# On curvature in noncommutative geometry

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A general definition of a bimodule connection in noncommutative geometry has been recently proposed. For a given algebra this definition is compared with the ordinary definition of a connection on a left module over the associated enveloping algebra. The corresponding curvatures are also compared. © *1996 American Institute of Physics.* [S0022-2488(96)02408-5]

## I. INTRODUCTION AND MOTIVATION

Recently a general definition has been given[1,2] of a linear connection in the context of noncommutative geometry, which makes essential use of the full bimodule structure of the differential forms. A preliminary version of the curvature of the connection was given,[3] which had the drawback of not being, in general, a linear map with respect to the right-module structure. It is in fact analogous to the curvature that is implicitly used by those authors,[4–7] who define a linear connection using the formula for a covariant derivative on an arbitrary left (or right) module.[8,9] Our purpose here is to present a modified definition of curvature that is bilinear. Let $\mathcal{A}$ be a general associative algebra (with unit element). This is what replaces in noncommutative geometry the algebra of smooth functions on a smooth (compact) manifold that is used in ordinary differential geometry. By "bilinear" we mean, here and in what follows, bilinear with respect to $\mathcal{A}$. In fact, we shall present two definitions of curvature. The first is valid in all generality and reduces to the ordinary definition of curvature in the commutative case. The second definition seems to be better adapted to "extreme" noncommutative cases, such as the one considered in Sec. V.

The definition of a connection as a covariant derivative was given an algebraic form in the Tata lectures by Koszul[10] and generalized to noncommutative geometry by Karoubi[8] and Connes.[9,11] We shall often use here the expressions "connection" and "covariant derivative" synonymously. In fact, we shall distinguish three different types of connections. A "left $\mathcal{A}$ connection" is a connection on a left $\mathcal{A}$ module; it satisfies a left Leibniz rule. A "bimodule $\mathcal{A}$ connection" is a connection on a general bimodule $\mathcal{M}$ that satisfies a left and right Leibniz rule. In the particular case where $\mathcal{M}$ is the module of one-forms, we shall speak of a "linear connection." The precise definitions are given below. A bimodule over an algebra $\mathcal{A}$ is also a left module over the tensor product $\mathcal{A}^e = \mathcal{A} \otimes_{\mathbb{C}} \mathcal{A}^{\mathrm{op}}$ of the algebra with its "opposite." So a bimodule can have a bimodule $\mathcal{A}$ connection as well as a left $\mathcal{A}^e$ connection. These two definitions are compared in Sec. II. In Sec. III we discuss the curvature of a bimodule connection. In Sec. IV we consider an algebra of forms based on derivations and we compare the left connections with the linear connections. We show that in a sense to be made precise the two definitions yield the same bilinear curvature. That is, the extra restriction that the bimodule structure seems to place on the linear connections does not in fact restrict the corresponding curvature. In Sec. V we consider a more abstract geometry whose differential calculus is not based on derivations. In Sec. VI a possible definition is given of the curvature of linear connections on braided-commutative alge-

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bras. In Sec. VII we examine the (left) projective structure of the one-forms of the Connes–Lott model.

Let $\mathcal{A}$ be an arbitrary algebra and $(\Omega^*(\mathcal{A}),d)$ a differential calculus over $\mathcal{A}$. One defines a left $\mathcal{A}$ connection on a left $\mathcal{A}$ module $\mathcal{H}$ as a covariant derivative,

$$\mathcal{H} \xrightarrow{D} \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{H}, \tag{1.1}$$

which satisfies the left Leibniz rule

$$D(f\psi) = df \otimes \psi + f\, D\psi, \tag{1.2}$$

for arbitrary $f \in \mathcal{A}$. This map.has a natural extension,

$$\Omega^*(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{H} \xrightarrow{\nabla} \Omega^*(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{H}, \tag{1.3}$$

given, for $\psi \in \mathcal{H}$ and $\alpha \in \Omega^n(\mathcal{A})$, by $\nabla \psi = D\psi$ and

$$\nabla(\alpha\psi) = d\alpha \otimes \psi + (-1)^n \alpha\, \nabla\psi.$$

The operator $\nabla^2$ is necessarily left linear. However, when $\mathcal{H}$ is a bimodule it is not, in general, right linear.

A covariant derivative on the module $\Omega^1(\mathcal{A})$ must satisfy (1.2). But $\Omega^1(\mathcal{A})$ also has a natural structure as a right $\mathcal{A}$ module, and one must be able to write a corresponding right Leibniz rule in order to construct a bilinear curvature. Quite generally, let $\mathcal{M}$ be an arbitrary bimodule. Consider a covariant derivative,

$$\mathcal{M} \xrightarrow{D} \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{M}, \tag{1.4}$$

which satisfies both a left and a right Leibniz rule. In order to define a right Leibniz rule that is consistent with the left one, it was proposed by Mourad,[1] by Dubois-Violette and Michor,[12] and by Dubois-Violette and Masson[13] to introduce a generalized permutation,

$$\mathcal{M} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \xrightarrow{\sigma} \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{M}. \tag{1.5}$$

The right Leibniz rule is given then as

$$D(\xi f) = \sigma(\xi \otimes df) + (D\xi) f, \tag{1.6}$$

for arbitrary $f \in \mathcal{A}$ and $\xi \in \mathcal{M}$. The purpose of the map $\sigma$ is to bring the differential to the left while respecting the order of the factors. It is necessarily bilinear.[2] We define a bimodule $\mathcal{A}$ connection to be the couple $(D,\sigma)$.

