On Finite Differential Calculi

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ABSTRACT. In the case of matrix algebras, a comparison is made between the
differential calculus based on derivations and that based on the Dirac operator.
In particular the appropriate Dirac operator is given which reproduces the
differential calculus based on derivations.

1. Introduction

The extension to noncommutative algebras of the definition of a differential
calculus has been given both without [6] and with [14] the use of the derivations
of the algebra. The former is of course a more general procedure since it can
be used even in the case of algebras which have no derivations, for example the
algebras of continuous or measurable functions on a differential manifold. At least
in the finite-dimensional case which we shall consider here a comparison between
the two approaches is readily made. In particular the differential calculus of an
algebra based on derivations can be easily put in the form of the differential calculus
introduced by Connes & Lott [10] which is based on a generalization of the Dirac
operator.

In Section 2 we shall give a brief description of the differential calculus based
on derivations. In Section 3 we shall do the same for the differential calculus
which is based on a generalization of the Dirac operator. In Section 4 we show the
relation between the two. In Section 5 we briefly recall how physical theories can
be constructed by considering in conjunction with these finite calculi the ordinary
differential calculus of space-time.

2. Derivations

A smooth global vector field on a manifold $V$ is equivalent to a derivation of
the algebra of smooth functions $C(V)$ defined on $V$ [29]. As such it has a natural
extension to noncommutative geometry [6, 7, 8]. The algebras $M_n$ of $n \times n$ complex
matrices furnish finite analogues of compact parallelizable manifolds. Let $\text{Der}(M_n)$
be the vector space of all derivations of $M_n$. It is an elementary fact of algebra that
all derivations of $M_n$ are inner. This means that every element $X$ of $\text{Der}(M_n)$ is of
the form $X = \text{ad} f$ for some $f$ in $M_n$. The derivation $X$ is real if and only if $f$ is

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anti-hermitian. The main difference with the commutative case lies in the fact that Der($M_n$) is not an $M_n$-module; if $X \in \text{Der}(M_n)$ and $f \in M_n$ then in general $fX$ is not in Der($M_n$). It is however a free module over the center of $M_n$, that is, over the complex numbers; it is a vector space of dimension $n^2 - 1$. This is the analogue of the fact that the global vector fields on a parallelizable manifold $V$ form a free $C(V)$-module.

Let $\lambda_a$, for $1 \leq a \leq n^2 - 1$, be an anti-hermitian basis of the Lie algebra of the special unitary group $SU_n$. The product $\lambda_a \lambda_b$ can be written in the form

\begin{equation}
\lambda_a \lambda_b = \frac{1}{2} C_{ab}^c \lambda_c + \frac{1}{2} D_{ab}^c \lambda_c - \frac{1}{n} g_{ab}.
\end{equation}

The $g_{ab}$ are the components of the Killing metric, a positive, nondegenerate inner product on the Lie algebra. We have suppressed the unit element of $M_n$ which should appear in the above formula as a factor of $g_{ab}$. The structure constants $C_{ab}^c$ are real. The Killing metric can be defined in terms of them by the equation

\begin{equation}
g_{ab} = \frac{1}{2n} C_{ad}^c C_{bc}^d.
\end{equation}

We shall lower and raise indices with $g_{ab}$ and its inverse $g^{ab}$. The tensor $C_{abc}$ is completely antisymmetric and $D_{abc}$ is completely symmetric and trace-free. We shall normalize the $\lambda_a$ so that $g_{ab} = \delta_{ab}$.

The set of $\lambda_a$ is a set of generators of $M_n$. It is not a minimal set but it is convenient because of the fact that the derivations

\begin{equation} e_a = \text{ad} \lambda_a \end{equation}

form a basis over the complex numbers for Der($M_n$). We recall that the adjoint action is defined by

\begin{equation} e_a f = \text{ad} \lambda_a (f) = [\lambda_a, f], \quad f \in M_n. \end{equation}

Any element $X$ of Der($M_n$) can be written as a linear combination $X = X^a e_a$ of the $e_a$ where the $X^a$ are complex numbers.

