Submanifolds and quotient manifolds in noncommutative geometry

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We define and study noncommutative generalizations of submanifolds and quotient manifolds using the derivation-based differential calculus introduced by M. Dubois-Violette and P. Michor. We give examples to illustrate these definitions. © *1996 American Institute of Physics.* [S0022-2488(96)00705-9]

I. INTRODUCTION

Various noncommutative generalizations of differential forms have been proposed as well as generalizations of vector bundles and connections. What is still missing is the concept of a submanifold and of a quotient manifold, that is, how the differential structure of a given algebra must be related to the differential structure of a subalgebra ("quotient manifold") or a quotient algebra ("submanifold"). In this paper, we propose a definition of a noncommutative submanifold and of a noncommutative quotient manifold within the context of the derivation-based differential calculus first introduced by M. Dubois-Violette,[1] and completed[2,3] with P. Michor.

In the first section, we recall various definitions related to this differential calculus. In the second section, we recall the definition of Hochschild cohomology and other cohomologies which will be used later. Submanifolds and quotient manifolds are defined respectively in Secs. IV and V.

II. NONCOMMUTATIVE DIFFERENTIAL STRUCTURES

In noncommutative geometry, the algebra of smooth functions on a manifold is replaced by a noncommutative algebra (see, for example, Refs. 4 and 5). Geometric objects are first expressed in terms of the algebra of functions and then they can be generalized to the noncommutative case. In this section, we recall the definition of differential forms, central bimodules, and connections as they are given in Refs. 1–3.

A. Noncommutative differential forms

Let $\mathscr{A}$ denote an associative algebra with unit. It is then the generalization of the algebra of smooth functions on a compact manifold. The center of the algebra will be denoted by $\mathscr{Z}(\mathscr{A})$. The differential forms we wish to introduce are based on derivations, the algebraic generalizations of vector fields on a manifold:

$$\mathrm{Der}(\mathscr{A}) = \{X: \mathscr{A} \to \mathscr{A}/X(ab) + X(a)b + aX(b)\};$$

$\mathrm{Der}(\mathscr{A})$ is naturally a $\mathscr{Z}(\mathscr{A})$-module and a Lie algebra.

The two noncommutative generalizations of the graded differential algebra of differential forms which we shall need[1,2] are constructed as follows. Let $C_{\mathscr{Z}(\mathscr{A})}(\mathrm{Der}(\mathscr{A}),\mathscr{A})$ be the graded algebra of antisymmetric $\mathscr{Z}(\mathscr{A})$-multilinear mappings from $\mathrm{Der}(\mathscr{A})$ to $\mathscr{A}$. Notice that this algebra is not graded commutative. In degree 0 we take $C^0_{\mathscr{Z}(\mathscr{A})}(\mathrm{Der}(\mathscr{A}),\mathscr{A}) = \mathscr{A}$. We introduce a differential d by the Koszul formula:

$$d\omega(X_1,\ldots,X_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} X_i \omega(X_1,\ldots \overset{\overset{i}{\vee}}{\cdot} \ldots,X_{n+1})$$

$$+ \sum_{1 \le i < j \le n+1} (-1)^{i+j} \omega([X_i,X_j],\ldots \overset{\overset{i}{\vee}}{\cdot} \ldots \overset{\overset{j}{\vee}}{\cdot} \ldots,X_{n+1}),$$

for any $\omega \in C^n_{\mathscr{Z}(\mathscr{A})}(\mathrm{Der}(\mathscr{A}),\mathscr{A})$ and any set of derivations X_i.

Now, we can introduce the first generalization of differential forms over $\mathscr{A}$. We define $\Omega_{\mathrm{Der}}(\mathscr{A})$ to be the smallest differential graded subalgebra of the algebra $C_{\mathscr{Z}(\mathscr{A})}(\mathrm{Der}(\mathscr{A}),\mathscr{A})$ which contains $\mathscr{A}$. Every element $\omega \in \Omega^n_{\mathrm{Der}}(\mathscr{A})$ can be written as a finite sum of elements of the type $a_0 da_1 \cdots da_n$, where $da \in \Omega^1_{\mathrm{Der}}(\mathscr{A})$ is the one-form $X \mapsto Xa \in \mathscr{A}$, and where the product is that of $C_{\mathscr{Z}(\mathscr{A})}(\mathrm{Der}(\mathscr{A}),\mathscr{A})$.

The second differential graded algebra of forms we shall use is the algebra $C_{\mathscr{Z}(\mathscr{A})}(\mathrm{Der}(\mathscr{A}),\mathscr{A})$ itself, denoted by $\underline{\Omega}_{\mathrm{Der}}(\mathscr{A})$. We refer the reader to Ref. 3 for the relation between $\Omega_{\mathrm{Der}}(\mathscr{A})$, $\mathrm{Der}(\mathscr{A})$ and $\underline{\Omega}_{\mathrm{Der}}(\mathscr{A})$ from the point of view of duality.

There is a canonical Cartan operation i_X of the Lie algebra $\mathrm{Der}(\mathscr{A})$ on $\Omega_{\mathrm{Der}}(\mathscr{A})$ and $\underline{\Omega}_{\mathrm{Der}}(\mathscr{A})$.[1] For any $X \in \mathrm{Der}(\mathscr{A})$, one defines the antiderivation of degree -1

$$i_X : \Omega^n_{\mathrm{Der}}(\mathscr{A}) \to \Omega^{n-1}_{\mathrm{Der}}(\mathscr{A})$$

by

$$(i_X\omega)(X_1,\ldots,X_{n-1}) = \omega(X,X_1,\ldots,X_{n-1}),$$

with $i_X a = 0$ for any $a \in \mathscr{A} = \Omega^0_{\mathrm{Der}}(\mathscr{A})$. It follows that $L_X = i_X d + d i_X$ is a derivation of degree 0 on $\Omega_{\mathrm{Der}}(\mathscr{A})$.