If, in particular,

$$\mathcal{M} = \Omega^1(\mathcal{A}), \tag{1.7}$$

then we shall refer to the bimodule $\mathcal{A}$ connection as a linear connection. Although here we shall be concerned principally with this case, we shall often consider more general situations. In any case we shall use the more general notation to be able to distinguish the two copies of $\Omega^1(\mathcal{A})$ on the right-hand side of (1.4).

Let $\Omega_u^*(\mathcal{A})$ be the universal differential calculus. Dubois-Violette and Masson[13] have shown that given an arbitrary left connection on a bimodule $\mathcal{M}$ there always exists a bimodule homomorphism,

$$\mathcal{M}\otimes_{\mathcal{A}}\Omega_u^1(\mathcal{A}) \xrightarrow{\sigma(D)} \Omega^1(\mathcal{A})\otimes_{\mathcal{A}}\mathcal{M},$$

such that

$$D(\xi f)=\sigma(D)(\xi\otimes d_u f)+(D\xi)f.$$

The notation $\sigma(D)$ is taken from the definition of the symbol of a differential operator. The condition (1.6) means then that $\sigma(D)$ factorizes as a composition of a $\sigma$ as above and the canonical homomorphism of $\mathcal{M}\otimes_{\mathcal{A}}\Omega_u^1(\mathcal{A})$ onto $\mathcal{M}\otimes_{\mathcal{A}}\Omega^1(\mathcal{A})$.

Using $\sigma$ one can also construct[1] an extension,

$$\Omega^1(\mathcal{A})\otimes_{\mathcal{A}}\Omega^1(\mathcal{A})\xrightarrow{D}\Omega^1(\mathcal{A})\otimes_{\mathcal{A}}\Omega^1(\mathcal{A})\otimes_{\mathcal{A}}\Omega^1(\mathcal{A}). \tag{1.8}$$

It can also be proved, in fact,[14] that this extension implies the existence of $\sigma$. The operator $D^2$ is not in general left linear. However, if we define $\pi$ to be the product in $\Omega^*(\mathcal{A})$ and set $\pi_{12}=\pi\otimes 1$, then $\pi_{12}D^2$ is left linear,

$$\pi_{12}D^2(f\xi)=f\pi_{12}\, D^2\xi, \tag{1.9}$$

provided the torsion vanishes and the map $\sigma$ satisfies the condition

$$\pi\circ(\sigma+1)=0. \tag{1.10}$$

The map $\nabla$ is related to $D$ on $\mathcal{H}=\Omega^1(\mathcal{A})$ by

$$\nabla^2=\pi_{12}\circ D^2. \tag{1.11}$$

The left-hand side of this equation is defined for a general $\mathcal{A}$ connection whereas the right-hand side is defined only in the case of a linear connection.

The torsion $T$ is defined to be the map

$$T=d-\pi\circ D, \tag{1.12}$$

from $\Omega^1$ into $\Omega^2$. The restriction (1.7) is essential here. It follows from the condition (1.10) that $T$ is bilinear. A metric can be defined and it can be required to be symmetric using the map $\sigma$. The standard condition that the covariant derivative be metric compatible can be also carried over to the noncommutative case. For more details we refer, for example, to Madore *et al.*[3]

## II. THE BIMODULE STRUCTURE

For any algebra $\mathcal{A}$ the enveloping algebra $\mathcal{A}^e$ is defined to be

$$\mathcal{A}^e=\mathcal{A}\otimes_{\mathbb{C}}\mathcal{A}^{\mathrm{op}}.$$

A bimodule $\mathcal{M}$ can also be considered then as a left $\mathcal{A}^e$ module. The differential calculus $\Omega^*(\mathcal{A})$ has a natural extension to a differential calculus $\Omega^*(\mathcal{A}^e)$, given by

$$\Omega^*(\mathcal{A}^e)=\Omega^*(\mathcal{A})\otimes\Omega^*(\mathcal{A}^{\mathrm{op}})=(\Omega^*(\mathcal{A}))^e, \tag{2.1}$$

with $d(a\otimes b)=da\otimes b+a\otimes db$. This is not the only choice. For example, if $\Omega^*(\mathcal{A})$ were the universal calculus over $\mathcal{A}$ then $\Omega^*(\mathcal{A}^e)$ would not be equal to the universal calculus over $\mathcal{A}^e$. Suppose that $\mathcal{M}$ has a left $\mathcal{A}^e$ connection,

$$\mathcal{M}\xrightarrow{D^e}\Omega^1(\mathcal{A}^e)\otimes_{\mathcal{A}^e}\mathcal{M}. \tag{2.2}$$

From the equality

$$\Omega^1(\mathcal{A}^e)=(\Omega^1(\mathcal{A})\otimes_{\mathbb{C}}\mathcal{A}^{\mathrm{op}})\oplus(\mathcal{A}\otimes_{\mathbb{C}}\Omega^1(\mathcal{A}^{\mathrm{op}})), \tag{2.3}$$

and using the identification

$$(\mathcal{A}\otimes_{\mathbb{C}}\Omega^1(\mathcal{A}^{\mathrm{op}}))\otimes_{\mathcal{A}^e}\mathcal{M}\simeq\mathcal{M}\otimes_{\mathcal{A}}\Omega^1(\mathcal{A}), \tag{2.4}$$

given by

$$(1\otimes\xi)\otimes\eta\mapsto\eta\otimes\xi,$$

we find that we have

$$\Omega^1(\mathcal{A}^e)\otimes_{\mathcal{A}^e}\mathcal{M}=(\Omega^1(\mathcal{A})\otimes_{\mathcal{A}}\mathcal{M})\oplus(\mathcal{M}\otimes_{\mathcal{A}}\Omega^1(\mathcal{A})). \tag{2.5}$$