The vector space Der($M_n$) has a Lie-algebra structure. In particular the derivations $e_a$ satisfy the commutation relations

\begin{equation} [e_a, e_b] = C_{ab}^c e_c. \end{equation}

We define the algebra of forms $\Omega^*(M_n)$ over $M_n$ just as we did in the commutative case, as the graded algebra of multilinear, completely antisymmetric maps of the derivations into the algebra. The product is defined also using the same formula [14]. First we set $\Omega^0(M_n)$ to be equal to $M_n$. Then we define $df$ for $f \in M_n$ by the equation

\begin{equation} df(e_a) = e_a (f). \end{equation}

This means in particular that

\begin{equation} d \lambda^a (e_b) = [\lambda_b, \lambda^a] = -C_{bc}^a \lambda^c. \end{equation}

We define the set of 1-forms $\Omega^1(M_n)$ to be the set of all elements of the form $f dg$ with $f$ and $g$ in $M_n$. So $\Omega^1(M_n)$ is a left $M_n$-module. We could have also defined $\Omega^1(M_n)$ to be the set of all elements of the form $(dg)f$ which would define $\Omega^1(M_n)$ as a right module. Since $M_n$ is not a commutative algebra one has to distinguish between left and right multiplication. Although $f dg$ and $(dg)f$ are not
equal the two definitions of $\Omega^1(M_n)$ coincide as a bimodule because of the relation $d(fg) = f(dg) + (df)g$. A differential form of order $p$ or $p$-form is defined exactly as in the commutative case, as are the product and the differential $d$. As in the commutative case we have the formulae

\begin{equation}
(2.8) \quad d(\alpha \beta) = (d\alpha)\beta + (-1)^p \alpha d\beta
\end{equation}

as well as

\begin{equation}
(2.9) \quad d^2 = 0.
\end{equation}

The algebra $\Omega^*(M_n)$ is a graded differential algebra. It is easy to see however that if $\alpha, \beta \in \Omega^*(M_n)$ then in general

\begin{equation}
(2.10) \quad \alpha \beta \neq \pm \beta \alpha.
\end{equation}

The set of $d\lambda^a$ constitutes a system of generators of $\Omega^1(M_n)$ as a left or right module but it is not a convenient one. For example $\lambda^a d\lambda^b \neq d\lambda^b \lambda^a$. There is a better system of generators completely characterized by the equations

\begin{equation}
(2.11) \quad \theta^a(e_b) = \delta^a_b.
\end{equation}

We have suppressed the unit matrix which should appear as a factor of $\delta^a_b$ on the right-hand side of this equation. The $\theta^a$ are related to the $d\lambda^a$ by the equations

\begin{equation}
(2.12) \quad d\lambda^a = C^{a}_{bc} \lambda^b \theta^c
\end{equation}

and their inverse

\begin{equation}
(2.13) \quad \theta^a = \lambda_b \lambda^a d\lambda^b.
\end{equation}

They form a basis of the 1-forms dual to the derivations 2.3. Equation 2.12 follows immediately from 2.7 and 2.11 but the proof of the inverse Equation 2.13 uses the identities [30]

\begin{align*}
C^{a}_{bc} C^{c}_{de} f_a &= -n C^{d}_{bdf}, \\
C^{a}_{bc} C^{c}_{de} D^e f_a &= -n D^{d}_{bdf}, \\
C^{a}_{bc} D^c_{de} D^e f_a &= -\frac{1}{n} (n^2 - 4) C_{bdf}
\end{align*}

between the structure constants and the tensor $D_{abc}$.