B. Central bimodules and connections

In ordinary differential geometry, vector bundles of finite rank can be considered from an algebraic point of view through their space of sections. In fact this space is a finite projective module over the algebra of smooth functions. In noncommutative geometry, the generalization of a vector bundle will then be such a module over the algebra. However, since $\mathscr{A}$ is noncommutative this can be a right module, or a bimodule.

In Ref. 2 it was proposed that this generalization should at least have the structure of a central bimodule. We recall that a central bimodule $\mathscr{M}$ is a bimodule over $\mathscr{A}$ and which is also a module over the center $\mathscr{Z}(\mathscr{A})$ of $\mathscr{A}$ in the commutative sense. That is, for any $z \in \mathscr{Z}(\mathscr{A})$ and $m \in \mathscr{M}$, one has $zm = mz$.

It is then easy to introduce the notion of a connection on a central bimodule. A connection on $\mathscr{M}$ is a linear mapping ∇ from $\mathrm{Der}(\mathscr{A})$ into the linear endomorphisms of $\mathscr{M}$ such that

$$\nabla_{zX} m = z \nabla_X m, \quad \nabla_X(amb) = (Xa)mb + a(\nabla_X m)b + am(Xb)$$

for any $X \in \mathrm{Der}(\mathscr{A})$, $z \in \mathscr{Z}(\mathscr{A})$, $a,b \in \mathscr{A}$, and $m \in \mathscr{M}$.

The curvature of this connection is defined by the usual formula

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

for any $X,Y \in \mathrm{Der}(\mathscr{A})$. Here $R(X,Y)$ is an $\mathscr{A}$-bimodule endomorphism of $\mathscr{M}$, antisymmetric and $\mathscr{Z}(\mathscr{A})$-linear in X,Y. We refer the reader to Ref. 3 for more properties on these connections.

III. HOCHSCHILD COHOMOLOGY AND RELATED COHOMOLOGIES

In this section we introduce a class of subcomplexes of the Hochschild complex of an associative algebra, and their cohomology. These cohomologies, in degree 1, will be useful in the next section.

A. Hochschild cohomology

We recall the definition of the ordinary Hochschild cohomology. Let $\mathscr{A}$ be an associative algebra with unit over $\mathbb{C}$, and $\mathscr{M}$ a bimodule over $\mathscr{A}$.

We define the complex $C(\mathscr{A};\mathscr{M})$ as follows: $C^n(\mathscr{A};\mathscr{M})$ is the linear space of $\mathbb{C}$-linear mappings from $\mathscr{A}^{\otimes^n}$ to $\mathscr{M}$. In degree 0, we set $C^0(\mathscr{A};\mathscr{M})=\mathscr{M}$. We set $C(\mathscr{A};\mathscr{M})=\oplus_{n\geqslant 0}C^n(\mathscr{A};\mathscr{M})$. Then we introduce the Hochschild differential δ on the space $C(\mathscr{A};\mathscr{M})$ by the formula

$$(\delta f)(a_1\otimes\cdots\otimes a_{n+1})=a_1 f(a_2\otimes\cdots\otimes a_{n+1})+\sum_{i=1}^{n}(-1)^i f(a_1\otimes\cdots\otimes a_i a_{i+1}\otimes\cdots\otimes a_{n+1})$$

$$+(-1)^{n+1}f(a_1\otimes\cdots\otimes a_n)a_{n+1}$$

for any $f\in C^n(\mathscr{A};\mathscr{M})$. Because $\mathscr{A}$ is an associative algebra, one has $\delta^2=0$. The cohomology of this differential complex is denoted by $H(\mathscr{A};\mathscr{M})$. It is the Hochschild cohomology of $\mathscr{A}$ with values in $\mathscr{M}$.

The bimodule of interest for our purpose is $\mathscr{A}$ itself. In this case, the complex $C(\mathscr{A};\mathscr{A})$ is an associative algebra (see Refs. 6 and 7 and references therein) and $H(\mathscr{A};\mathscr{A})$ inherits a structure of graded commutative algebra.

Let us consider now the previous case with $n=1$. Then $Z^1(\mathscr{A};\mathscr{A})=\operatorname{Im}\delta\cap C^1(\mathscr{A};\mathscr{A})$ is the Lie algebra $\operatorname{Der}(\mathscr{A})$ of derivations of $\mathscr{A}$, and $B^1(\mathscr{A};\mathscr{A})=\operatorname{Ker}\delta\cap C^1(\mathscr{A};\mathscr{A})$ is the Lie subalgebra $\operatorname{Int}(\mathscr{A})$ of $\operatorname{Der}(\mathscr{A})$ of inner derivations of $\mathscr{A}$. This is an ideal of $\operatorname{Der}(\mathscr{A})$, so $H^1(\mathscr{A};\mathscr{A})$ is a Lie algebra, denoted $\operatorname{Out}(\mathscr{A})$.