The covariant derivative $D^e$ splits then as the sum of two terms,

$$D^e=D_L+D_R. \tag{2.6}$$

From the identifications it is obvious that $D_L$ ($D_R$) satisfies a left (right) Leibniz rule and is right (left) $\mathcal{A}$ linear. Such covariant derivatives have been considered by Cuntz and Quillen[15] and by Bresser *et al.*[14]

One can write a (noncommutative) triangular diagram,

$$\begin{tikzcd} & \mathcal{M} \arrow[dl, "D_L"'] \arrow[dr, "D_R"] & \\ \Omega^1(\mathcal{A})\otimes_{\mathcal{A}}\mathcal{M} & & \mathcal{M}\otimes_{\mathcal{A}}\Omega^1(\mathcal{A}), \arrow[ll, "\sigma"'] \end{tikzcd} \tag{2.7}$$

from which one sees that given an arbitrary bimodule homomorphism (1.5) and a covariant derivative (2.2) one can construct a covariant derivative (1.4) by the formula

$$D=D_L+\sigma\circ D_R, \tag{2.8}$$

which satisfies both (1.2) and (1.6).

Suppose further that the differential calculus is such that the differential $d$ of an element $f\in\mathcal{A}$ is of the form

$$df=-[\theta,f], \tag{2.9}$$

for some element $\theta\in\Omega^1$. Then, obviously particular choices for $D_L$ and $D_R$ are the expressions

$$D_L\xi=-\theta\otimes\xi,\quad D_R\xi=\xi\otimes\theta. \tag{2.10}$$

Let $\tau$ be a bimodule homomorphism from $\mathcal{M}$ into $\Omega^1(\mathcal{A}^e)\otimes_{\mathcal{A}^e}\mathcal{M}$ and decompose

$$\tau=\tau_L+\tau_R, \tag{2.11}$$

according to the decomposition (2.5). The most general $D_L$ and $D_R$ are of the form

$$D_L\xi=-\theta\otimes\xi+\tau_L(\xi),\quad D_R\xi=\xi\otimes\theta+\tau_R(\xi). \tag{2.12}$$

Using (2.8), we can construct a covariant derivative,

$$D\xi=-\theta\otimes\xi+\sigma(\xi\otimes\theta), \tag{2.13}$$

from (2.10). In Sec. IV we shall study a differential calculus for which this is the only possible $D$.

From the formula (2.9) we know that there is a bimodule projection of $\mathcal{A}^e$ onto $\Omega^1(\mathcal{A})$. Suppose that $\Omega^1(\mathcal{A})$ is a projective bimodule and let $P$ be the corresponding projector. We can then identify $\Omega^1(\mathcal{A})$ as a subbimodule of the free $\mathcal{A}^e$ module of rank 1:

$$\Omega^1(\mathcal{A})=\mathcal{A}^eP.$$

A left $\mathcal{A}^e$ connection on $\mathcal{A}^e$ as a left $\mathcal{A}^e$ module is a covariant derivative of the form (1.4) with $\mathcal{M}=\mathcal{A}^e$. The ordinary differential $d^e$ on $\mathcal{A}^e$,

$$\mathcal{A}^e\xrightarrow{d^e}\Omega^1(\mathcal{A}^e), \tag{2.14}$$

is clearly a covariant derivative in this sense. The right-hand side can be written using (2.5) as

$$\Omega^1(\mathcal{A}^e)=(\Omega^1(\mathcal{A})\otimes_{\mathcal{A}}\mathcal{A}^e)\oplus(\mathcal{A}^e\otimes_{\mathcal{A}}\Omega^1(\mathcal{A})), \tag{2.15}$$

and so we can split $d^e$ as the sum of two terms $d_L$ and $d_R$. Let $a\otimes b$ be an element of $\mathcal{A}^e$. Then we have

$$d_L(a\otimes b)=-[\theta,a]\otimes b=-\theta\otimes(a\otimes b)+(a\otimes b)(\theta\otimes 1). \tag{2.16}$$

In the first term on the right-hand side the first tensor product is over the algebra and the second is over the complex numbers; in the second term the first tensor product is over the complex numbers and the second is over the algebra.

A general element of $\Omega^1(\mathcal{A})$ can be written as a sum of elements of the form $\xi=(a\otimes b)P=aPb$. We have then

$$d_L\xi=-\theta\otimes\xi+\xi(\theta\otimes 1).$$

Define $D_L$ by

$$D_L\xi=(d_L\xi)P. \tag{2.17}$$

Then we obtain the first of Eqs. (2.12) with

$$\tau_L(\xi)=\xi(\theta\otimes P). \tag{2.18}$$

Here, on the right-hand side, the tensor product is over the algebra and $\theta\otimes P$ is an element of $\Omega^1(\mathcal{A})\otimes\Omega^1(\mathcal{A})$. This is a left $\mathcal{A}^e$ module. Similarly, one can construct a $D_R$ and a $D^e$ by Eq. (2.6):

$$D^e\xi=(d^e\xi)P. \tag{2.19}$$

In the case of ordinary geometry with $\mathcal{A}$ equal to the algebra $\mathcal{C}^\infty(V)$ of smooth functions on a smooth manifold $V$ the algebra $\mathcal{A}^e$ is the algebra of smooth functions in two variables. If $\Omega^*(\mathcal{A})$ is the algebra of de Rham differential forms, the only possible $\sigma$ is the permutation and the left and right Leibniz rules are identical. In this case $D^e$ cannot exist. In fact, $D_L$ would satisfy a left Leibniz rule and be left linear since the left and right multiplication are equal. In general, let $\mathcal{M}$ be the $\mathcal{A}$ module of smooth sections of a vector bundle over $V$. Then $\mathcal{M}$ is a $\mathcal{A}^e$ module. It is important to notice that although it is projective as an $\mathcal{A}$ module it is never projective as an $\mathcal{A}^e$ module since a projective $\mathcal{A}^e$ module consists of two-point functions.

