi - t)$, with λ given by (2.8), gives rise to a curve $t \mapsto z(t)$ which is a semicircle linking two rational numbers as end-points on the real axis, together with a point z_0 located on this semicircle and satisfying $\lambda(z_0) = 1 + \exp\chi$. The inversion of this map does not result in a unique semicircle because if $t \mapsto z(t)$ is a solution then, for any $G \in \Gamma(2)$, $t \mapsto G(z(t))$ is also a solution (whose end-points on the real axis are therefore the images by $G \in \Gamma(2)$ of the end-points of the initial solution). However, it can be easily realized that the choice of such a solution from among all those possible is completely equivalent to the choice of z_0 such that $\lambda(z_0) = 1 + \exp\chi$, which can be regarded as the initial conditions for the RG equation.

To illustrate the previous discussion, a representative example of the resulting flow is depicted in figure 2 where only semicircle line flows are considered. The plateau–insulator transition $0 \to 1$ corresponds to the (full curve) large semicircle linking 0 and 1. As it should now be clear, the action of successive $\Gamma(2)$ transformations on this semicircle (which may be viewed as a template for the transitions) gives rise to the other (full curve) semicircles appearing in this figure. For instance, the following transformation $G(z) = \frac{3z-2}{2z-1} \in \Gamma(2)$, maps the semicircle for the $0 \to 1$ transition into the semicircle connecting 1 and 2, which therefore corresponds to a direct transition between the Hall plateaus with integer filling factors $\nu = 1$ and $\nu = 2$, whereas $G(z) = \frac{z}{2z+1} \in \Gamma(2)$ maps the $0 \to 1$ semicircle into the one associated with the transitions $0 \to \frac{1}{3}$. Therefore, if the present framework is correct, it is expected that such a semicircle law should be experimentally observed for all other permitted transitions. This is already the case for the observed $0 \to \frac{1}{3}$, $0 \to 1$ and $1 \to 2$ transitions that have been recently studied in experiment, e.g., some Si MOSFET devices in the quantum Hall regime [20, 25], for which the data on σ_{xx} and σ_{xy} fit well with the semicircle law expected from the present $\Gamma(2)$-based framework. Note that selection rules for the permitted direct plateau–plateau and plateau–insulator transitions can obviously be extracted from the flow diagram depicted in figure 2 by simply observing that each permitted transition is rigidly linked with a semicircle whose end-points on the real axis correspond to the filling factors labelling the transition.

† Other real value for α corresponds to simple rescaling of t so that the ensuing analysis is not modified.

The above analysis can be easily adapted to the case where χ is a complex number. In the latter situation, the trajectories $t \mapsto z(t)$ (each of which still connects two rational numbers on the real axis) are no longer semicircles, as depicted in figure 3.

As stated at the beginning of this section, there are *a priori* possible transitions of the type insulator$\rightarrow$ even-denominator (metallic) state as well as odd-denominator $\rightarrow$ metallic state, namely $0 \rightarrow \frac{1}{2}$, $1 \rightarrow \frac{1}{2}$ and the corresponding images by $\Gamma(2)$. Some representative examples of these transitions are indicated in figures 2 and 3 by dashed curves†. The associated trajectories are, in fact, unstable as any small deviation from a given semicircle will give rise to a quite different transition. For instance, it can be easily deduced from figure 3 that a small perturbation of the flow line associated with the $0 \rightarrow \frac{1}{2}$ transition will give rise to either the $0 \rightarrow 1$ or $0 \rightarrow \frac{1}{3}$ transition, whereas any plateau–plateau as well as plateau–insulator transition is stable against perturbation.

The explicit expressions for σ_{xx} and σ_{xy} as functions of t, which are expected to provide the qualitative behaviour of the crossover for a transition, can be easily extracted from (3.2)–(3.4). The method is standard and is essentially similar to the one used in [10]. It is interesting to plot the resulting expression numerically. This is done in figure 4 (resp. figure 6) for the $1 \rightarrow 0$ (resp. $1 \rightarrow 2$) transition, whereas figure 5 and 7 represent the t-dependance of the corresponding resistivities. It can be seen that the predicted behaviour of the conductivities (as well as the resistivities) is in good qualitative agreement with the experimental one reported, in particular, in [19, 20]. Note that the 'almost-linear' asymptotic shape of $\rho_{xx}(t)$ for $t > 0$ that we obtain fits well with a recent experimental measurement of the corresponding quantity for the $1 \rightarrow 0$ transition [20]. Note, also, that somewhat similar results for the crossovers have been obtained in [10] where the symmetry $\Gamma_0(2)$ is considered instead of $\Gamma(2)$.