B. Relative Hochschild cohomology

We follow here the exposition in Ref. 6 (see also Ref. 7). Let $\mathscr{S}$ denote a subalgebra of $\mathscr{A}$, and $\mathscr{M}$ a bimodule over $\mathscr{A}$. The complex $C(\mathscr{A},\mathscr{S};\mathscr{M})$ is defined by

$$C^0(\mathscr{A},\mathscr{S};\mathscr{M})=\mathscr{M}^{\mathscr{S}}=\{m\in\mathscr{M}/sm=ms\,\forall s\in\mathscr{S}\}$$

and $C^n(\mathscr{A},\mathscr{S};\mathscr{M})$ is the linear space of n-linear mappings $f:\mathscr{A}\otimes\cdots\otimes\mathscr{A}\to\mathscr{M}$ such that

$$f(sa_1\otimes\cdots\otimes a_n)=sf(a_1\otimes\cdots\otimes a_n),\quad f(a_1\otimes\cdots\otimes a_n s)=f(a_1\otimes\cdots\otimes a_n)s,$$

$$f(a_1\otimes\cdots\otimes a_i s\otimes a_{i+1}\otimes\cdots\otimes a_n)=f(a_1\otimes\cdots\otimes a_i\otimes sa_{i+1}\otimes\cdots\otimes a_n)$$

for any $a_i\in\mathscr{A}$ and $s\in\mathscr{S}$. Then f is a $\mathscr{S}$-bimodule homomorphism $\mathscr{A}\otimes_{\mathscr{S}}\cdots\otimes_{\mathscr{S}}\mathscr{A}\to\mathscr{M}$.

The Hochschild differential δ maps $C^n(\mathscr{A},\mathscr{S};\mathscr{M})$ into $C^{n+1}(\mathscr{A},\mathscr{S};\mathscr{M})$, and then defines a cohomology $H(\mathscr{A},\mathscr{S};\mathscr{M})$. This is the relative Hochschild cohomology of $\mathscr{A}$ in $\mathscr{M}$ for $\mathscr{S}$. This cohomology can be calculated on a subcomplex of $C(\mathscr{A},\mathscr{S};\mathscr{M})$.[6] Let us denote by $\bar{C}(\mathscr{A},\mathscr{S};\mathscr{M})$ the linear subspace of $C(\mathscr{A},\mathscr{S};\mathscr{M})$ of elements f such that f vanishes when at least one of its arguments is in $\mathscr{S}$. This is the normalized complex of the relative Hochschild cohomology. These two complexes have the same cohomology.

Let us now consider the case where $\mathscr{S}=\mathscr{Z}(\mathscr{A})$. Then the relative Hochschild cohomology is well adapted to study central bimodules. In degree 0, one has $C^0(\mathscr{A},\mathscr{Z}(\mathscr{A});\mathscr{M})=\mathscr{M}$ for $\mathscr{M}$ a

central bimodule. In higher degrees, one can remark that $\mathscr{A} \otimes_{\mathscr{Z}(\mathscr{A})} \cdots \otimes_{\mathscr{Z}(\mathscr{A})} \mathscr{A}$ is a central bimodule, and then the normalized relative complex is a set of homomorphisms of central bimodules.

For future use, consider the case $\mathscr{S} = \mathscr{Z}(\mathscr{A})$, $\mathscr{M} = \mathscr{A}$, and $n = 1$. Then $Z^1(\mathscr{A}, \mathscr{Z}(\mathscr{A}); \mathscr{A})$ is exactly the Lie algebra of derivations of $\mathscr{A}$ which vanish on the center $\mathscr{Z}(\mathscr{A})$. Remark that $B^1(\mathscr{A}, \mathscr{Z}(\mathscr{A}); \mathscr{A})$ is equal to $B^1(\mathscr{A}; \mathscr{A})$. So, one has the two left exact sequences:

$$0 \rightarrow Z^1(\mathscr{A}, \mathscr{Z}(\mathscr{A}); \mathscr{A}) \rightarrow \mathrm{Der}(\mathscr{A}) \rightarrow \mathrm{Der}(\mathscr{Z}(\mathscr{A})),$$

$$0 \rightarrow H^1(\mathscr{A}, \mathscr{Z}(\mathscr{A}); \mathscr{A}) \rightarrow \mathrm{Out}(\mathscr{A}) \rightarrow \mathrm{Der}(\mathscr{Z}(\mathscr{A})),$$

which are not short exact sequences in general. The condition $H^1(\mathscr{A}, \mathscr{Z}(\mathscr{A}); \mathscr{A}) = 0$; which means that any derivation of $\mathscr{A}$ which vanishes on $\mathscr{Z}(\mathscr{A})$ is an inner derivation, gives the injectivity of the canonical homomorphism $\mathrm{Out}(\mathscr{A}) \rightarrow \mathrm{Der}(\mathscr{Z}(\mathscr{A}))$.

C. Constrained Hochschild cohomology

Let us now introduce a new subcomplex of the Hochschild complex. As before, $\mathscr{A}$ is an associative algebra with unit and $\mathscr{M}$ is a bimodule over $\mathscr{A}$. Let $\mathscr{C}$ be an ideal in $\mathscr{A}$ and $\mathscr{N}$ a sub-bimodule of $\mathscr{M}$ such that $cm, mc \in \mathscr{N}$ for any $c \in \mathscr{C}$ and $m \in \mathscr{M}$. This is equivalent to saying that $\mathscr{C}$ is included in the two side ideal

$$\mathscr{T}_{\mathscr{N}} = \{a \in \mathscr{A} | a\mathscr{M} \subset \mathscr{N} \text{ and } \mathscr{M}a \subset \mathscr{N}\}.$$

We define the subcomplex $C(\mathscr{A}, \mathscr{C}; \mathscr{M}, \mathscr{N})$ of $C(\mathscr{A}; \mathscr{M})$ of the mappings $f : \mathscr{A} \otimes \cdots \otimes \mathscr{A} \rightarrow \mathscr{M}$ such that $f(a_1 \otimes \cdots \otimes a_n) \in \mathscr{N}$ if at least one of the a_i is in $\mathscr{C}$. In degree 0, $C^0(\mathscr{A}, \mathscr{C}; \mathscr{M}, \mathscr{N}) = \mathscr{M}$. It is easy to see that this subcomplex is stable by the Hochschild differential δ. So one has a cohomology $H(\mathscr{A}, \mathscr{C}; \mathscr{M}, \mathscr{N})$. This is the constrained cohomology of $\mathscr{A}$ in $\mathscr{M}$ by $(\mathscr{C}, \mathscr{N})$. One has then the following lemma:

Lemma III.1: In the above situation, one has a canonical mapping of graded vector spaces

$$H(\mathscr{A}, \mathscr{C}; \mathscr{M}, \mathscr{N}) \rightarrow H(\mathscr{A}/\mathscr{C}; \mathscr{M}/\mathscr{N}),$$

where the second cohomology is the ordinary Hochschild cohomology of the bimodule $\mathscr{M}/\mathscr{N}$ over the algebra $\mathscr{A}/\mathscr{C}$.