Because of the relation 2.11 we have

\begin{equation}
(2.14) \quad \theta^a \wedge \theta^b = -\theta^b \wedge \theta^a, \quad \lambda^a \theta^b = \theta^b \lambda^a.
\end{equation}

The $\theta^a$ satisfy the same structure equations as the components of the Maurer-Cartan form on the special unitary group $SU_n$:

\begin{equation}
(2.15) \quad d\theta^a = -\frac{1}{2} C^{a}_{bc} \theta^b \wedge \theta^c.
\end{equation}

The product on the right-hand side of this formula is the product in $\Omega^*(M_n)$. Although this product is not in general antisymmetric, the subalgebra $\wedge^n$ of $\Omega^*(M_n)$ generated by the $\theta^a$ is an exterior algebra. Formula 2.15 means that it is a differential subalgebra. Since the $\theta^a$ commute with the elements of $M_n$ we can identify $\Omega^*(M_n)$ with the tensor product of $M_n$ and $\wedge^n$:

\begin{equation}
(2.16) \quad \Omega^*(M_n) = M_n \otimes \wedge^n.
\end{equation}

The interior product and the Lie derivative are defined using the same formulae as in the commutative case.
From the generators $\theta^a$ we can construct the 1-form
\[(2.17)\quad \theta = -\lambda_a \theta^a\]
in $\Omega^1(M_n)$. From 2.13 we see that it can also be written as
\[(2.18)\quad \theta = -\frac{1}{n} \lambda_a d\lambda^a = -\frac{1}{n} d\lambda_a \lambda^a.\]
Using $\theta$ we can rewrite 2.13 as
\[(2.19)\quad \theta^a = C^a_{bc} \lambda^b d\lambda^c - n \lambda^a \theta = -C^a_{bc} \lambda^b \theta \lambda^c.
\]
Apart from the second term on the right-hand side this equation is related to 2.12 by an interchange of $d\lambda^a$ and $\theta^a$. From 2.12 and 2.15 one sees that $\theta$ satisfies the condition
\[(2.20)\quad d\theta + \theta^2 = 0.\]
It satisfies with respect to the algebraic exterior derivative the same equation which the Maurer-Cartan form satisfies with respect to ordinary exterior derivation on the group $SU_n$.

It follows directly from the definitions that the exterior derivative $df$ of an element of $M_n$ can be written in terms of a commutator with $\theta$:
\[(2.21)\quad df = -[\theta, f].\]
This is not true however for an arbitrary element of $\Omega^*(M_n)$ and this accounts for the difference between the differential calculus we are here considering and that of the next section.

There is a map of the trace-free elements of $M_n$ onto the derivations of $M_n$ given by $f \mapsto X_f = iad f$. The factor $i$ has been included to make $X_f$ real when $f$ is hermitian. The 1-form $\theta$ can be defined, without any reference to the $\theta^a$, as the inverse map:
\[(2.22)\quad \theta(X_f) = -if.\]

The complete set of all derivations of $M_n$ is the natural analogue of the space of all smooth vector fields $\text{Der}(C(V))$ on a manifold $V$. If $V$ is parallelizable then $\text{Der}(C(V))$ is a free module over $C(V)$ with a set of generators $e_a$ which is closed under the Lie bracket and which has the property that if $e_a f = 0$ for all $e_a$ then $f$ is a constant function. Consider the Lie algebra 2.5 with the $e_a$ defined by 2.3. We have supposed that the $\lambda_a$ are elements of the fundamental representation of $su_m$, the Lie algebra of $SU_n$. We could equally well have assumed that they lie in an $n$-dimensional representation of $su_m$ for $m < n$. If the representation is irreducible they will generate $M_n$ as an algebra and the corresponding set of derivations $\text{Der}_m(M_n)$ will be a Lie subalgebra of the Lie algebra $\text{Der}(M_n)$ of all the derivations of $M_n$. This freedom of representation allows one to introduce an extra integer parameter into some of the theories which we shall mention in the last section.