At this point of the discussion, let us summarize the main hypotheses underlying our analysis. Assuming that: (i) $\Gamma(2)$ commutes with the flow, (ii) the β-function is holomorphic and (iii) β has no other critical points than the fixed points of $\Gamma(2)$, we have obtained an ansatz for β successfully reproducing the main experimental features of the QHE. In particular, the observed Hall plateaus are associated with the fixed points 0 and 1 of $\Gamma(2)$ and their corresponding images; furthermore, the qualitative behaviour of the crossover for the various transitions is found to be in good agreement with the present experimental data.

However, the $\Gamma(2)$ symmetry does not provide a direct constraint on the position of the crossing point appearing in any transition, since none of the fixed points of $\Gamma(2)$ (which manifest themselves as (critical) fixed points of the flow) can be a candidate for such a crossing point in the present framework. Presumably, such a point should appear as a zero of a β-function. A possible phenomenological way to take into account the occurence of a crossing point is to relax hypothesis (iii). Thus, one may assume that β given in (2.7) has another zero at some point, z_c, located on the semicircle for the $0 \rightarrow 1$ transition in order to reproduce the semicircle law that is observed experimentally. At the present time, the situation concerning the crossing point is not quite clear. As far as the value of the conductivity tensor at the transition is concerned, there is little agreement between experiment and theory. While some theoretical arguments seem to favour $\sigma_{xy} = \frac{1}{2}$ and $\sigma_{xx} = \frac{1}{2}$, that is, a crossing point located at the uppermost point on the semicircle for the $0 \rightarrow 1$ transition, the experimental situation is not so clear [24]. In particular, some deviation from the uppermost location seems to have been observed recently in the experiment performed in [20]. Notice that in this experiment the corresponding candidates for the crossing points in the $0 \rightarrow 1$ and $0 \rightarrow \frac{1}{3}$ transitions can be related to each other (within, say, $\sim$10%) by a $\Gamma(2)$ transformation. Besides, it is believed that

† Notice that there also exist, as a consequence of the present construction, transitions of the type $0 \rightarrow \infty$ or $1 \rightarrow \infty$ and the corresponding $\Gamma(2)$ images. The state $\sigma_{xx} = \infty$ would represent some (yet unobserved) superconducting state.