Proof: Let $pr : \mathscr{M} \rightarrow \mathscr{M}/\mathscr{N}$ denote the projection from the bimodule $\mathscr{M}$ over $\mathscr{A}$ on the bimodule $\mathscr{M}/\mathscr{N}$ over $\mathscr{A}/\mathscr{C}$, and $a \rightarrow [a]$ the projection $\mathscr{A} \rightarrow \mathscr{A}/\mathscr{C}$. Then one has $pr(am) = [a]pr(m)$ for any $a \in \mathscr{A}$ and $m \in \mathscr{M}$, and a similar formula for ma.

Any $f \in C(\mathscr{A}, \mathscr{C}; \mathscr{M}, \mathscr{N})$ can be mapped into $\chi(f) \in C(\mathscr{A}/\mathscr{C}; \mathscr{M}/\mathscr{N})$ by the definition

$$\chi(f)([a_1] \otimes \cdots \otimes [a_n]) = (pr \circ f)(a_1 \otimes \cdots \otimes a_n).$$

Then it is easy to see that

$$pr \circ \delta = \bar{\delta} \circ pr,$$

where δ is the Hochschild differential on $C(\mathscr{A}, \mathscr{C}; \mathscr{M}, \mathscr{N})$ and $\bar{\delta}$ the Hochschild differential on $C(\mathscr{A}/\mathscr{C}; \mathscr{M}/\mathscr{N})$. $\qquad \square$

A simpler situation occurs when one takes $\mathscr{M} = \mathscr{A}$ and $\mathscr{N} = \mathscr{C}$. Then the subcomplex is a subalgebra of $C(\mathscr{A}; \mathscr{A})$, but not an ideal. We denote it by $C_{\mathscr{C}}(\mathscr{A}; \mathscr{A})$, and its cohomology by $H_{\mathscr{C}}(\mathscr{A}; \mathscr{A})$. In degree 1, one has obviously $B^1_{\mathscr{C}}(\mathscr{A}; \mathscr{A}) = B^1(\mathscr{A}; \mathscr{A})$. Here $Z^1_{\mathscr{C}}(\mathscr{A}; \mathscr{A})$ is the Lie algebra of derivations of $\mathscr{A}$ which preserve $\mathscr{C}$ and $B^1_{\mathscr{C}}(\mathscr{A}; \mathscr{A})$ is an ideal in this Lie algebra. Then $H^1_{\mathscr{C}}(\mathscr{A}; \mathscr{A})$ is a Lie algebra.

IV. NONCOMMUTATIVE SUBMANIFOLDS

In this section, we introduce a noncommutative generalization of the notion of submanifold of a manifold.

A. The commutative case

We first recall the situation in the commutative case. Let M be a smooth compact manifold, and let $N \subset M$ be a closed submanifold. Any smooth function $f : M \to \mathbb{R}$ can be restricted to N. Thus one has a mapping

$$\mathcal{F}(M) \overset{p}{\to} \mathcal{F}(N),$$

where $\mathcal{F}(M)$ is the algebra of smooth functions on M. This mapping is in fact surjective, and there exists a short exact sequence

$$0 \to \mathcal{C} \to \mathcal{F}(M) \overset{p}{\to} \mathcal{F}(N) \to 0,$$

where $\mathcal{C}$ is the ideal of $\mathcal{F}(M)$ of functions vanishing on N.

A vector field $X \in \Gamma(M)$ on M, which satisfies $Xf \in \mathcal{C}$ for any $f \in \mathcal{C}$, can be restricted to a vector field $\overline{X}$ on N. Thus one has a homomorphism of Lie algebras

$$\Gamma_{\mathcal{C}}(M) \overset{\pi}{\to} \Gamma(N),$$

where $\Gamma_{\mathcal{C}}(M) = \{X \in \Gamma(M) / X \mathcal{C} \subset \mathcal{C}\}$. This mapping is surjective, and there exists a short exact sequence of Lie algebras:

$$0 \to \Gamma_{\mathcal{F}} \to \Gamma_{\mathcal{C}}(M) \overset{\pi}{\to} \Gamma(N) \to 0,$$

where $\Gamma_{\mathcal{F}} = \{X \in \Gamma(M) / X \mathcal{F}(M) \subset \mathcal{C}\}$ is an ideal of the Lie algebra $\Gamma_{\mathcal{C}}(M)$.

B. The noncommutative case

Now we can generalize these notions to the framework of noncommutative geometry. Let $\mathcal{A}$ be an associative algebra over $\mathbb{C}$ with unit and let $\mathcal{C}$ be an ideal in $\mathcal{A}$. We denote by $\mathcal{Q} = \mathcal{A}/\mathcal{C}$ the quotient algebra and by $p : \mathcal{A} \to \mathcal{Q}$ the quotient mapping.

