Contemporary Mathematics
Volume 203, 1997

On Finite Differential Calculi

D. KASTLER, J. MADORE, and T. MASSON

ABSTRACT. In the case of matrix algebras, a comparison is made between the differential calculus based on derivations and that based on the Dirac operator. In particular the appropriate Dirac operator is given which reproduces the differential calculus based on derivations.

1. Introduction

The extension to noncommutative algebras of the definition of a differential calculus has been given both without [**6**] and with [**14**] the use of the derivations of the algebra. The former is of course a more general procedure since it can be used even in the case of algebras which have no derivations, for example the algebras of continuous or measurable functions on a differential manifold. At least in the finite-dimensional case which we shall consider here a comparison between the two approaches is readily made. In particular the differential calculus of an algebra based on derivations can be easily put in the form of the differential calculus introduced by Connes & Lott [**10**] which is based on a generalization of the Dirac operator.

In Section 2 we shall give a brief description of the differential calculus based on derivations. In Section 3 we shall do the same for the differential calculus which is based on a generalization of the Dirac operator. In Section 4 we show the relation between the two. In Section 5 we briefly recall how physical theories can be constructed by considering in conjunction with these finite calculi the ordinary differential calculus of space-time.

2. Derivations

A smooth global vector field on a manifold V is equivalent to a derivation of the algebra of smooth functions $\mathcal{C}(V)$ defined on V [**29**]. As such it has a natural extension to noncommutative geometry [**6, 7, 8**]. The algebras M_n of $n \times n$ complex matrices furnish finite analogues of compact parallelizable manifolds. Let $\mathrm{Der}(M_n)$ be the vector space of all derivations of M_n. It is an elementary fact of algebra that all derivations of M_n are inner. This means that every element X of $\mathrm{Der}(M_n)$ is of the form $X = \mathrm{ad}\, f$ for some f in M_n. The derivation X is real if and only if f is

1991 *Mathematics Subject Classification.* Primary 47C10; Secondary 14A22 .
Supported in part by the CNRS.

anti-hermitian. The main difference with the commutative case lies in the fact that $\mathrm{Der}(M_n)$ is not an M_n-module; if $X \in \mathrm{Der}(M_n)$ and $f \in M_n$ then in general fX is not in $\mathrm{Der}(M_n)$. It is however a free module over the center of M_n, that is, over the complex numbers; it is a vector space of dimension $n^2 - 1$. This is the analogue of the fact that the global vector fields on a parallelizable manifold V form a free $\mathcal{C}(V)$-module.

Let λ_a, for $1 \leq a \leq n^2 - 1$, be an anti-hermitian basis of the Lie algebra of the special unitary group SU_n. The product $\lambda_a \lambda_b$ can be written in the form

$$\lambda_a \lambda_b = \frac{1}{2} C^c{}_{ab} \lambda_c + \frac{1}{2} D^c{}_{ab} \lambda_c - \frac{1}{n} g_{ab}. \tag{2.1}$$

The g_{ab} are the components of the Killing metric, a positive, nondegenerate inner product on the Lie algebra. We have suppressed the unit element of M_n which should appear in the above formula as a factor of g_{ab}. The structure constants $C^c{}_{ab}$ are real. The Killing metric can be defined in terms of them by the equation

$$g_{ab} = -\frac{1}{2n} C^c{}_{ad} C^d{}_{bc}. \tag{2.2}$$

We shall lower and raise indices with g_{ab} and its inverse g^{ab}. The tensor C_{abc} is completely antisymmetric and D_{abc} is completely symmetric and trace-free. We shall normalize the λ_a so that $g_{ab} = \delta_{ab}$.

The set of λ_a is a set of generators of M_n. It is not a minimal set but it is convenient because of the fact that the derivations

$$e_a = \mathrm{ad}\, \lambda_a \tag{2.3}$$

form a basis over the complex numbers for $\mathrm{Der}(M_n)$. We recall that the adjoint action is defined by

$$e_a f = \mathrm{ad}\, \lambda_a(f) = [\lambda_a, f], \quad f \in M_n. \tag{2.4}$$

Any element X of $\mathrm{Der}(M_n)$ can be written as a linear combination $X = X^a e_a$ of the e_a where the X^a are complex numbers.