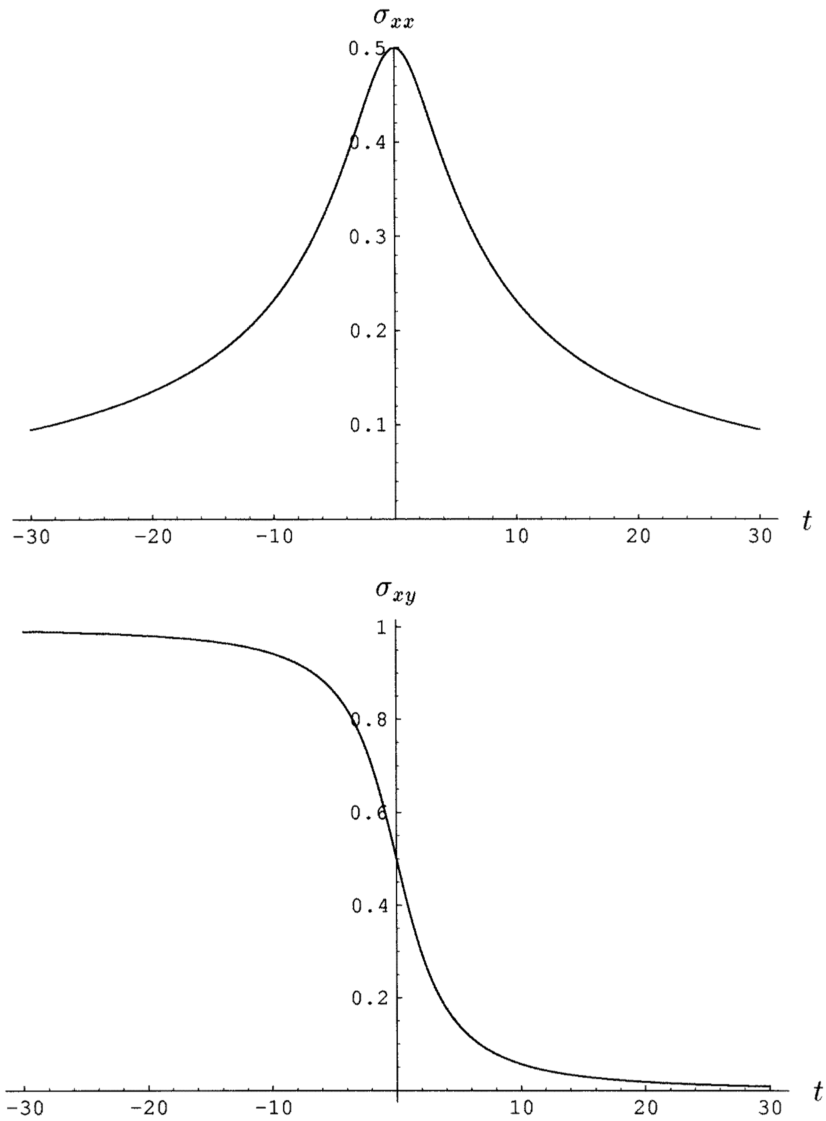

Figure 4. σ_{xx} and σ_{xy} versus scale parameter t for the transition $1 \to 0$.

the crossing point is associated with a continuous quantum phase transition [26]. Taking the above discussion into account, a construction similar to the one presented in section 2 gives

$$\beta(z) = \frac{\alpha}{\lambda'(z)}\lambda(z)^p(\lambda(z)-1)^q(\lambda(z)-\lambda(z_c))^r \tag{3.5}$$

where $p, q \in \mathbb{Z}$, $q \geqslant 1$, $r \geqslant 1$, $p+q+r-1 \leqslant 0$ and λ still given by (2.8). The resulting flow has been studied numerically for $p=-1$, $q=r=1$†, $\alpha=-1$ and $z_c = \frac{1+\mathrm{i}}{2}$, and is found to have a structure quite similar to the one depicted in figure 2‡. The numerical integration of the RG equations gives rise to the t-dependance of the conductivities which is similar to the

† Corresponding to the fastest approach to the fixed points by the flow.
‡ Any other value for z_c on the semicircle would not modify the structure of the flow.

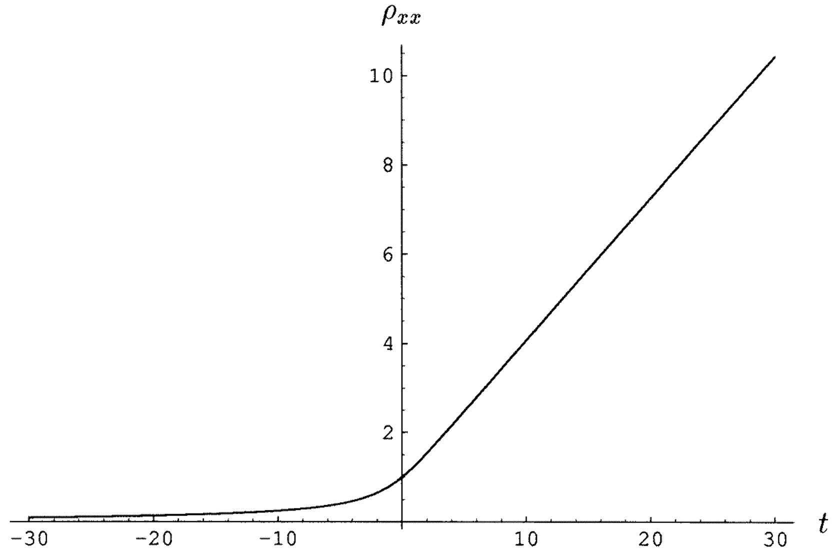

Figure 5. ρ_{xx} versus scale parameter t for the transition $1 \to 0$.