3. The Dirac Operator

In the previous section we used derivations to define differential forms and the $d$ operator but we then saw in that we could have defined $d$ as acting on elements of $M_n$ by the equation $df = -[\theta, f]$. There is here no explicit reference to the set of derivations. The algebra of forms $\Omega^*(M_n)$ has a $\mathbb{Z}_2$ grading and $\theta$ is an odd element. Let $\Omega_S$ be the matrix algebra $M_n$ itself with a $\mathbb{Z}_2$ grading. That is, $\Omega_S$
can be expressed as the direct sum $\Omega_S = \Omega_S^+ \oplus \Omega_S^-$ where $\Omega_S^+$ ($\Omega_S^-$) are the even (odd) elements of $\Omega_S$. It is possible to consider $\Omega_S$ as an algebra of forms. One can define on $\Omega_S$ a graded derivation $\hat{d}$ by the formula

\begin{equation}
\hat{d} f = -[\eta, f],
\end{equation}

where $\eta$ is an arbitrary anti-hermitian odd element of $\Omega_S$ and the commutator is taken as a graded commutator. We find that $\hat{d}\eta = -2\eta^2$ and for any $\alpha \in \Omega_S$,

\begin{equation}
\hat{d}^2\alpha = [\eta^2, \alpha].
\end{equation}

The $\mathbb{Z}_2$ grading can be induced from a decomposition $\mathbb{C}^n = \mathbb{C}^l \oplus \mathbb{C}^{n-l}$ for some integer $l$. The elements of $\Omega_S^+$ are diagonal with respect to the decomposition; the elements of $\Omega_S^-$ are off-diagonal. If $n$ is even and $l = n/2$ then it is possible to impose the condition

\begin{equation}
\eta^2 = -1.
\end{equation}

The unit on the right-hand side of this equation is the unit in $\Omega_S$. From 3.2 we see that $\hat{d}^2 = 0$. The map 3.1 is a differential. In this case we shall write

\begin{equation}
\hat{d} = \hat{d}.
\end{equation}

When 3.3 is satisfied the algebra $\Omega_S$ is a differential algebra with a $\mathbb{Z}_2$ grading. Notice that 2.20 is not satisfied for $\eta$. We have instead

\begin{equation}
\hat{d}\eta + \eta^2 = 1.
\end{equation}

The reason for the difference is that initially a relation of the form 3.1 was valid only for elements of $\Omega^0(M_n)$. In defining the second differential $d$ we have imposed it on all elements of $\Omega_S$.

If $n$ is not even or, in general, if $\eta^2$ is not proportional to the unit element of $M_n$ then $\hat{d}^2$ given by 3.2 will not vanish and $\Omega_S$ will not be a differential algebra. Consider now an arbitrary grading of $M_n$ and let $\eta$ be an arbitrary odd element. It is possible to construct over $M_n^+$ an integer-graded differential algebra $\Omega^+ = \Omega^0(M_n^+)$ based on Formula 3.1. Let $\Omega^0 = M_n^+$ and let $\Omega^1 = \hat{d}\Omega^0 \subset M_n^-$ be the $M_n^+$-bimodule generated by the image of $\Omega^0$ in $M_n^-$ under $\hat{d}$. Define

\begin{equation}
\Omega^0 \xrightarrow{d_\eta} \Omega^1
\end{equation}

using directly 3.1: $d_\eta = \hat{d}$. Let $\hat{d}\Omega^1$ be the $M_n^+$-module generated by the image of $\Omega^1$ in $M_n^+$ under $\hat{d}$. It would be natural to try to set $\Omega^2 = \hat{d}\Omega^1$ and define

\begin{equation}
\Omega^1 \xrightarrow{d_\eta} \Omega^2
\end{equation}

using once again 3.1. Every element of $\Omega^1$ can be written as a sum of elements of the form $f_0\hat{d}f_1$. If we attempt to define an application 3.7 using again directly 3.1,

\begin{equation}
d_\eta(f_0\hat{d}f_1) = \hat{d}f_0\hat{d}f_1 + f_0\hat{d}^2f_1,
\end{equation}