The vector space $\mathrm{Der}(M_n)$ has a Lie-algebra structure. In particular the derivations e_a satisfy the commutation relations

$$[e_a, e_b] = C^c{}_{ab}\, e_c. \tag{2.5}$$

We define the algebra of forms $\Omega^*(M_n)$ over M_n just as we did in the commutative case, as the graded algebra of multilinear, completely antisymmetric maps of the derivations into the algebra. The product is defined also using the same formula [14]. First we set $\Omega^0(M_n)$ to be equal to M_n. Then we define df for $f \in M_n$ by the equation

$$df(e_a) = e_a(f). \tag{2.6}$$

This means in particular that

$$d\lambda^a(e_b) = [\lambda_b, \lambda^a] = -C^a{}_{bc} \lambda^c. \tag{2.7}$$

We define the set of 1-forms $\Omega^1(M_n)$ to be the set of all elements of the form fdg with f and g in M_n. So $\Omega^1(M_n)$ is a left M_n-module. We could have also defined $\Omega^1(M_n)$ to be the set of all elements of the form $(dg)f$ which would define $\Omega^1(M_n)$ as a right module. Since M_n is not a commutative algebra one has to distinguish between left and right multiplication. Although fdg and $(dg)f$ are not equal the two definitions of $\Omega^1(M_n)$ coincide as a bimodule because of the relation $d(fg) = f(dg) + (df)g$. A differential form of order p or p-form is defined exactly as in the commutative case, as are the product and the differential d. As in the commutative case we have the formulae

$$d(\alpha\beta) = (d\alpha)\beta + (-1)^p \alpha d\beta \tag{2.8}$$

as well as

$$d^2 = 0. \tag{2.9}$$

The algebra $\Omega^*(M_n)$ is a graded differential algebra. It is easy to see however that if $\alpha, \beta \in \Omega^*(M_n)$ then in general

$$\alpha\beta \neq \pm\beta\alpha. \tag{2.10}$$

The set of $d\lambda^a$ constitutes a system of generators of $\Omega^1(M_n)$ as a left or right module but it is not a convenient one. For example $\lambda^a d\lambda^b \neq d\lambda^b \lambda^a$. There is a better system of generators completely characterized by the equations

$$\theta^a(e_b) = \delta^a_b. \tag{2.11}$$

We have suppressed the unit matrix which should appear as a factor of δ^a_b on the right-hand side of this equation. The θ^a are related to the $d\lambda^a$ by the equations

$$d\lambda^a = C^a{}_{bc}\lambda^b\theta^c \tag{2.12}$$

and their inverse

$$\theta^a = \lambda_b\lambda^a d\lambda^b. \tag{2.13}$$

They form a basis of the 1-forms dual to the derivations 2.3. Equation 2.12 follows immediately from 2.7 and 2.11 but the proof of the inverse Equation 2.13 uses the identities [30]

$$\begin{aligned}
C^a{}_{bc}C^c{}_{de}C^e{}_{fa} &= -nC_{bdf}, \\
C^a{}_{bc}C^c{}_{de}D^e{}_{fa} &= -nD_{bdf}, \\
C^a{}_{bc}D^c{}_{de}D^e{}_{fa} &= -\frac{1}{n}(n^2-4)C_{bdf}
\end{aligned}$$

between the structure constants and the tensor D_{abc}.

Because of the relation 2.11 we have

$$\theta^a \wedge \theta^b = -\theta^b \wedge \theta^a, \qquad \lambda^a\theta^b = \theta^b\lambda^a. \tag{2.14}$$

The θ^a satisfy the same structure equations as the components of the Maurer-Cartan form on the special unitary group SU_n:

$$d\theta^a = -\frac{1}{2}C^a{}_{bc}\,\theta^b \wedge \theta^c. \tag{2.15}$$

The product on the right-hand side of this formula is the product in $\Omega^*(M_n)$. Although this product is not in general antisymmetric, the subalgebra $\bigwedge^*$ of $\Omega^*(M_n)$ generated by the θ^a is an exterior algebra. Formula 2.15 means that it is a differential subalgebra. Since the θ^a commute with the elements of M_n we can identify $\Omega^*(M_n)$ with the tensor product of M_n and $\bigwedge^*$:

$$\Omega^*(M_n) = M_n \otimes_{\mathbb{C}} \bigwedge^*. \tag{2.16}$$

The interior product and the Lie derivative are defined using the same formulae as in the commutative case.