We can consider the two following Lie subalgebras of $\mathrm{Der}(\mathcal{A})$:

$$\mathcal{G}_{\mathcal{C}} = \{X \in \mathrm{Der}(\mathcal{A}) / X \mathcal{C} \subset \mathcal{C}\}$$

and

$$\mathcal{G}_{\mathcal{A}} = \{X \in \mathrm{Der}(\mathcal{A}) / X \mathcal{A} \subset \mathcal{C}\}.$$

One sees that $\mathcal{G}_{\mathcal{A}}$ is an ideal in $\mathcal{G}_{\mathcal{C}}$. One has a mapping $\mathcal{G}_{\mathcal{C}} \overset{\pi}{\to} \mathrm{Der}(\mathcal{Q})$ defined by $\pi(X)p(a) = p(Xa)$ for any $a \in \mathcal{A}$ and $X \in \mathcal{G}_{\mathcal{C}}$. The kernel of this mapping is exactly $\mathcal{G}_{\mathcal{A}}$.

Definition IV.1: The quotient algebra $\mathcal{Q} = \mathcal{A}/\mathcal{C}$ will be called a submanifold algebra of $\mathcal{A}$ if π is surjective. The ideal $\mathcal{C}$ of $\mathcal{A}$ is called the constraint ideal for $\mathcal{Q}$.

In this situation, one has the short exact sequence of Lie algebras

$$0 \to \mathcal{G}_{\mathcal{A}} \to \mathcal{G}_{\mathcal{C}} \overset{\pi}{\to} \mathrm{Der}(\mathcal{Q}) \to 0. \tag{1}$$

The condition of the definition imposes a strong relation between the differential structure on $\mathscr{A}$ and the differential structure on $\mathscr{Q}$. This strong relation is revealed in the following proposition:

Proposition IV.1: There exists a short exact sequence of graded differential algebras

$$0 \to \Omega_{\mathrm{Der},\mathscr{C}} \to \Omega_{\mathrm{Der}}(\mathscr{A}) \overset{p}{\to} \Omega_{\mathrm{Der}}(\mathscr{Q}) \to 0. \tag{2}$$

Proof: Let $\overline{X} = \pi(X) \in \mathrm{Der}(\mathscr{Q})$ for any $X \in \mathscr{G}_{\mathscr{C}}$ and let $\overline{a} = p(a) \in \mathscr{Q}$ for any $a \in \mathscr{A}$. Then one has over $\mathscr{Q}$

$$d\overline{a}(\overline{X}) = \overline{X}\overline{a} = p(Xa) = p(da(X)).$$

One then extends p in a mapping $\Omega_{\mathrm{Der}}^n(\mathscr{A}) \to \Omega_{\mathrm{Der}}^n(\mathscr{Q})$ by the relation

$$p(a_0 da_1 \cdots da_n) = p(a_0) dp(a_1) \cdots dp(a_n),$$

and then one has

$$d \circ p = p \circ d$$

and

$$i_{\overline{X}} \circ p = p \circ i_X.$$

It is easy to see that p is surjective; so we obtain the short exact sequence (2). $\qquad\square$

Remark 1: In the short exact sequence (2), one has

$$\Omega_{\mathrm{Der},\mathscr{C}}^n = \{\omega \in \Omega_{\mathrm{Der}}^n(\mathscr{A}) / \forall X \in \mathscr{G}_{\mathscr{C}}, i_X \omega \in \Omega_{\mathrm{Der},\mathscr{C}}^{n-1}\}$$

with $\Omega_{\mathrm{Der},\mathscr{C}}^0 = \mathscr{C}$. For example, for any $a \in \mathscr{C}, da \in \Omega_{\mathrm{Der},\mathscr{C}}^1$.

Remark 2: In general there are no relations between $\Omega_{\mathrm{Der}}(\mathscr{A})$ and $\Omega_{\mathrm{Der}}(\mathscr{Q})$.

Let us now study the derivations of $\mathscr{Q}$. Any inner derivation of $\mathscr{A}$ is obviously in $\mathscr{G}_{\mathscr{C}}$. In the quotient homomorphism $\mathscr{G}_{\mathscr{C}} \overset{\pi}{\to} \mathrm{Der}(\mathscr{Q})$, these inner derivations are mapped on inner derivations, from the very definition of π. It is easy to see that π restricted to inner derivations is surjective on inner derivations of $\mathscr{Q}$ (even if π does not satisfy the condition of Definition IV.1, i.e., π is not surjective) and one has $\pi(ad(a)) = ad(p(a))$ for any $a \in \mathscr{A}$. So, the kernel of π contains $ad(\mathscr{C}) = \{ad(c)/c \in \mathscr{C}\} \subset \mathrm{Der}(\mathscr{A})$.

Lemma IV.1: If $\mathscr{Q} = \mathscr{A}/\mathscr{C}$ has only inner derivations, then the mapping $\mathscr{G}_{\mathscr{C}} \to \mathrm{Der}(\mathscr{Q})$ is surjective. Then $\mathscr{Q}$ is a submanifold algebra.

Proof: This is a direct consequence of the previous discussion about inner derivations. $\qquad\square$

It is now interesting to say something about the other derivations of $\mathscr{Q}$, that is, about the first Hochschild cohomology of $\mathscr{Q}$ with values in itself. One has the following lemma:

Lemma IV.2: One has a surjective homomorphism of Lie algebras

$$H_{\mathscr{C}}^1(\mathscr{A};\mathscr{A}) \to H^1(\mathscr{Q};\mathscr{Q}).$$

Proof: This is a direct consequence of Lemma III.1 (with $\mathscr{M} = \mathscr{A}$ and $\mathscr{N} = \mathscr{C}$), the previous remark about inner derivations, and the surjectivity of π from Definition IV.1. $\qquad\square$

One can say something about the kernel of this mapping, if one imposes a supplementary condition on the ideal $\mathscr{C}$.