## III. CURVATURE

Consider a covariant derivative (1.4) that satisfies the left Leibniz rule (1.2). We can define a right-linear curvature by factoring out in the image of $\nabla^2$ all those elements ($J=$"junk") that do not satisfy the desired condition. Define $J$ as the vector space,

$$J=\left\{\sum_i (\nabla^2(\xi_i f_i)-\nabla^2(\xi_i)f_i)\,\middle|\,\xi_i\in\mathcal{M}, f_i\in\mathcal{A}\right\}. \tag{3.1}$$

In fact, $J$ is a subbimodule of $\Omega^2(\mathcal{A})\otimes_{\mathcal{A}}\mathcal{M}$. It is obviously a left submodule. Consider the element $\alpha=\nabla^2(\xi g)-\nabla^2(\xi)g\in J$ and let $f\in\mathcal{A}$. We can write

$$\alpha f=(\nabla^2(\xi g f)-\nabla^2(\xi)gf)-(\nabla^2(\xi g f)-\nabla^2(\xi g)f).$$

Therefore $\alpha f\in J$ and $J$ is also a right submodule.

Let $p$ be the projection

$$\Omega^2(\mathcal{A})\otimes_{\mathcal{A}}\mathcal{M}\xrightarrow{p}\Omega^2(\mathcal{A})\otimes_{\mathcal{A}}M/J. \tag{3.2}$$

We shall define the curvature of $D$ as the combined map,

$$\mathrm{Curv}=-p\circ\nabla^2. \tag{3.3}$$

In the case of a linear connection, we can write

$$\mathrm{Curv}=-p\circ\pi_{12}\circ D^2.$$

By construction, Curv is left and right linear:

$$\mathrm{Curv}(f\xi)=f\,\mathrm{Curv}(\xi),\quad \mathrm{Curv}(\xi f)=\mathrm{Curv}(\xi)f. \tag{3.4}$$

In the next section we shall present an example that illustrates the role that the right Leibniz rule (1.6) plays in this construction.

Consider the covariant derivative (2.2). One can define a bilinear curvature as the map

$$\mathrm{Curv}_L=-\nabla_L^2, \tag{3.5}$$

from $\mathcal{M}$ into $\Omega^2(\mathcal{A})\otimes_{\mathcal{A}}\mathcal{M}$. It is bilinear because by construction it is trivially right linear. In the case where the differential $d$ is given by (2.9) and $D_L$ is given by (2.10), we find that $\mathrm{Curv}_L$ is given by the formula

$$\mathrm{Curv}_L(\xi)=(d\theta+\theta^2)\otimes\xi. \tag{3.6}$$

From this expression it is obvious that $\mathrm{Curv}_L$ is right linear; it is easy to verify directly that it is also left linear because of the fact that the two-form $d\theta+\theta^2$ commutes with the elements of $\mathcal{A}$:

$$[d\theta+\theta^2,f]=d[\theta,f]=-d^2f=0. \tag{3.7}$$

The covariant derivative (2.2) also has a bilinear curvature two-form,

$$\mathrm{Curv}^e = -\nabla^{e2}, \tag{3.8}$$

which naturally decomposes into three terms, all of which are bilinear. One of these terms corresponds to the covariant derivative of Sec. I with $\sigma$ set equal to zero. It takes its values in a space that can be naturally identified with $\Omega^2(\mathcal{A})\otimes_{\mathcal{A}}\mathcal{M}$. However, because the second action of $D^e$ does not commute with that of $\sigma$, the corresponding term of $\mathrm{Curv}^e$ does not necessarily coincide with the image of Curv. We shall discuss this in an example in Sec. V. The curvature of the particular connection (2.19) can be written in terms of the projector $P$:

$$\mathrm{Curv}^e\ \xi = -\nabla^{e2}\xi = -\xi((d^eP)(d^eP)P). \tag{3.9}$$

The extension of $D$ to the tensor product of $n$ copies of $\Omega^1(\mathcal{A})$ defines a covariant derivative on the left module,

$$\mathcal{H} = (\Omega^1(\mathcal{A}))^{\otimes n}. \tag{3.10}$$

The curvature is given by (1.11). In the commutative case and, more generally, in the case of a derivation-based differential calculus, this curvature can be expressed in terms of the curvature of the covariant derivative (1.4). For a general differential calculus this will not be the case.

The same remarks can be made concerning the torsion (1.12). In general, let $\pi$ be the product map of $(\Omega^1(\mathcal{A}))^{\otimes n}$ into $\Omega^n(\mathcal{A})$. Then one can also define a module homomorphism,

$$(\Omega^1(\mathcal{A}))^{\otimes n} \xrightarrow{T_n} \Omega^{n+1}(\mathcal{A}), \tag{3.11}$$

given by

$$T_n = d\pi - \pi\circ D. \tag{3.12}$$

These maps are all left-module homomorphisms. If $\xi\in\Omega^1(\mathcal{A})$ and $\nu\in(\Omega^1(\mathcal{A}))^{\otimes n}$, then we have

$$T_{n+1}(\xi\otimes\nu) = T_1(\xi)\pi(\nu) - \xi T_n(\nu) + \pi\circ((\sigma+1)\otimes 1)\xi\otimes\nabla\nu. \tag{3.13}$$

In order for the last term in the previous equation to vanish, it is necessary and sufficient that (1.10) be satisfied. In this case one sees by iteration that the $T_n$ can all be expressed in terms of $T_1$ and therefore that all of them are bimodule homomorphisms.