From the generators θ^a we can construct the 1-form

$$\theta = -\lambda_a \theta^a \tag{2.17}$$

in $\Omega^1(M_n)$. From 2.13 we see that it can also be written as

$$\theta = -\frac{1}{n}\lambda_a d\lambda^a = \frac{1}{n} d\lambda_a \lambda^a. \tag{2.18}$$

Using θ we can rewrite 2.13 as

$$\theta^a = C^a{}_{bc}\lambda^b d\lambda^c - n\lambda^a\theta = -C^a{}_{bc}\lambda^b\theta\lambda^c. \tag{2.19}$$

Apart from the second term on the right-hand side this equation is related to 2.12 by an interchange of $d\lambda^a$ and θ^a. From 2.12 and 2.15 one sees that θ satisfies the condition

$$d\theta + \theta^2 = 0. \tag{2.20}$$

It satisfies with respect to the algebraic exterior derivative the same equation which the Maurer-Cartan form satisfies with respect to ordinary exterior derivation on the group SU_n. It follows directly from the definitions that the exterior derivative df of an element of M_n can be written in terms of a commutator with θ:

$$df = -[\theta, f]. \tag{2.21}$$

This is not true however for an arbitrary element of $\Omega^*(M_n)$ and this accounts for the difference between the differential calculus we are here considering and that of the next section.

There is a map of the trace-free elements of M_n onto the derivations of M_n given by $f \mapsto X_f = i\operatorname{ad} f$. The factor i has been included to make X_f real when f is hermitian. The 1-form θ can be defined, without any reference to the θ^a, as the inverse map:

$$\theta(X_f) = -if. \tag{2.22}$$

The complete set of all derivations of M_n is the natural analogue of the space of all smooth vector fields $\mathrm{Der}(\mathcal{C}(V))$ on a manifold V. If V is parallelizable then $\mathrm{Der}(\mathcal{C}(V))$ is a free module over $\mathcal{C}(V)$ with a set of generators e_α which is closed under the Lie bracket and which has the property that if $e_\alpha f = 0$ for all e_α then f is a constant function. Consider the Lie algebra 2.5 with the e_a defined by 2.3. We have supposed that the λ_a are elements of the fundamental representation of $\underline{su}_n$, the Lie algebra of SU_n. We could equally well have assumed that they lie in an n-dimensional representation of $\underline{su}_m$ for $m < n$. If the representation is irreducible they will generate M_n as an algebra and the corresponding set of derivations $\mathrm{Der}_m(M_n)$ will be a Lie subalgebra of the Lie algebra $\mathrm{Der}(M_n)$ of all the derivations of M_n. This freedom of representation allows one to introduce an extra integer parameter into some of the theories which we shall mention in the last section.

3. The Dirac Operator

In the previous section we used derivations to define differential forms and the d operator but we then saw in that we could have defined d as acting on elements of M_n by the equation $df = -[\theta, f]$. There is here no explicit reference to the set of derivations. The algebra of forms $\Omega^*(M_n)$ has a $\mathbb{Z}_2$ grading and θ is an odd element. Let Ω_S be the matrix algebra M_n itself with a $\mathbb{Z}_2$ grading. That is, Ω_S

can be expressed as the direct sum $\Omega_S = \Omega_S^+ \oplus \Omega_S^-$ where Ω_S^+ (Ω_S^-) are the even (odd) elements of Ω_S. It is possible to consider Ω_S as an algebra of forms. One can define on Ω_S a graded derivation $\hat{d}$ by the formula

$$\hat{d}f = -[\eta, f], \tag{3.1}$$

where η is an arbitrary anti-hermitian odd element of Ω_S and the commutator is taken as a graded commutator. We find that $\hat{d}\eta = -2\eta^2$ and for any $\alpha \in \Omega_S$,