Proposition IV.2: If the constraint ideal $\mathscr{C}$ for the submanifold algebra $\mathscr{Q}$ satisfies

$$ad(\mathscr{C}) = \{ad(a)/a \in \mathscr{A} \text{ and } [a,\mathscr{A}] \subset \mathscr{C}\}.$$

or, equivalently, if $\mathrm{Ker}\,\pi \cap \mathrm{Int}(\mathscr{A}) = ad(\mathscr{C})$, then one has the short exact sequence of Lie algebras

$$0 \to H^1(\mathscr{A};\mathscr{C}) \to H^1_{\mathscr{C}}(\mathscr{A};\mathscr{A}) \to H^1(\mathcal{Q};\mathcal{Q}) \to 0. \tag{3}$$

In $H^1(\mathscr{A};\mathscr{C})$, $\mathscr{C}$ is considered as a bimodule over $\mathscr{A}$.

Proof: The condition on $\mathscr{C}$ means, in fact, that one has the short exact sequence of Lie algebras

$$0 \to B^1(\mathscr{A};\mathscr{C}) \to B^1_{\mathscr{C}}(\mathscr{A};\mathscr{A}) = B^1(\mathscr{A};\mathscr{A}) \to B^1(\mathcal{Q};\mathcal{Q}) \to 0.$$

The new information is the exactness at $B^1_{\mathscr{C}}(\mathscr{A};\mathscr{A})$. If one associates this short exact sequence with the short exact sequence (1) written as

$$0 \to Z^1(\mathscr{A};\mathscr{C}) \to Z^1_{\mathscr{C}}(\mathscr{A};\mathscr{A}) \to Z^1(\mathcal{Q};\mathcal{Q}) \to 0,$$

then one obtains the exactness of (3). □

In algebraic geometry (Ref. 8 and references therein), one works with the commutative algebra with unit $\mathscr{A} = \mathbb{C}[X_1,\dots,X_n]$ of complex polynomials of n variables. The geometric objects are considered as zero sets of polynomials. An ideal $\mathscr{C}$ represents the set of points $V(\mathscr{C}) = \{x \in \mathbb{C}^n / P(x) = 0 \,\forall P \in \mathscr{C}\}$. From the point of view of the ''duality'' *set of points↔algebra of functions*, the set $V(\mathscr{C})$ is represented by the algebra $\mathcal{Q} = \mathbb{C}[X_1,\dots,X_n]/\mathrm{Rad}(\mathscr{C})$, where $\mathrm{Rad}(\mathscr{C}) = \{P \in \mathbb{C}[X_1,\dots,X_n]/\exists k \in \mathbb{N}\ P^k \in \mathscr{C}\}$ is the radical of $\mathscr{C}$. If $\mathcal{Q}$ admits ideals, then the set $V(\mathscr{C})$ admits subsets. But if $\mathcal{Q}$ does not have any ideal, then the set $V(\mathscr{C})$ can be considered as a *point*. This is equivalent to the fact that $\mathscr{C}$ is a maximal ideal in $\mathscr{A}$. Notice that from the point of view of ordinary geometry, points are the minimal sets. The only maximal ideals of $\mathscr{A}$ are generated by n polynomials $X_i - a_i$ where $a_i \in \mathbb{C}$. The point represented by this ideal is obviously $(a_i) \in \mathbb{C}^n$. Notice that maximal ideals are in one-to-one correspondence with the characters of $\mathscr{A}$. The quotient mapping $\mathscr{A} \to \mathscr{A}/\mathscr{C}$ is the restriction at the set of points represented by $\mathscr{C}$. If $\mathscr{C}$ is maximal, the restriction of an element $P \in \mathscr{A}$ at $\mathscr{C}$ is exactly the value of this polynomial at the point of $\mathbb{C}^n$ represented by $\mathscr{C}$.

This correspondence *points↔maximal ideals* is also used in the theory of commutative Banach algebras and commutative C^*-algebras.[9] In this context, maximal ideals are also in one-to-one correspondence with characters. If $\mathscr{C}$ is a maximal ideal in the commutative Banach algebra with unit $\mathscr{A}$, then the quotient $\mathscr{A}/\mathscr{C}$ is isomorphic to $\mathbb{C}$. (The quotient mapping $\mathscr{A} \to \mathscr{A}/\mathscr{C}$ is a character). In the theory of commutative C^*-algebras, by the Gel'fand transformation, the set of characters is exactly the set of points on which it is possible to put a canonical topology. One says that a point takes its values in the quotient $\mathscr{A}/\mathscr{C} \simeq \mathbb{C}$. So, in those two situations, points are maximal ideals, and take their values in the field $\mathbb{C}$.

In noncommutative geometry, an ideal $\mathscr{C}$ of a given complex algebra with unit $\mathscr{A}$ can also be interpreted as a ''subspace'' of the noncommutative ''space'' dually represented by $\mathscr{A}$. This subspace can be considered as a ''submanifold'' if the differential structure of $\mathscr{A}/\mathscr{C}$ is compatible with the differential structure of $\mathscr{A}$. One of the compatibility conditions one can take is Definition IV.1.

Now, if the ideal is maximal, then the quotient algebra is simple. It is then a ''point,'' in the sense that it cannot have ''proper subspaces.'' However, considering the quotient $\mathcal{Q} = \mathscr{A}/\mathscr{C}$, one sees that points take their values in (*a priori* noncommutative) simple algebras and not in fields as in the commutative case. There is then a residual structure of purely noncommutative origin. See Example 5 below for applications in physics.