## IV. LINEAR CONNECTIONS ON MATRIX GEOMETRIES

As a first example we present the case of the algebra $M_n$ of $n\times n$ matrices[16,17] with a differential calculus based on derivations.[18,19] Let $\lambda_r$ be a set of generators of the Lie algebra of the special linear group $SL_n$. Then the derivations $e_r = \mathrm{ad}\,\lambda_r$ is a basis for the derivations of $M_n$ and the dual one-forms $\theta^r$ commute with the elements of $M_n$. The set of one-forms $\Omega^1(M_n)$ is a free left (or right) module of rank $n^2-1$. The natural map $\sigma$ that we shall use is given[3] by

$$\sigma(\theta^r\otimes\theta^s) = \theta^s\otimes\theta^r. \tag{4.1}$$

Quite generally, for any algebra $\mathcal{A}$ with a differential calculus that is based on derivations there is a natural map $\sigma$ given by a permutation of the arguments in the forms. Let $X$ and $Y$ be derivations. Then one can define $\sigma$ by

$$\sigma(\xi\otimes\eta)(X,Y) = \xi\otimes\eta(Y,X).$$

A general left $M_n$ connection can be defined by the covariant derivative

$$D\theta^r = -\omega^r{}_{st}\theta^s \otimes \theta^t, \tag{4.2}$$

with $\omega^r{}_{st}$ an arbitrary element of $M_n$ for each value of the indices $r$, $s$, $t$. We write

$$\omega^r{}_{st} = \Gamma^r{}_{st} + J^r{}_{st}, \tag{4.3}$$

where the $\Gamma^r_{st}$ are proportional to the identity in $M_n$ and the $J^r{}_{st}$ are trace-free. If we require that the torsion vanish then we have[3]

$$\Gamma^r_{[st]} = C^r{}_{st}, \tag{4.4}$$

where the $C^r{}_{st}$ are $SL_n$ structure constants.

If we impose the right Leibniz rule, we find that

$$0 = D([f,\theta^r]) = [f, D\theta^r] = -[f, J^r{}_{st}]\theta^s \otimes \theta^t, \tag{4.5}$$

for arbitrary $f \in M_n$ and so we see that if the connection is a linear connection then

$$J^r{}_{st} = 0. \tag{4.6}$$

Consider now the curvature of the left $M_n$ connection and write

$$\nabla^2 \theta^r = -\Omega^r{}_{stu}\theta^t\theta^u \otimes \theta^s. \tag{4.7}$$

Then since the elements of the algebra commute with the generators $\theta^r$, we have

$$\nabla^2(\theta^r f) - (\nabla^2\theta^r)f = \nabla^2(f\theta^r) - (\nabla^2\theta^r)f = -[f, \Omega^r{}_{stu}]\theta^t\theta^u \otimes \theta^s. \tag{4.8}$$

Since $f$ is arbitrary, it follows then that we have

$$\mathrm{Curv}(\theta^r) = \tfrac{1}{2}R^r{}_{stu}\theta^t\theta^u \otimes \theta^s, \tag{4.9}$$

where the $R^r{}_{stu}$ are defined uniquely in terms of the $\Gamma^r_{st}$:

$$R^r{}_{stu} = \Gamma^r_{tp}\Gamma^p_{us} - \Gamma^r_{up}\Gamma^p_{ts} - \Gamma^r_{ps}C^p{}_{tu}. \tag{4.10}$$

That is, $R^r{}_{stu}$ does not depend on $J^r{}_{st}$.

We conclude then that even had we not required the right Leibniz rule and had admitted an extra term of the form $J^r{}_{st}$ in the expression for the covariant derivative then we would find that the curvature map Curv would remain unchanged. The extra possible terms are eliminated under the projection $p$ of (3.2).

There is a covariant derivative that is of the form (2.8) with $D_L$ and $D_R$ given by (2.10). For this covariant derivative, one has

$$\omega^r{}_{st} \equiv 0. \tag{4.11}$$

This covariant derivative has obviously vanishing curvature but it is not torsion-free. If we use the ambiguity (2.11) we can write any covariant derivative (4.2) in the form (2.8).

The generators $\theta^r$ are no longer independent if one considers the bimodule structure. In fact, one finds that

$$\theta^r = -C^r{}_{st}\lambda^s\theta\lambda^t, \quad \theta = -\lambda_r\theta^r, \tag{4.12}$$

and as a bimodule $\Omega^1(M_n)$ is generated by $\theta$ alone. For dimensional reasons $\Omega^1(M_n)$ cannot be of rank one. In fact, the free $M_n$ bimodule of rank one is of dimension $n^4$ and the dimension of $\Omega^1(M_n)$ is equal to $(n^2-1)n^2<n^4$. With the normalization that we have used for the generators $\lambda_r$, the element

$$\zeta=\frac{1}{n^2}1\otimes 1-\frac{1}{n}\lambda_r\otimes\lambda^r,$$

is a projector in $M_n\otimes M_n$, which commutes with the elements of $M_n$. This can be written as

$$d(M_n)\zeta=0.$$

We have the direct-sum decomposition

$$M_n\otimes M_n=\Omega^1(M_n)\oplus M_n\zeta. \tag{4.13}$$

As in Sec. I one can define $M_n^e = M_n \otimes {}_{\mathbb{C}}M_n^{\text{op}}$. The prescription (2.19) with

$$P=1\otimes 1-\zeta$$

yields then a covariant derivative of the form (4.2) whose curvature vanishes.

## V. LINEAR CONNECTIONS ON THE CONNES–LOTT MODEL

Consider the algebra $M_3$ with the grading defined by the decomposition $\mathbb{C}^3=\mathbb{C}^2\oplus\mathbb{C}$. Define[20,21] $\Omega^0(M_3^+)=M_3^+=M_2\times M_1$, $\Omega^1(M_3^+)=M_3^-$, $\Omega^2(M_3^+)=M_1$, and $\Omega^p(M_3^+)=0$ for $p\geq 3$. A differential $d$ can be defined by[9,11]

$$df=-[\theta,f], \tag{5.1}$$

where $\theta\in\Omega^1(M_3^+)$.