$$\hat{d}^2\alpha = [\eta^2, \alpha]. \tag{3.2}$$

The $\mathbb{Z}_2$ grading can be induced from a decomposition $\mathbb{C}^n = \mathbb{C}^l \oplus \mathbb{C}^{n-l}$ for some integer l. The elements of Ω_S^+ are diagonal with respect to the decomposition; the elements of Ω_S^- are off-diagonal. If n is even and $l = n/2$ then it is possible to impose the condition

$$\eta^2 = -1. \tag{3.3}$$

The unit on the right-hand side of this equation is the unit in Ω_S. From 3.2 we see that $\hat{d}^2 = 0$. The map 3.1 is a differential. In this case we shall write

$$\hat{d} = d. \tag{3.4}$$

When 3.3 is satisfied the algebra Ω_S is a differential algebra with a $\mathbb{Z}_2$ grading. Notice that 2.20 is not satisfied for η. We have instead

$$d\eta + \eta^2 = 1. \tag{3.5}$$

The reason for the difference is that initially a relation of the form 3.1 was valid only for elements of $\Omega^0(M_n)$. In defining the second differential d we have imposed it on all elements of Ω_S.

If n is not even or, in general, if η^2 is not proportional to the unit element of M_n then $\hat{d}^2$ given by 3.2 will not vanish and Ω_S will not be a differential algebra. Consider now an arbitrary grading of M_n and let η be an arbitrary odd element. It is possible to construct over M_n^+ an integer-graded differential algebra $\Omega_\eta^* = \Omega_\eta^*(M_n^+)$ based on Formula 3.1. Let $\Omega_\eta^0 = M_n^+$ and let $\Omega_\eta^1 \equiv \overline{d\Omega_\eta^0} \subset M_n^-$ be the M_n^+-bimodule generated by the image of Ω_η^0 in M_n^- under $\hat{d}$. Define

$$\Omega_\eta^0 \xrightarrow{d_\eta} \Omega_\eta^1 \tag{3.6}$$

using directly 3.1: $d_\eta = \hat{d}$. Let $\overline{d\Omega_\eta^1}$ be the M_n^+-module generated by the image of Ω_η^1 in M_n^+ under $\hat{d}$. It would be natural to try to set $\Omega_\eta^2 = \overline{d\Omega_\eta^1}$ and define

$$\Omega_\eta^1 \xrightarrow{d_\eta} \Omega_\eta^2 \tag{3.7}$$

using once again 3.1. Every element of Ω_η^1 can be written as a sum of elements of the form $f_0\hat{d}f_1$. If we attempt to define an application 3.7 using again directly 3.1,

$$d_\eta(f_0\hat{d}f_1) = \hat{d}f_0\hat{d}f_1 + f_0\hat{d}^2 f_1, \tag{3.8}$$

then we see that in general d_η^2 does not vanish. To remedy this problem we eliminate simply the unwanted terms. Let $\operatorname{Im}\hat{d}^2$ be the submodule of $\overline{d\Omega_\eta^1}$ consisting of those elements which contain a factor which is the image of $\hat{d}^2$ and define Ω_η^2 by

$$\Omega_\eta^2 = \overline{d\Omega_\eta^1}/\operatorname{Im}\hat{d}^2. \tag{3.9}$$

Then by construction the second term on the right-hand side of 3.8 vanishes as an element of Ω^2_η and we have a well defined map 3.7 with $d^2_\eta = 0$. This procedure can be continued to arbitrary order by iteration. For each $p \geq 2$ we let $\mathrm{Im}\, \hat{d}^2$ be the submodule of $\overline{d\Omega^{p-1}_\eta}$ defined as above and we define Ω^p_η by

$$\Omega^p_\eta = \overline{d\Omega^{p-1}_\eta}/\mathrm{Im}\, \hat{d}^2. \tag{3.10}$$

Since $\Omega^p_\eta \Omega^q_\eta \subset \Omega^{p+q}_\eta$ the complex Ω^*_η is a differential algebra. The Ω^p_η need not vanish for large values of p. In fact if $\eta^2 \propto 1$ we see that $\mathrm{Im}\, \hat{d}^2 = 0$ and $\Omega^p_\eta \simeq M^+_n (M^-_n)$ for p even (odd). An equivalent construction of Ω^*_η can be given [**10, 25**] using what is known as the universal differential calculus [**6, 7, 8, 27**].