To any ideal $\mathscr{C}$ in $\mathscr{A}$, one can construct $\mathscr{G}_{\mathscr{A}} \subset \mathrm{Der}(\mathscr{A})$. If $\mathscr{C}$ is a maximal ideal, the quotient of linear space $T_{\mathscr{C}} = \mathrm{Der}(\mathscr{A})/\mathscr{G}_{\mathscr{A}}$ can be considered as the ''tangent space'' at the point $\mathscr{C}$ in the ''manifold'' represented by $\mathscr{A}$. The value of a derivation X at the ''point'' $\mathscr{C}$ is the image of X by

the quotient mapping $\text{Der}(\mathscr{A}) \rightarrow T_\mathscr{C}$. One can also take the value of a one-form $\alpha \in \Omega^1(\mathscr{A})$ at $\mathscr{C}$ by the definition $\alpha_\mathscr{C}: T_\mathscr{C} \rightarrow \mathcal{Q}, \alpha_\mathscr{C}(X_\mathscr{C}) = p \circ \alpha(X)$ for any $X \in \text{Der}(\mathscr{A})$ whose value at $\mathscr{C}$ is $X_\mathscr{C}$. This can be generalized for any n-form in $\Omega^n(\mathscr{A})$.

C. Examples

Example 1: The commutative case.

In the commutative case, any smooth closed submanifold of a smooth compact manifold gives a submanifold algebra: the algebra of smooth functions on this submanifold.

Example 2: The tensor algebra.

Let $\mathscr{A}$ be the free algebra with unit over $\mathbb{C}$ generated by n elements $x^1,\ldots,x^n$, with $n \geqslant 2$.

Any derivation of $\mathscr{A}$ is given by n elements $P^i(x^1,\ldots,x^n) \in \mathscr{A}$. We denote it by $D = (P^i)_{1 \leqslant i \leqslant n}$. The value of this derivation on any element of $\mathscr{A}$ is obtained by the Leibniz rule and the definition $D(x^i) = P^i(x^1,\ldots,x^n)$.

If one takes $\mathscr{C}$ the ideal in $\mathscr{A}$ generated by x^1, then the algebra $\mathcal{Q}$ is the free algebra with unit over $\mathbb{C}$ generated by $x^2,\ldots,x^n$, and one has $\mathscr{A} = \mathscr{C} \oplus \mathcal{Q}$ as vector spaces.

Any derivation in $\mathscr{G}_\mathscr{C}$ is the sum of two kinds of derivation: $(P^i)_{1 \leqslant i \leqslant n}$, with $P^i \in \mathcal{Q}$ and $P^1 = 0$ ($\mathcal{Q}$ is considered here as a subalgebra of $\mathscr{A}$), and $(P^i)_{1 \leqslant i \leqslant n}$, with $P^i \in \mathscr{C}$. Any derivation in $\mathscr{G}_\mathscr{A}$ is of the second kind, and the Lie algebra of derivation of $\mathcal{Q}$ is the set of the first kind derivations in $\mathscr{G}_\mathscr{C}$. So the condition of the Definition IV.1 is fulfilled. Thus $\mathcal{Q}$ is a submanifold algebra of $\mathscr{A}$. In this case, one has $\mathscr{G}_\mathscr{C} = \mathscr{G}_\mathscr{A} \oplus \text{Der}(\mathcal{Q})$.

As maximal ideals of $\mathscr{A}$, one has the ideals generated by the n elements $x^i - a_i$, where $a_i \in \mathbb{C}$. Then the point associated to such an ideal is a point in $\mathbb{C}^n$, with values in $\mathbb{C}$. This situation is analogous to the situation of the polynomial algebra generated by the n variables x^i, for which there are only those maximal ideals. It is not difficult to see that such an ideal contains the ideals generated by the expressions $x^i x^j - x^j x^i$. In the case of the tensor algebra $\mathscr{A}$, there are other interesting maximal ideals, as the following examples show.

Example 3: The Heisenberg algebra.

Let $\mathscr{A}$ be the free algebra with unit generated by two elements x, y. Consider in $\mathscr{A}$ the ideal generated by $xy - yx - i\mathbb{1}$. Then the quotient algebra is the Heisenberg algebra $\mathscr{H}$, generated by two elements p, q and the relation $pq - qp = i\mathbb{1}$. It is well known that this algebra is simple. The ideal is maximal. In the quotient, we take $x \mapsto p$ and $y \mapsto q$.

Now let us consider derivations. If we denote by $D = (X, Y)$ the derivation $D(x) = X$ and $D(y) = Y$, then one has

$$\mathscr{G}_\mathscr{C} = \{(X,Y)/[X,\, y] + [x,\, Y] \in \mathscr{C}\},$$

where $X, Y \in \mathscr{A}$, and

$$\mathscr{G}_\mathscr{A} = \{(X,Y)/X, Y \in \mathscr{C}\}.$$

On the other hand, one knows that $\mathscr{H}$ has only inner derivations (see Ref. 5, for instance), so

$$\text{Der}(\mathscr{H}) = \{([A,\, p],[A,\, q])/A \in \mathscr{H}\}$$

with the same notations as above. It is easy to prove that the mapping $\mathscr{G}_\mathscr{C} \rightarrow \text{Der}(\mathscr{H})$ (the quotient by $\mathscr{G}_\mathscr{A}$) is surjective (one can use Lemma IV.1, but the direct calculation shows in this particular case how that works). Indeed, take $A \in \mathscr{H}$ and let $\overline{A} \in \mathscr{A}$ be such that $\overline{A} \mapsto A$ in the quotient mapping $\mathscr{A} \rightarrow \mathscr{H}$. Then the derivation $([\overline{A},\, x],[\overline{A},\, y])$ maps to $([A,\, p],[A,\, q]) \in \text{Der}(\mathscr{H})$. One must then show that this derivation is indeed in $\mathscr{G}_\mathscr{C}$. This is equivalent to showing $[[\overline{A},\, x],y] + [x,[\overline{A},\, y]] \in \mathscr{C}$. However, this expression equals $-[[x,\, y],\, \overline{A}]$, which is obviously in the kernel of the mapping $\mathscr{A} \rightarrow \mathscr{H}$. So, it is in $\mathscr{C}$.