then we see that in general $d_\eta^2$ does not vanish. To remedy this problem we eliminate simply the unwanted terms. Let $\text{Im} \hat{d}^2$ be the submodule of $\hat{d}\Omega^1$ consisting of those elements which contain a factor which is the image of $\hat{d}^2$ and define $\Omega^2$ by

\begin{equation}
\Omega^2 = \frac{\hat{d}\Omega^1}{\text{Im} \hat{d}^2}.
\end{equation}
Then by construction the second term on the right-hand side of 3.8 vanishes as an element of $\Omega_\eta^2$ and we have a well defined map 3.7 with $d_\eta^2 = 0$. This procedure can be continued to arbitrary order by iteration. For each $p \geq 2$ we let $\text{Im} \, d^2$ be the submodule of $d\Omega_\eta^{p-1}$ defined as above and we define $\Omega_\eta^p$ by

$$\Omega_\eta^p = \frac{d\Omega_\eta^{p-1}}{\text{Im} \, d^2}.$$  

(3.10)

Since $\Omega_\eta^n \Omega_\eta^q \subset \Omega_\eta^{n+q}$ the complex $\Omega_\eta^n$ is a differential algebra. The $\Omega_\eta^p$ need not vanish for large values of $p$. In fact if $\eta^2 \propto 1$ we see that $\text{Im} \, d^2 = 0$ and $\Omega_\eta^p \simeq M_1^+(M_n^-)$ for $p$ even (odd). An equivalent construction of $\Omega_\eta^n$ can be given [10, 25] using what is known as the universal differential calculus [6, 7, 8, 27].

Let $n = 2$. Then $\Omega_S = M_2$ with a $\mathbb{Z}_2$ grading. The most general element of $\Omega_S$ can be expanded using the 4 matrices $\{\lambda_i, 1\}$ with $\lambda_1$ and $\lambda_2$ odd and $\lambda_3$ and the unit even. The most general possible form for $\eta$ is a linear combination of $\lambda_1$ and $\lambda_2$ and it can be normalized so that 3.3 is satisfied. Then $\Omega_\eta$ is a differential algebra of dimension 4. The algebra of forms $\Omega^*(M_2)$ constructed using the differential of the previous section is a $4 \times 8 = 32$ dimensional algebra. Over $M_2$ we have constructed then two differential calculi, one with an algebra of dimension 32 and a second of dimension 4, although more properly the algebra $\Omega_S$ should be considered as an algebra of forms over its even elements $M_1^+$. Using $\Omega_S$ one can construct a differential calculus over $C(\mathbb{R}^4) \times C(\mathbb{R}^4)$, the algebra of smooth functions on a double-sheeted space-time. This is one of the models [9] which we shall mention in the last section.

Let $n = 3$. Then $\Omega_S = M_3$ with a $\mathbb{Z}_2$ grading defined by the decomposition $\mathbb{C}^3 = \mathbb{C}^2 \oplus \mathbb{C}$. The most general possible form for $\eta$ is

$$\eta = \begin{pmatrix} 0 & 0 & a_1 \\ 0 & 0 & a_2 \\ -a_1 & -a_2 & 0 \end{pmatrix}$$  

(3.11)

For no values of the parameters can 3.3 be satisfied. The general construction yields $\Omega_\eta^0 = M_1^+ = M_2 \times M_1$ and $\Omega_\eta^1 = M_1^-$ as in the previous example but after that the quotient by elements of the form $\text{Im} \, d^2$ reduces the dimensions. One finds $\Omega_\eta^n = M_1$ and $\Omega_\eta^p = 0$ for $p \geq 3$. This differential calculus is closely related to that used in one of the models [10] which we shall mention in the last section.