The vector space of one-forms is of dimension 4 over the complex numbers. The dimension of $\Omega^1(M_3^+)\otimes_{\mathbb{C}}\Omega^1(M_3^+)$ is equal to 16 but the dimension of the tensor product $\Omega^1(M_3^+)\otimes_{M_3^+}\Omega^1(M_3^+)$ is equal to 5 and we can make the identification

$$\Omega^1(M_3^+)\otimes_{M_3^+}\Omega^1(M_3^+)=M_3^+. \tag{5.2}$$

To define a linear connection we must first define the map $\sigma$ of (1.5) with $\mathscr{M}=\Omega^1(M_3^+)$. Because of the identification (5.2), it can be considered as a map from $M_3^+$ into itself and because of the bilinearity it is necessarily of the form

$$\sigma=\begin{pmatrix}\mu & 0 & 0\\ 0 & \mu & 0\\ 0 & 0 & -1\end{pmatrix},$$

where $\mu\in\mathbb{C}$. The $-1$ in the lower right corner is imposed by the condition (1.10).

It can be shown[3] that for each such $\sigma$ there is a unique linear connection given by the covariant derivative (2.13). That is, necessarily $\tau\equiv 0$. Let $e$ be the unit in $M_1$ considered as a generator of $\Omega^2(M_3^+)$. The expression $d\theta+\theta^2$ is given by

$$d\theta+\theta^2=e.$$

Therefore we have

$$\mathrm{Curv}_L(\xi) = e \otimes \xi. \tag{5.3}$$

To construct $J$ it is convenient to fix a vector-space basis for $\Omega^1(M_3^+)$. We introduce the (unique) upper-triangular matrices $\eta_1$ and $\eta_2$ such that $\theta = \eta_1 - \eta_1^*$ and such that

$$\eta_i \eta_j^* = 0, \quad \eta_i^* \eta_j = \delta_{ij} e.$$

We find[3] that

$$\begin{aligned} &\nabla^2 \eta_1 = 0, \quad \nabla^2 \eta_2 = 0, \\ &\nabla^2 \eta_1^* = -(\mu+1) e \otimes \eta_1^*, \quad \nabla^2 \eta_2^* = -e \otimes \eta_2^*. \end{aligned} \tag{5.4}$$

Since there is an element $u$ of the algebra such that $\eta_2 = u \eta_1$, it is obvious that the map $\nabla^2$ is right linear only in the degenerate case $\mu=0$. In this case $J=0$ and

$$\mathrm{Curv} = \mathrm{Curv}_L. \tag{5.5}$$

Otherwise it is easy to see that

$$J = \Omega^2(M_3^+) \otimes_{M_3^+} \Omega^1(M_3^+),$$

and therefore that

$$\mathrm{Curv} \equiv 0. \tag{5.6}$$

It is difficult to appreciate the meaning of this result since there is only one connection for each value of $\mu$. However, (5.4) does not appear to define a curvature that is any less flat for generic $\mu$ than for the special value $\mu=0$. As we have defined it Curv does not perhaps contain enough information to characterize a general noncommutative geometry.

From (5.4) one sees that for all values of $\mu$ there is a subalgebra of $M_3^+$ with respect to which $\nabla^2$ is right linear. It consists of those elements that leave invariant the vector subspaces of $\Omega_1(M_3^+)$ defined by $\eta_1$ and $\eta_2$. That is, it is the algebra

$$M_1 \times M_1 \times M_1 \subset M_3^+.$$

As in Sec. I we define $M_3^{+e} = M_3^+ \otimes_{\mathbb{C}} M_3^{+\mathrm{op}}$. A general element $\xi$ of $\Omega^1(M_3^+)$ can be written in the form

$$\xi = \begin{pmatrix} 0 & 0 & \xi_{13} \\ 0 & 0 & \xi_{23} \\ \xi_{31} & \xi_{32} & 0 \end{pmatrix}, \tag{5.7}$$

where the $\xi_{ij}$ are arbitrary complex numbers. The map

$$\xi \mapsto \begin{pmatrix} 0 & \xi_{13} & 0 \\ 0 & \xi_{23} & 0 \\ 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 \\ \xi_{31} & \xi_{32} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{5.8}$$

identifies $\Omega^1(M_3^+)$ as a sub-bimodule of the free $M_3^{+e}$ module of rank 1 and the projector

$$P=\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}+\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\otimes\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}\tag{5.9}$$

projects $M_3^{+e}$ onto $\Omega^1(M_3^+)$. One immediately sees that by multiplication of $P$ on the right and left by elements of $M_3^+$ one obtains all elements of $\Omega^1(M_3^+)$. The construction of Sec. II can be used to construct by projection a covariant derivative (2.19). In the present case we find that

$$P(\theta\otimes P)=0,$$

as it must be since we have already noticed that in the present case $\tau\equiv 0$. The covariant derivative (2.19) is identical to that given by (2.10).