Let $n = 2$. Then $\Omega_S = M_2$ with a $\mathbb{Z}_2$ grading. The most general element of Ω_S can be expanded using the 4 matrices $\{\lambda_a, 1\}$ with λ_1 and λ_2 odd and λ_3 and the unit even. The most general possible form for η is a linear combination of λ_1 and λ_2 and it can be normalized so that 3.3 is satisfied. Then Ω_S is a differential algebra of dimension 4. The algebra of forms $\Omega^*(M_2)$ constructed using the differential of the previous section is a $4 \times 8 = 32$ dimensional algebra. Over M_2 we have constructed then two differential calculi, one with an algebra of dimension 32 and a second of dimension 4, although more properly the algebra Ω_S should be considered as an algebra of forms over its even elements M^+_2. Using Ω_S one can construct a differential calculus over $\mathcal{C}(\mathbb{R}^4) \times \mathcal{C}(\mathbb{R}^4)$, the algebra of smooth functions on a double-sheeted space-time. This is one of the models [**9**] which we shall mention in the last section.

Let $n = 3$. Then $\Omega_S = M_3$ with a $\mathbb{Z}_2$ grading defined by the decomposition $\mathbb{C}^3 = \mathbb{C}^2 \oplus \mathbb{C}$. The most general possible form for η is

$$\eta = \begin{pmatrix} 0 & 0 & a_1 \\ 0 & 0 & a_2 \\ -a_1^* & -a_2^* & 0. \end{pmatrix} \tag{3.11}$$

For no values of the parameters can 3.3 be satisfied. The general construction yields $\Omega^0_\eta = M^+_3 = M_2 \times M_1$ and $\Omega^1_\eta = M^-_3$ as in the previous example but after that the quotient by elements of the form $\mathrm{Im}\, \hat{d}^2$ reduces the dimensions. One finds $\Omega^2_\eta = M_1$ and $\Omega^p_\eta = 0$ for $p \geq 3$. This differential calculus is closely related to that used in one of the models [**10**] which we shall mention in the last section.

4. The comparison

Now we show that the differential calculus of Section 2 can be put in the form of the language of Section 3. Set $N = n^2 2^{n^2-1}$. Then $\Omega^*(M_n)$ is a complex N-dimensional vector space with a natural $\mathbb{Z}_2$ grading. The algebra M_N has therefore also a natural $\mathbb{Z}_2$ grading. There is an imbedding of $\Omega^*(M_n)$ in M_N by left multiplication. We designate the image of a form α in M_N by $\hat{\alpha}$. The differential d has an extension $\hat{d}$ to all elements of M_N. Let L be an element of M_N with parity $|L|$ and α an arbitrary element of $\Omega^*(M_n)$. Define $\hat{d}L$ by

$$(\hat{d}L)\alpha = d(L\alpha) - (-1)^{|L|} L d\alpha. \tag{4.1}$$

Using the graded commutator this can be written in the form

$$\hat{d}L = [d, L]. \tag{4.2}$$

That is, $\hat{d}$ is just the commutator of d considered as an element of M_N. In particular $\hat{d}\hat{\alpha} = [d, \hat{\alpha}] = \widehat{d\alpha}$. We have seen that when acting on the elements of $\Omega^0(M_n) = M_n$ the exterior derivative d can be expressed as a commutator with the element θ. Although in general it is not possible to write the right-hand side of 4.2 as the image of a commutator in $\Omega^*(M_n)$, there is a sense in which this can almost be done. That is for any $\alpha \in \Omega^*(M_n)$ we can write

$$\hat{d}\hat{\alpha} = -[\eta, \hat{\alpha}] \tag{4.3}$$

where η is almost the image of an element of $\Omega^*(M_n)$.