one obtained previously. Note that one has some freedom in the choice of the location of z_c on the semicircle which may now be fitted with some further experimental measurement. In this respect, the $\Gamma_0(2)$ symmetry is much more restrictive since one of its fixed points, $z = \frac{1+i}{2}$, may be identified with the crossing point for the $0 \to 1$ transition, as proposed in [10], which is, therefore, rigidly fixed in the conductivity plane. Obviously, a crossing point experimentally found to show definite deviation from the uppermost location on the semicircle would cause problems for $\Gamma_0(2)$ symmetry.

Although the above method of incorporating the existence of a crossing point into the present framework is the most direct one, there exists an alternative possibility which preserves hypothesis (iii), and allows one to make a closer contact with experiment. This may be done through the introduction of an additional hypothesis on the scale parameter t (assuming again the validity of hypotheses (i)–(iii)), which is motivated by a two-parameter scaling description [24]. One therefore assumes that t can be parametrized as

$$t = f(T, B) = f\left(\frac{1}{T^{\gamma}}(B - B_c)^{\delta}\right) \tag{3.6}$$

where f is a regular monotonic function, T is the temperature and B_c is a critical value for the magnetic field B. In (3.6), the exponents γ and δ will be discussed in a while. Some explicit expression for f in terms of T and B should definitely come from a specific model. Observe now, that for $B = B_c$, $t_c \equiv f(0)$ is independent of the temperature as well as $z_c \equiv z(t_c)$, where $t \mapsto z(t)$ is any solution of the RG equations stemming from (3.1) that have been discussed previously. Any other point of the curve $B \mapsto z(f(T, B))$ actually depends on T. Then, it is obvious to deduce that $B \mapsto \mathrm{Re}\, z(f(T, B)) = \sigma_{xy}(T, B)$ and $B \mapsto \mathrm{Im}\, z(f(T, B)) = \sigma_{xx}(T, B)$ define two families of curves parametrized by T, each one having a fixed (T-independent) point at $B = B_c$ (basically, all the curves intersect at this point when $B = B_c$). For the $0 \to 1$ transition, the above curves have a shape similar to the ones for the plots depicted in figure 4, thanks to the monotonic character of f. Then, it can be easily realized that these families of curves are in good qualitative agreement with the experimental plots (see e.g. [20]). The point z_c can then be naturally identified with the crossing point.