## VI. BRAIDED-COMMUTATIVE ALGEBRAS

As an example of a braided-commutative differential calculus, we consider the quantum plane with its $SL_q(2)$-covariant differential calculus $\Omega^*$. It has been found[2] that there is a unique one-parameter family of linear connections given by the covariant derivative

$$D\xi=\mu^4x^i(x\eta-qy\xi)\otimes(x\eta-qy\xi),\tag{6.1}$$

with $\mu$ a complex number. The corresponding $\sigma$ is uniquely defined in terms of the $R$ matrix. There are other linear connections if we extend the algebra to include additional elements $x^{-1}$ and $y^{-1}$. For example, consider the construction of Sec. II based on the formula (2.8). For arbitrary complex number $c$ define, for $q\neq\pm 1$,

$$\theta=\frac{1}{1-q^{-2}}\,(y^{-1}\eta+cxy^{-2}\xi).\tag{6.2}$$

We have then

$$\xi^i=dx^i=-[\theta,x^i].\tag{6.3}$$

If $c=0$ then the differential $d$ is given on the entire algebra of forms as a graded commutator with $\theta$. Using $\theta$ we can define $D_L$ and $D_R$ by (2.10) and a covariant derivative by (2.13). As in Sec. IV the curvature of this covariant derivative vanishes. In fact, for arbitrary $c$ we have $d\theta+\theta^2=0$. This construction can be used for any generalized permutation that satisfies the condition (1.10). There are many such $\sigma$. For example, if $i$ is a bimodule injection of $\Omega^2$ into $\Omega^1\otimes_{\mathcal{A}}\Omega^1$ that satisfies the condition $\pi\circ i=1$ then a generalized permutation is given by the formula $\sigma=1-2i\circ\pi$.[1]

A linear connection has also been constructed on $GL_q(n)$.[22] The differential calculus is constructed using a one-form $\theta$ and a linear connection is given by the formula (2.13).

The construction of a bilinear curvature based on the projection (3.2) is not interesting in the general braided-commutative case. In this case the right-module structure of $\Omega^1(\mathcal{A})$ is determined in terms of its left-module structure, even though the forms do not commute with the algebra. The construction can be modified, however, using the braiding. There is then a morphism $\rho$ of the algebra such that the vector space,

$$J_\rho=\left\{\sum_i\,(\nabla^2(\xi_if_i)-\nabla^2(\xi_i)\rho(f_i))\,\middle|\,\xi_i\in\Omega^1(\mathcal{A}),f_i\in\mathcal{A}\right\},$$

vanishes identically. The curvature $\mathrm{Curv}_\rho$ defined by the obvious modification of (3.3) is therefore left linear and right $\rho$ linear:

$$\mathrm{Curv}_\rho(f\xi)=f\,\mathrm{Curv}_\rho(\xi),\quad \mathrm{Curv}_\rho(\xi f)=\mathrm{Curv}_\rho(\xi)\rho(f). \tag{6.4}$$

In general, for each automorphism $\rho$ of the algebra, a curvature $\mathrm{Curv}_\rho$ can be defined; it would be of interest, however, only in the case when $J_\rho$ vanishes.

## VII. THE PROBLEM OF CURVATURE INVARIANTS

In Sec. IV we considered a geometry with a module of one-forms that was free of rank $n^2-1$ as a left (and right) module. We noticed also in (4.13) that it can be written as a direct summand in a free bimodule of rank 1. In Sec. V we considered the projective bimodule structure of $\Omega^1(M_3^+)$. In this section we shall examine the projective structure of $\Omega^1(M_3^+)$ as a left (and right) module in order to see to what extent it is possible to express the geometry of Sec. V in the language of Sec. IV. If it were possible to do this it would be possible to define curvature invariants as in Sec. IV.

Consider the element (5.7) of $\Omega^1(M_3^+)$. The map

$$\xi\mapsto\begin{pmatrix}0&0&0\\0&0&0\\0&0&\xi_{31}\end{pmatrix},\quad\begin{pmatrix}0&0&0\\0&0&0\\0&0&\xi_{32}\end{pmatrix},\quad\begin{pmatrix}0&\xi_{13}&0\\0&\xi_{23}&0\\0&0&0\end{pmatrix} \tag{7.5}$$

identifies $\Omega^1(M_3^+)$ as a submodule of the free module,

$$\mathscr{M}\equiv(M_3^+)^3=M_3^+\oplus M_3^+\oplus M_3^+,$$

of rank 3. This imbedding respects the left-module structure of $M_3^+$. It also respects the right-module structure if we identify

$$f\mapsto\rho(f)=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\otimes f. \tag{7.6}$$

That is, under the right action by $M_3^+$ the element $f$ acts on the row vector and not on the matrix entries.

Define the projectors

$$P_1=P_2=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix},\quad P_3=\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix} \tag{7.7}$$

in $M_3^+$ and the projector

$$P=P_1\otimes\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}+P_2\otimes\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}+P_3\otimes\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix} \tag{7.8}$$

in $M_3(M_3^+)$. Then $\xi P=\xi$. Let $\alpha$ be a general element of $\mathscr{M}$, that is, a triplet of elements of $M_3^+$ written as in (7.5) as a row vector. Then $\alpha P\in\Omega^1(M_3^+)$ and all elements $\xi\in\Omega^1(M_3^+)$ can be obtained in this way; the module $\Omega^1(M_3^+)$ is a projective left $M_3^+$ module:

$$\Omega^1(M_3^+)=\mathscr{M}P. \tag{7.9}$$

This defines a projection,

$$\mathcal{M} \xrightarrow{p} \Omega^1(M_3^+), \tag{7.10}$$

which is a left inverse of the imbedding (7.5).

Let $\theta^r$ be the canonical basis of $\mathcal{M}$:

$$\theta^1 = (1,0,0), \quad \theta^2 = (0,1,0), \quad \theta^3 = (0,0,1),$$

where the unit is the unit in $M_3^+$. We use a notation here that parallels that of Sec. IV (with $n=2$). In general, however, for $f \in M_3^+$,

$$f\theta^r \neq \theta^r \rho(f) \equiv \theta^s (\rho(f))^r_s.$$

This is an essential difference with the geometry of Sec. IV.