Let $c^a = \hat{\theta}^a$ be the element of M_N which corresponds to left-multiplication by the element θ^a of $\Omega^1(M_n)$ and let $\bar{c}_a$ be the element of M_N which corresponds to the interior product by e_a. Both c^a and $\bar{c}_a$ are odd elements of M_N and their (graded) commutation rules are

$$[c^a, c^b] = 0, \qquad [\bar{c}_a, \bar{c}_b] = 0, \qquad [\bar{c}_a, c^b] = \delta^b_a. \tag{4.4}$$

They commute with the elements $\hat{\lambda}^a$. Let $\hat{\theta} = -\hat{\lambda}_a c^a$ be the image of θ in M_N and define η as

$$\eta = \hat{\theta} + \frac{1}{2} C^a{}_{bc} c^b c^c \bar{c}_a. \tag{4.5}$$

It is easy to verify that 4.3 is satisfied. It follows immediately from the definitions that

$$\hat{d}c^a = -[\eta, c^a] = -\frac{1}{2} C^a{}_{bc} c^b c^c. \tag{4.6}$$

Let L_a be the Lie derivative with respect to e_a. It is interesting to note also that

$$\hat{d}\bar{c}_a = L_a, \qquad \hat{d}L_a = 0. \tag{4.7}$$

But because $\bar{c}_a$ is not in the image of $\Omega^*(M_n)$ in M_N the expression $\hat{d}\bar{c}_a + [\eta, \bar{c}_a]$ does not vanish. It commutes however with all elements of the image of $\Omega^*(M_n)$ in M_N. We have

$$\eta^2 = 0, \tag{4.8}$$

and therefore $\hat{d}^2 = 0$. Consider the algebra Ω^*_η constructed from the η defined in 4.5. There is a natural imbedding of differential algebras

$$\Omega^*(M_n) \overset{i}{\hookrightarrow} \Omega^*_\eta. \tag{4.9}$$

We construct the map i by iteration. First of all there is the imbedding of $\Omega^0(M_n) = M_n$ into $\Omega^0_\eta = M_N^+$ given by left multiplication. We extend i to $\Omega^1(M_n)$ by setting

$$i(df) = -[\eta, \hat{f}] = d_\eta i(f). \tag{4.10}$$

Since both $\Omega^*(M_n)$ and Ω^*_η are graded differential algebras 4.9 is uniquely defined. The notation in this example is adapted from the standard notation used in the application of ghosts in the quantization of gauge fields.

In the above example we imbedded a differential algebra in a larger algebra such that in the latter the differential was given by a commutator. The larger algebra was the algebra of homomorphisms of the original algebra considered as a vector space. This construction can be generalized to arbitrary algebras. There is also another construction which uses a different extension. Let $\mathcal{A}$ be any (graded) algebra and d a (graded) derivation. It is possible to imbed $\mathcal{A}$ into an algebra $\mathcal{B}$

such that the derivation on the image is given by a commutator. We construct the larger algebra by simply adding to the original one an (odd) element η and defining the product ηa and $a\eta$ such that the (graded) commutator is given by $da = -[\eta, a]$. In general d need not be defined on η. In the previous example we added in fact n elements $\bar{c}_a$ to the original algebra. The larger algebra constructed was contained in M_N.

Connes has introduced **[6, 7, 8]** the noncommutative generalization of a K-cycle or spectral triple $(\mathcal{A}, \mathcal{H}, D)$. This consists of an algebra $\mathcal{A}$, a representation of $\mathcal{A}$ on a graded Hilbert space $\mathcal{H}$ and an odd operator D which is a generalization of the Dirac operator. From the spectral triple a differential calculus can be constructed. We have defined two examples of spectral triples. The first, $(M_n, \mathbb{C}^n, \eta)$ was used to construct the differential calculus used in the Connes-Lott model **[9, 10]**. The second, $(M_N, \mathbb{C}^N, \eta)$ was implicit in the construction of Dubois-Violette *et al.* **[15, 16, 17, 18]**.