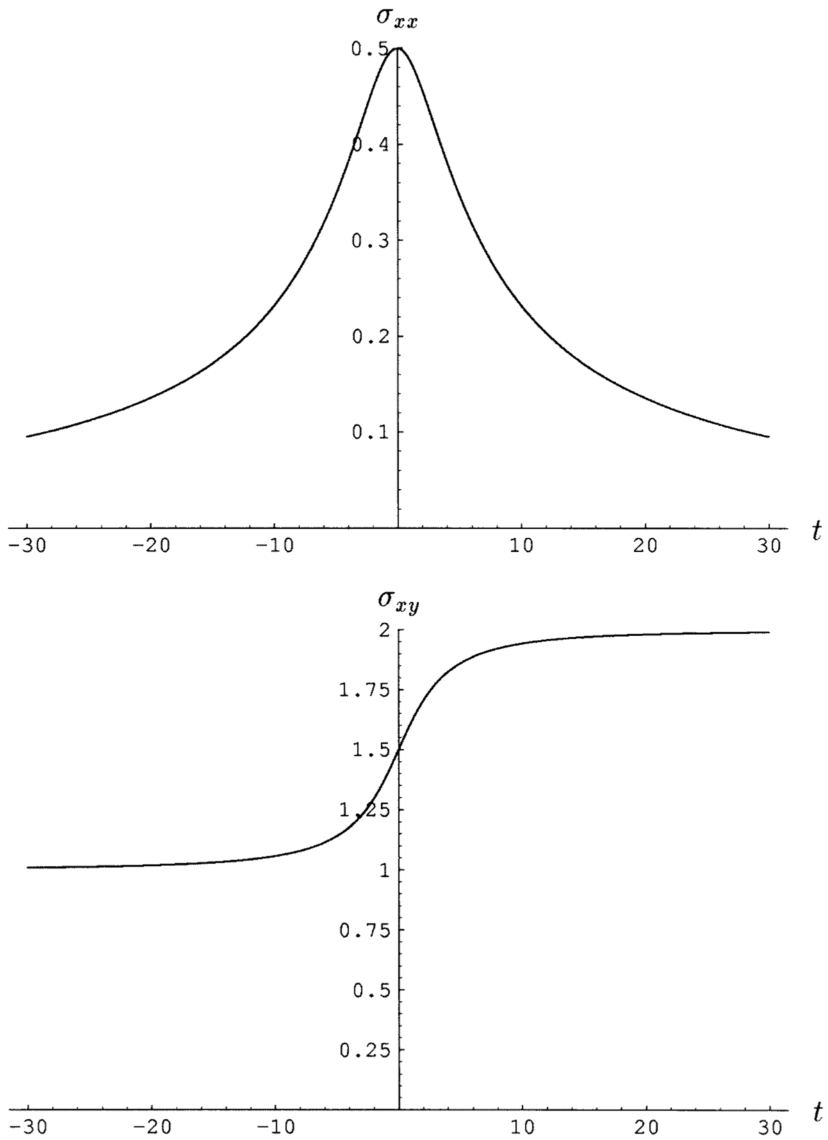

Figure 6. σ_{xx} and σ_{xy} versus scale parameter t for the transition $1 \to 2$.

Furthermore, notice that for $T \neq 0$ one has

$$\frac{\partial z}{\partial B} = \frac{\delta}{T^\gamma}(B - B_c)^{\delta-1} f'(T^{-\gamma}(B - B_c)^\delta) z'(t) \tag{3.7}$$

which can be derived using (3.6). Experimentally (see e.g. [15]), it is observed that $\frac{\mathrm{d}\sigma_{xy}}{\mathrm{d}B}$ and/or $\frac{\mathrm{d}\sigma_{xx}}{\mathrm{d}B}$ do not vanish at $B = B_c$, so that consistency requires $\frac{\partial z}{\partial B}$ to be non-zero at $B = B_c$, which implies that $\delta = 1$. From this, it follows that $t = f(\frac{1}{T^\gamma}(B - B_c))$, a form which is usually assumed in the literature [26] (see, also, [24]), where now γ may be identified with the inverse of the usual νz exponent. We point out that the salient feature of this second alternative is that the crossing point does not appear as a zero of the β-function at non-zero temperature.