4. The comparison

Now we show that the differential calculus of Section 2 can be put in the form of the language of Section 3. Set $N = n^22^n2^{-1}$. Then $\Omega^*(M_n)$ is a complex $N$-dimensional vector space with a natural $\mathbb{Z}_2$ grading. The algebra $M_N$ has therefore also a natural $\mathbb{Z}_2$ grading. There is an imbedding of $\Omega^*(M_n)$ in $M_N$ by left multiplication. We designate the image of a form $\alpha$ in $M_N$ by $\hat{\alpha}$. The differential $d$ has an extension $\hat{d}$ to all elements of $M_N$. Let $L$ be an element of $M_N$ with parity $|L|$ and $\alpha$ an arbitrary element of $\Omega^*(M_n)$. Define $\hat{d}L$ by

$$\hat{d}L\alpha = d(L\alpha) - (-1)^{|L|}Ld\alpha.$$  

(4.1)

Using the graded commutator this can be written in the form

$$\hat{d}L = [d, L].$$  

(4.2)
That is, \( \hat{d} \) is just the commutator of \( d \) considered as an element of \( M_N \). In particular \( \hat{d} \alpha = [d, \alpha] = \overrightarrow{d} \alpha \). We have seen that when acting on the elements of \( \Omega^0(M_n) = M_n \) the exterior derivative \( d \) can be expressed as a commutator with the element \( \theta \). Although in general it is not possible to write the right-hand side of 4.2 as the image of a commutator in \( \Omega^*(M_n) \), there is a sense in which this can almost be done. That is for any \( \alpha \in \Omega^*(M_n) \) we can write

\[
\hat{d} \alpha = -[\eta, \alpha]
\]

where \( \eta \) is almost the image of an element of \( \Omega^*(M_n) \).

Let \( c^a = \hat{\theta}^a \) be the element of \( M_N \) which corresponds to left-multiplication by the element \( \theta^a \) of \( \Omega^1(M_n) \) and let \( \overline{c}_a \) be the element of \( M_N \) which corresponds to the interior product by \( e_a \). Both \( c^a \) and \( \overline{c}_a \) are odd elements of \( M_N \) and their (graded) commutation rules are

\[
[c^a, c^b] = 0, \quad [\overline{c}_a, \overline{c}_b] = 0, \quad [\overline{c}_a, c^b] = \delta^b_a.
\]

They commute with the elements \( \hat{\lambda}^a \). Let \( \hat{\theta} = -\hat{\lambda}_a c^a \) be the image of \( \theta \) in \( M_N \) and define \( \eta \) as

\[
\eta = \hat{\theta} + \frac{1}{2} C^a_{b c} c^b c^c \overline{c}_a.
\]

It is easy to verify that 4.3 is satisfied. It follows immediately from the definitions that

\[
\hat{d} c^a = -[\eta, c^a] = -\frac{1}{2} C^a_{b c} c^b c^c.
\]

Let \( L_a \) be the Lie derivative with respect to \( e_a \). It is interesting to note also that

\[
\hat{d} \overline{c}_a = L_a, \quad \hat{d} L_a = 0.
\]

But because \( \overline{c}_a \) is not in the image of \( \Omega^*(M_n) \) in \( M_N \) the expression \( \hat{d} \overline{c}_a + [\eta, \overline{c}_a] \) does not vanish. It commutes however with all elements of the image of \( \hat{\Omega}^*(M_n) \) in \( M_N \). We have

\[
\eta^2 = 0,
\]

and therefore \( \hat{d}^2 = 0 \). Consider the algebra \( \hat{\Omega}^*_\eta \) constructed from the \( \eta \) defined in 4.5. There is a natural imbedding of differential algebras

\[
\Omega^*(M_n) \hookrightarrow \hat{\Omega}^*_\eta.
\]

We construct the map \( i \) by iteration. First of all there is the imbedding of \( \Omega^0(M_n) = M_n \) into \( \Omega^0_\eta = M_N^\eta \) given by left multiplication. We extend \( i \) to \( \Omega^1(M_n) \) by setting

\[
i(df) = -[\eta, f] = d_\eta i(f).
\]

Since both \( \Omega^*(M_n) \) and \( \Omega^*_\eta \) are graded differential algebras 4.9 is uniquely defined. The notation in this example is adapted from the standard notation used in the application of ghosts in the quantization of gauge fields.