5. Extended Space-Time

The first attempts to introduce extra dimensions in order to unify the gravitational field with electromagnetism were made by Kaluza and Klein. It was suggested by Einstein & Bergmann that at sufficiently small scales what appears as a point will in fact be seen as a circle. Later, with the advent of more elaborate gauge fields, it was proposed that this internal manifold could be taken as a compact Lie group or even as a general compact manifold **[1, 2]**. The disadvantage of these extra dimensions is that they introduce even more divergences in the quantum theory and lead to an infinite spectrum of new particles. One can take this as motivation for introducing a modification of Kaluza-Klein theory with an internal structure which is described by a finite noncommutative geometry **[16, 31, 3, 33]**.

Closely related to Kaluza-Klein theory is the study of Maxwell's equations in higher-dimensions. As has been shown by Forgacs & Manton **[35, 21]** and others **[20, 4, 22]** if one considers the Maxwell action on a principle fibre bundle over space-time with group G then the resulting effective action is that of a Yang-Mills theory unified with a Higgs field which takes its value in the Lie algebra of G. The quartic potential appears naturally since the Yang-Mills action is quartic in the gauge potentials. Like all theories in higher dimensions, these theories include an infinite tower of massive states over each massless state. A rather artificial truncation procedure must be introduced to eliminate them. At least in the cases where the group G is one of the unitary groups U_n, equivalent theories can be reproduced without truncation by using a finite noncommutative extension of space-time based on the algebras M_n of complex $n \times n$ matrices **[15, 16, 17, 18]** and by using the differential calculus based on derivations **[14]** which we reviewed in Section 2. The resulting physical theories **[31, 32]** contain only one free parameter and are not realistic. In order to reproduce the standard model with a set of Higgs fields in the fundamental representation of the gauge group it is necessary to use the more abstract differential calculi which we reviewed in Section 3, either the $\mathbb{Z}_2$-based calculus **[9, 12, 13]** or its $\mathbb{Z}$-based extension **[10, 12, 23, 28]** For more details of this we refer to the numerous review articles on the subject **[27, 37, 24]**.

6. Acknowledgement

The authors would like to thank M. Dubois-Violette for enlightening conversations.