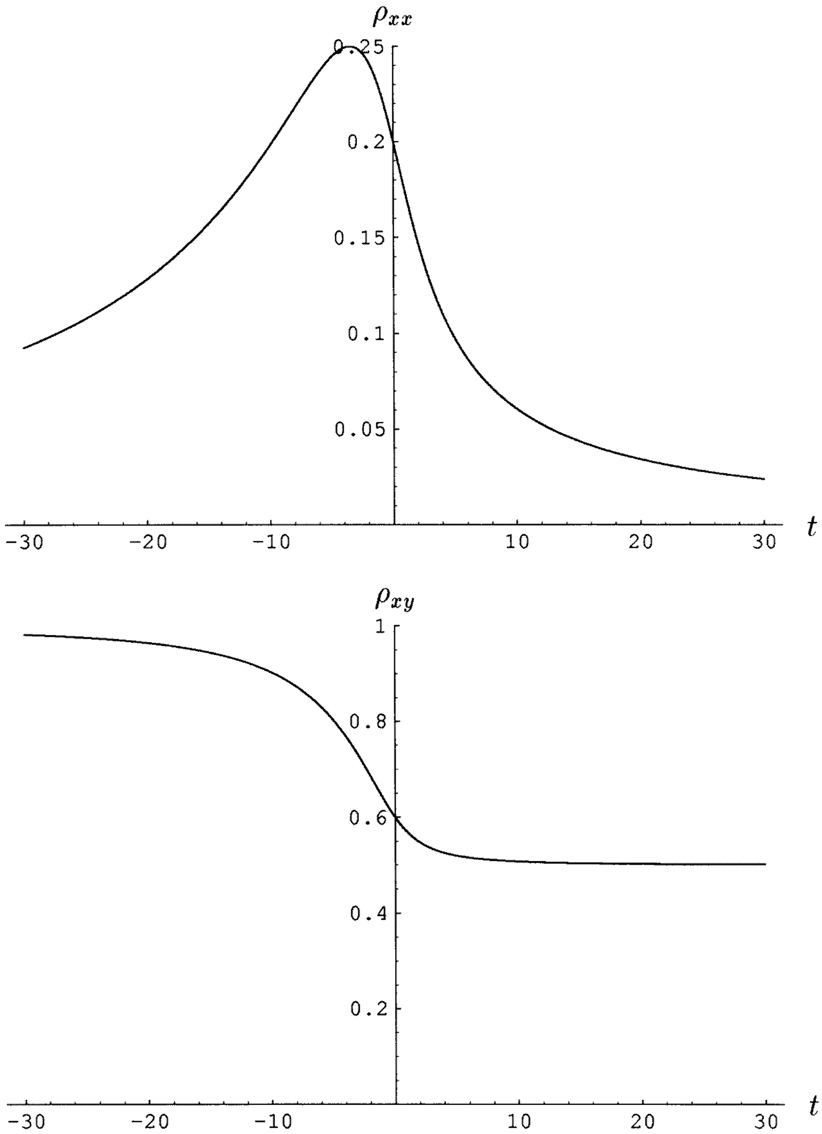

Figure 7. ρ_{xx} and ρ_{xy} versus scale parameter t for the transition $1 \rightarrow 2$.

Nevertheless, this interpretation of the scale parameter t combined with the β-function given in (3.1) provides a phenomenological picture which is in good (qualitative) agreement with the experimental data.

4. Conclusion

Let us summarize, in a physical sense, the main results of this paper. We have proposed a (family of) holomorphic β-function(s) whose RG flow preserves the $\Gamma(2)$ modular symmetry and which is consistant with the observed stability of the Hall plateaus. The semicircle law relating the longitudinal and Hall conductivities that has been observed experimentally for the $0 \to 1$, $0 \to \frac{1}{3}$ and $1 \to 2$ transitions is obtained from the integration of the RG equations for these transitions and, in fact, must hold in the present framework for any permitted transition which can be easily identified from the selection rules encoded in the flow diagram.The generic scale dependance of the conductivities has been verified to agree qualitatively with the present experimental data. In the present framework, the trajectories in the conductivity plane involving an even-denominator filling factor are found to be unstable. Although we do not have a clear interpretation (if any) of this, one might expect that the corresponding (observed) transitions are unfavourable.

It is interesting to note that the flow preserving $\Gamma(2)$ (which is a rather small symmetry compared with the one already considered) encodes numerous physical features of the QHE. However, the $\Gamma(2)$ symmetry, in itself, does not furnish any direct constraint on the crossing point appearing in any transition since there is basically no candidate for it among the fixed points of $\Gamma(2)$. On might consider the $\Gamma_0(2)$ symmetry as being more suitable for the QHE since it involves $z = \frac{(1+i)}{2}$ as a candidate for the crossing point for the $0 \to 1$ transition, however, in this latter case the location of the crossing point in the conductivity plane is rigidly determined. This prediction may prove difficult to reconcile with experiment if the recently observed deviation from this value in the $0 \to 1$ transition is confirmed, which seems to be the case as recently reported in a new experimental work [22]. It appears that $\Gamma(2)$ offers more freedom to incorporate a crossing point whose position may be fitted with further experimental determination. In this respect, we have proposed two ways of dealing with the occurence of a crossing point. The first one is obtained by imposing that β has a zero at some point z_c whose position is consistent with the experimental data, which produces a modified ansatz for the β-function still in agreement with experiment. The second (more phenomenological) method is based on a combination of the main ansatz (3.1) with a parametrization for t inspired by a two-parameter scaling framework. In this last picture, the temperature-independent point (where all the conductivity curves intersect when varying the magnetic field, as is observed experimentally) is identified with the crossing point but does not appear as a zero of the β-function.