In the above example we imbedded a differential algebra in a larger algebra such that in the latter the differential was given by a commutator. The larger algebra was the algebra of homomorphisms of the original algebra considered as a vector space. This construction can be generalized to arbitrary algebras. There is also another construction which uses a different extension. Let \( A \) be any (graded) algebra and \( d \) a (graded) derivation. It is possible to imbed \( A \) into an algebra \( B \)
such that the derivation on the image is given by a commutator. We construct the larger algebra by simply adding to the original one an (odd) element \( \eta \) and defining the product \( \eta a \) and \( a \eta \) such that the (graded) commutator is given by \( \{ \eta, a \} \). In general \( d \) need not be defined on \( \eta \). In the previous example we added in fact \( n \) elements \( \bar{e}_a \) to the original algebra. The larger algebra constructed was contained in \( M_N \).

Connes has introduced [6, 7, 8] the noncommutative generalization of a K-cycle or spectral triple \((\mathcal{A}, \mathcal{H}, D)\). This consists of an algebra \( \mathcal{A} \), a representation of \( \mathcal{A} \) on a graded Hilbert space \( \mathcal{H} \) and an odd operator \( D \) which is a generalization of the Dirac operator. From the spectral triple a differential calculus can be constructed. We have defined two examples of spectral triples. The first, \((M_n, \mathbb{C}^n, \eta)\) was used to construct the differential calculus used in the Connes-Lott model [9, 10]. The second, \((M_N, \mathbb{C}^N, \eta)\) was implicit in the construction of Dubois-Violette et al. [15, 16, 17, 18].

5. Extended Space-Time

The first attempts to introduce extra dimensions in order to unify the gravitational field with electromagnetism were made by Kaluza and Klein. It was suggested by Einstein & Bergmann that at sufficiently small scales what appears as a point will in fact be seen as a circle. Later, with the advent of more elaborate gauge fields, it was proposed that this internal manifold could be taken as a compact Lie group or even as a general compact manifold [1, 2]. The disadvantage of these extra dimensions is that they introduce even more divergences in the quantum theory and lead to an infinite spectrum of new particles. One can take this as motivation for introducing a modification of Kaluza-Klein theory with an internal structure which is described by a finite noncommutative geometry [16, 31, 3, 33].

Closely related to Kaluza-Klein theory is the study of Maxwell's equations in higher-dimensions. As has been shown by Forgacs & Manton [35, 21] and others [20, 4, 22] if one considers the Maxwell action on a principle fibre bundle over space-time with group \( G \) then the resulting effective action is that of a Yang-Mills theory unified with a Higgs field which takes its value in the Lie algebra of \( G \). The quartic potential appears naturally since the Yang-Mills action is quartic in the gauge potentials. Like all theories in higher dimensions, these theories include an infinite tower of massive states over each massless state. A rather artificial truncation procedure must be introduced to eliminate them. At least in the cases where the group \( G \) is one of the unitary groups \( U_n \), equivalent theories can be reproduced without truncation by using a finite noncommutative extension of space-time based on the algebras \( M_n \) of complex \( n \times n \) matrices [15, 16, 17, 18] and by using the differential calculus based on derivations [14] which we reviewed in Section 2. The resulting physical theories [31, 32] contain only one free parameter and are not realistic. In order to reproduce the standard model with a set of Higgs fields in the fundamental representation of the gauge group it is necessary to use the more abstract differential calculi which we reviewed in Section 3, either the \( \mathbb{Z}_2 \)-based calculus [9, 12, 13] or its \( \mathbb{Z} \)-based extension [10, 12, 23, 28] For more details of this we refer to the numerous review articles on the subject [27, 37, 24].
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