Define

$$\theta^r_P = \theta^r P.$$

By this we mean that $\theta^r_P$ is the image of the triplet $\theta^r$ under the projection (7.10) that we again identify as an element of $\mathcal{M}$ by (7.5). An extension $\tilde{\sigma}$ of $\sigma$ is a map,

$$\Omega^1(M_3^+) \otimes_{M_3^+} \mathcal{M} \xrightarrow{\tilde{\sigma}} \mathcal{M} \otimes_{M_3^+} \Omega^1(M_3^+),$$

given by the action $\tilde{\sigma}(\theta^r_P \otimes \theta^s)$. It is clear that this $\tilde{\sigma}$ will not be a simple permutation as in (4.1). The covariant derivative on $\mathcal{M}$ will be defined by

$$\tilde{D}\theta^r = -\omega^r_{st}\theta^s_P \otimes \theta^t, \tag{7.11}$$

analogous to (4.2) but here with the $\omega^r{}_{st}$ arbitrary elements of $M_3^+$.

Using the projection $p$ we can define in particular a covariant derivative $\tilde{D}$ on $\mathcal{M}$ that coincides with the image of $D$ on $\Omega^1(M_3^+)$ by the requirement that the diagram

$$\begin{array}{ccc} \Omega^1(M_3^+) & \xrightarrow{D} & \Omega^1(M_3^+) \otimes_{M_3^+} \Omega^1(M_3^+) \\ p\uparrow & & \downarrow \\ \mathcal{M} & \xrightarrow{\tilde{D}} & \Omega^1(M_3^+) \otimes_{M_3^+} \mathcal{M} \end{array} \tag{7.12}$$

be commutative. The down arrow on the right is an injection defined by (7.5). The covariant derivative $\tilde{D}\theta^r$ is then defined by

$$\tilde{D}\theta^r = D\theta^r_P. \tag{7.13}$$

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[1] J. Mourad, ''Linear connections in non-commutative geometry,'' Class. Quantum Grav. **12**, 965 (1995).
[2] M. Dubois-Violette, J. Madore, T. Masson, and J. Mourad, ''Linear connections on the quantum plane,'' Lett. Math. Phys. **35**, 351 (1995).
[3] J. Madore, T. Masson, and J. Mourad, ''Linear connections on matrix geometries,'' Class. Quantum Grav. **12**, 1429 (1995).
[4] A. H. Chamseddine, G. Felder, and J. Fröhlich, ''Gravity in non-commutative geometry,'' Commun. Math. Phys. **155**, 205 (1993).
[5] A. Sitarz, ''Gravity from noncommutative geometry,'' Class. Quantum Grav. **11**, 2127 (1994).
[6] C. Klimčík, A. Pompoš, and V. Souček, ''Grading of spinor bundles and gravitating matter in noncommutative geometry,'' Lett. Math. Phys. **30**, 259 (1994).
[7] G. Landi, A. Viet Nguyen, and K. C. Wali, ''Gravity and electromagnetism in noncommutative geometry,'' Phys. Lett. B **326**, 45 (1994).
[8] M. Karoubi, ''Connections, courbures et classes caractéristiques en $K$-théorie algébrique,'' *Current Trends in Algebraic Topology*, Part I, London, Ontario, 1981 (American Mathematical Society, Providence, RI, 1982).
[9] A. Connes, ''Non-commutative differential geometry,'' Publ. Inst. Hautes Etudes Sci. **62**, 257 (1986).
[10] J. L. Koszul, *Lectures on Fibre Bundles and Differential Geometry* (Tata Institute of Fundamental Research, Bombay, 1960).
[11] A. Connes, *Noncommutative Geometry* (Academic Press, New York, 1994).
[12] M. Dubois-Violette and P. Michor, ''Connections on central bimodules,'' J. Geom. Phys. (in press).
[13] M. Dubois-Violette and T. Masson, ''On the first-order operators in bimodules,'' q-alg/9507028, Lett. Math. Phys. (in press).
[14] K. Bresser, F. Müller-Hoissen, A. Dimakis, and A. Sitarz, ''Noncommutative geometry of finite groups,'' J. Phys. A (to be published).
[15] A. Cuntz and D. Quillen, ''Algebra extensions and nonsingularity,'' J. Am. Math. Soc. **8**, 251 (1995).
[16] M. Dubois-Violette, R. Kerner, and J. Madore, ''Gauge bosons in a noncommutative geometry,'' Phys. Lett. B **217**, 485 (1989); ''Classical bosons in a noncommutative geometry,'' Class. Quantum Grav. **6**, 1709 (1989).
[17] M. Dubois-Violette, R. Kerner, and J. Madore, ''Noncommutative differential geometry of matrix algebras,'' J. Math. Phys. **31**, 316 (1990).
[18] M. Dubois-Violette, ''Dérivations et calcul différentiel non-commutatif,'' C. R. Acad. Sci. Paris **307**, I 403 (1988).
[19] M. Dubois-Violette and P. Michor, ''Dérivations et calcul différentiel non-commutatif II,'' C. R. Acad. Sci. Paris **319**, 927 (1994).
[20] A. Connes and J. Lott, ''Particle models and noncommutative geometry,'' in *Recent Advances in Field Theory*, Nucl. Phys. Proc. Suppl. B **18**, 29 (1990).
[21] A. Connes and J. Lott, ''The metric aspect of non-commutative geometry,'' *Proceedings of the 1991 Cargèse Summer School* (Plenum, New York, 1992).
[22] Y. Georgelin, J. Madore, T. Masson, and J. Mourad, ''On the non-commutative Riemannian geometry of $GL_q(n)$'', preprint LPTHE Orsay 95/51, q-alg/9507002, 1995.