References

[1] Appelquist T., Chodos A., Freund P.G.O. 1987, *Modern Kaluza-Klein Theory and Applications*, Benjamin-Cummings.
[2] Bailin D., Love A. 1987, *Kaluza-Klein theories*, Rep. Prog. Phys. **50** 1087.
[3] Chamseddine A.H., Felder G., Fröhlich J. 1993, *Gravity in Non-Commutative Geometry*, Commun. Math. Phys. **155** 205.
[4] Chapline G., Manton N. S. 1980, *The Geometrical Significance of Certain Higgs Potentials: An approach to grand unification*, Nucl. Phys. **B184** 391.
[5] Cho Y. M. 1985, *Geometric Symmetry Breaking*, Phys. Rev. Lett. **55** 2932.
[6] Connes A. 1986, *Non-Commutative Differential Geometry*, Publications of the Inst. des Hautes Etudes Scientifiques. **62** 257.
[7] Connes A. 1990, *Géométrie noncommutative*, InterEditions, Paris.
[8] Connes A. 1994, *Noncommutative Geometry*, Academic Press.
[9] Connes A., Lott J. 1990, *Particle Models and Noncommutative Geometry*, in 'Recent Advances in Field Theory', Nucl. Phys. Proc. Suppl. **B18** 29.
[10] Connes A., Lott J. 1992, *The metric aspect of non-commutative geometry*, Proceedings of the 1991 Cargèse Summer School, Plenum Press.
[11] Connes A., Rieffel M.A. 1987, *Yang-Mills for non-commutative two-tori*, Contemp. Math. **62** 237.
[12] Coquereaux R. 1989, *Noncommutative geometry and theoretical physics*, J. Geom. Phys. **6** 425.
[13] Coquereaux R., Esposito-Farèse G., Vaillant G. 1991, *Higgs Fields as Yang-Mills Fields and Discrete Symmetries*, Nucl. Phys. **B353** 689.
[14] Dubois-Violette M. 1988, *Dérivations et calcul différentiel non-commutatif*, C. R. Acad. Sci. Paris **307** Série I 403.
[15] Dubois-Violette M., Kerner R., Madore J. 1989, *Gauge bosons in a noncommutative geometry*, Phys. Lett. **B217** 485.
[16] Dubois-Violette M., Kerner R., Madore J. 1989, *Classical bosons in a noncommutative geometry*, Class. Quant. Grav. **6** 1709.
[17] Dubois-Violette M., Kerner R., Madore J. 1990, *Noncommutative differential geometry of matrix algebras*, J. Math. Phys. **31** 316.
[18] Dubois-Violette M., Kerner R., Madore J. 1990, *Noncommutative differential geometry and new models of gauge theory*, J. Math. Phys. **31** 323.
[19] Dubois-Violette M., Kerner R., Madore J. 1991, *Super Matrix Geometry*, Class. Quant. Grav. **8** 1077.
[20] Fairlie, D.B. 1979, *The Interpretation of Higgs Fields as Yang Mills Fields*, Geometrical and Topological Methods in Gauge Theories, Proceedings, Montreal, Lecture Notes in Physics **129** 45.
[21] Forgács P., Manton N.S. 1980, *Space-Time Symmetries in Gauge Theories*, Commun. Math. Phys. **72** 15.
[22] Harnad J., Shnider S., Tafel J. 1980, *Group actions on principal bundles and dimensional reduction*, Lett. in Math. Phys. **4** 107.
[23] Iochum B., Schücker T. 1994, *A Left-Right Symmetric Model à la Connes-Lott*, Lett. Math. Phys. **32** 153.
[24] Iochum B., Schücker T. 1994, *Yang-Mills-Higgs versus Connes-Lott*, Preprint, CPT-94/P.3090.
[25] Kalau W., Papadopoulos N.A., Plass J., Warzecha J.-M. 1995, *Differential Algebras in Non-Commutative Geometry*, Jour. of Geom. and Phys. (to appear).
[26] Kastler D. 1988, *Cyclic cohomology within the differential envelope*, Hermann, Paris.
[27] Kastler D. 1993, *A detailed account of Alain Connes' version of the standard model in non-commutative geometry*, Rev. Math. Phys. **5** 477.
[28] Kastler D., Schücker T. 1994, *The Standard Model à La Connes-Lott*, Preprint, CPT-94/P.3091.

[29] Koszul J.L. 1960, *Lectures on Fibre Bundles and Differential Geometry*, Tata Institute of Fundamental Research, Bombay.

[30] Macfarlane A.J., Sudbery A., Weisz P.H. 1968, *On Gell-Mann's λ-Matrices, d- and f-Tensors, Octets, and Parametrizations of $SU(3)$*, Commun. Math. Phys. **11** 77.

[31] Madore J. 1991, *Algebraic Structure and Particle Spectra*, Int. J. Mod. Phys. **A6** 1287.

[32] Madore J. 1993, *On a Lepton-Quark Duality*, Phys. Lett. **B305** 84.

[33] Madore J., Mourad J. 1993, *Algebraic-Kaluza-Klein Cosmology*, Class. Quant. Grav. **10** 2157.

[34] Madore J., Masson T., Mourad J. 1995, *Linear Connections on Matrix Geometries*, Class. Quant. Grav. **12** 1429.

[35] Manton N.S. 1979, *A New Six-Dimensional Approach to the Weinberg-Salam Model*, Nucl. Phys. **B158** 141.

[36] Mourad. J. 1995, *Linear Connections in Non-Commutative Geometry*, Class. Quant. Grav. **12** 965.

[37] Várilly, J.C., Gracia-Bondía J.M. 1993, *Connes' noncommutative differential geometry and the Standard Model*, Jour. of Geom. and Phys. **12** 223.

Centre de Physique Théorique CNRS, Luminy, Case 907, F-13288 Marseille 9.
E-mail address: kastler@@cpt.univ-mrs.fr

Laboratoire de Physique Théorique et Hautes Energies, Bât. 211, Université de Paris Sud, F-91405 Orsay
E-mail address: madore@@qcd.th.u-psud.fr

Laboratoire de Physique Théorique et Hautes Energies, Bât. 211, Université de Paris Sud, F-91405 Orsay
E-mail address: masson@@qcd.th.u-psud.fr