References

[1] Prange R E and Girvin S M (ed) 1990 *The Quantum Hall Effect* 2nd edn (New York: Springer)
See also Sarma S D and Pinczuk A (ed) 1997 *Perspectives in Quantum Hall Effect* (New York: Wiley)

[2] von Klitzing K, Dorda G and Pepper M 1980 *Phys. Rev. Lett.* **45** 494

[3] Tsui D C, Störmer H L and Gossard A C 1982 *Phys. Rev. Lett.* **48** 1559

[4] Laughlin R B 1983 *Phys. Rev. Lett.* **50** 1385
Haldane F D 1983 *Phys. Rev. Lett.* **51** 605
Halperin B I 1984 *Phys. Rev. Lett.* **52** 1583

[5] Kivelson S, Lee D H and Zhang S C 1992 *Phys. Rev.* B **46** 2223

[6] Lütken C A and Ross G G 1992 *Phys. Rev.* B **45** 11837
Lütken C A and Ross G G 1993 *Phys. Rev.* B **48** 2500
Fradkin E and Kivelson S 1996 *Nucl. Phys.* B **474** 543

[7] Shapere A and Wilczek F 1989 *Nucl. Phys.* B **320** 669

[8] Lütken C A 1993 *Nucl. Phys.* B **396** 670

[9] Georgelin Y, Masson T and Wallet J C 1997 *J. Phys. A: Math. Gen.* **30** 5065

[10] Dolan B P 1999 *Nucl. Phys.* B **554** 487
(Dolan B P 1998 Modular invariance universality and crossover in the QHE *Preprint* cond-mat/9809294)
Dolan B P 1999 *J. Phys. A: Math. Gen.* **32** L243
(Dolan B P 1998 Duality and the modular group in QHE *Preprint* cond-mat/9805171)

[11] Cardy J 1982 *Nucl. Phys.* B **203** 17
Cardy J and Rabinovici E 1982 *Nucl. Phys.* B **203** 1

[12] Carpentier D 1999 *J. Phys. A: Math. Gen.* **32** 3865 and references therein
[13] Seiberg N and Witten E 1994 *Nucl. Phys.* B **426** 19
Latorre J I and Lütken C A 1988 *Phys. Lett.* B **421** 217
Ritz A 1998 *Phys. Lett.* B **434** 54
[14] Levy L P *et al* 1998 Magnetization oscillations at fractional filling factors *Institute for Theoretical Physics Conf. on Disorder and Interactions in Quantum Hall and Mesoscopic Systems (Santa Barbara, CA, 1998)*
[15] Dykhne A M and Ruzin I M 1994 *Phys. Rev.* B **50** 2369
Ruzin I M and Feng S 1995 *Phys. Rev. Lett.* **74** 154
[16] Rankin R A 1977 *Modular Forms and Functions* (Cambridge: Cambridge University Press)
[17] Georgelin Y and Wallet J C 1997 *Phys. Lett.* A **224** 303
[18] Jain J K 1997 *Phys. Rev.* B **41** 7653
[19] Hilke M *et al* 1998 *Nature* **395** 675
[20] Hilke M *et al* 1999 *Europhys. Lett.* **46** 775
(Hilke M *et al* 1998 Semi circle: An exact relation in the integer or fractional quantum Hall effect *Preprint* cond-mat/9810217)
[21] Burgess C P and Lütken C A 1999 *Phys. Lett.* B **451** 365
(Burgess C P and Lütken C A 1998 On the implication of discrete symmetries for the β-function of quantum Hall system *Preprint* cond-mat/9812396)
Taniguchi N 1998 Nonperturbative renormalization group function for quantum Hall plateaus transitions imposed by global symmetries *Preprint* cond-mat/9810334
[22] Pruisken A M M 1985 *Phys. Rev.* B **32** 2636
Pruisken A M M 1987 *Nucl. Phys.* B **285** 61
[23] Khmel'niskii D E 1983 *Pisma Zh. Eksp. Teor. Fiz.* **38** 454
[24] For a general discussion see Huckestein B 1995 *Rev. Mod. Phys.* **67** 357
[25] Shahar D *et al* 1997 *Phys. Rev. Lett.* **79** 479
[26] Sondhi S L, Girvin S M, Carini J P and Shahar D 1997 *Rev. Mod. Phys.* **69** 315