The Heisenberg algebra is then a submanifold algebra, which can be regarded, from the point of view of algebraic geometry, as a point in the free algebra with unit $\mathscr{A}$. Its tangent space is the linear space $\mathscr{H}\oplus\mathscr{H}$.

Example 4: The matrix algebra.

Let $\mathscr{A}$ denote as above the free algebra with unit generated by two elements x,y. Let $q\in\mathbb{C}$ be an nth unit root, $q^n=1$. Let $\mathscr{C}$ denote the ideal in $\mathscr{A}$ generated by the relations

$$xy-qyx,\quad x^n-\mathbb{1},\quad y^n-\mathbb{1},$$

and denote by U and V the images of x and y in the quotient mapping $\mathscr{A}\to\mathscr{Q}$. Let us show that this algebra is the matrix algebra $M(n,\mathbb{C})$. Any element of $\mathscr{Q}$ can be written as

$$\sum_{0\leqslant k,l\leqslant n-1} a_{k,l}U^kV^l,$$

so dim $\mathscr{Q}\leqslant n^2$. Now, the following two matrices in $M(n,\mathbb{C})$,

$$U=\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & q & 0 & \cdots & 0 \\ 0 & 0 & q^2 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & q^{n-1} \end{pmatrix},\quad V=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

satisfy the relations of the algebra $\mathscr{Q}$, and then generate a subalgebra of $M(n,\mathbb{C})$. Because the only matrices which commute with this subalgebra are the multiple of identity, this is the full matrix algebra.

It is well known that the matrix algebra has only inner derivations. By Lemma IV.1, $M(n,\mathbb{C})$ can be considered as a submanifold algebra of the tensor algebra. Notice that this algebra is simple, and so can be considered as a ''point'' in the tensor algebra.

Example 5: The matrix-valued functions.

Consider, as in Ref. 10, the algebra $\mathscr{A}=C^\infty(V)\otimes M(n,\mathbb{C})$ of matrix-valued functions on a manifold V. Let $p\in V$ any point of the manifold. Take $\mathscr{C}$ to be the ideal of functions vanishing at p. This is obviously a maximal ideal. It has been shown in Ref. 10 that $\mathrm{Der}(\mathscr{A})=[\mathrm{Der}(C^\infty(V))\otimes\mathbb{1}]\oplus[C^\infty(V)\otimes\mathrm{Der}(M(n,\mathbb{C}))]$. Then a simple calculation shows that $\mathscr{Q}=\mathscr{A}/\mathscr{C}$ is the matrix algebra $M(n,\mathbb{C})$ and is a submanifold algebra of $\mathscr{A}$. The ''tangent space'' at p is $T_pV\oplus\mathrm{Der}(M(n,\mathbb{C}))$, where T_pV is the ordinary tangent space of V at p.

The physical interpretation of this situation is the following: from the point of view of non-commutative differential geometry, each point of space–time is a matrix, instead of $\mathbb{C}$ (or $\mathbb{R}$) in ordinary differential geometry. The structure looks like a fiber bundle, the fiber being a matrix algebra, but the differential structure is different, because the purely noncommutative differential structure of the matrix algebra (which is far from being trivial) is taken into account at each point of V. This supplementary differential structure of points has important consequences for gauge fields theory, as has been shown in Ref. 10.

This situation can be modified without many changes, by taking the algebra of sections of bundle over M, with fiber $M(n,\mathbb{C})$.

V. NONCOMMUTATIVE QUOTIENT MANIFOLDS

In this section, we introduce a generalization to the noncommutative framework of the notion of quotient manifold. We then introduce the generalization of the action of a group on a manifold which gives a way to construct such quotient manifolds. We give examples and we examine the possible relations with connections on central bimodules.

A. Quotient manifold algebra

Let $\mathscr{A}$ be an associative algebra with unit. Let $\mathscr{B}$ be a subalgebra of $\mathscr{A}$. Then we define the Lie subalgebras of $\mathrm{Der}(\mathscr{A})$:

$$\mathscr{Z}_{\mathrm{Der}}(\mathscr{B}) = \{X \in \mathrm{Der}(\mathscr{A})/X.\mathscr{B} = 0\}$$

and

$$\mathscr{N}_{\mathrm{Der}}(\mathscr{B}) = \{X \in \mathrm{Der}(\mathscr{A})/X.\mathscr{B} \subset \mathscr{B}\}.$$

Notice that $\mathscr{Z}_{\mathrm{Der}}(\mathscr{B})$ is an ideal in $\mathscr{N}_{\mathrm{Der}}(\mathscr{B})$, i.e., $[\mathscr{N}_{\mathrm{Der}}(\mathscr{B}), \mathscr{Z}_{\mathrm{Der}}(\mathscr{B})] \subset \mathscr{Z}_{\mathrm{Der}}(\mathscr{B})$. One can call $\mathscr{Z}_{\mathrm{Der}}(\mathscr{B})$ the centralizer of $\mathscr{B}$ in $\mathrm{Der}(\mathscr{A})$ and $\mathscr{N}_{\mathrm{Der}}(\mathscr{B})$ the normalizer of $\mathscr{B}$ in $\mathrm{Der}(\mathscr{A})$.

One has a natural homomorphism of Lie algebras $\rho : \mathscr{N}_{\mathrm{Der}}(\mathscr{B}) \to \mathrm{Der}(\mathscr{B})$, $X \mapsto \widetilde{X}$, the restriction of X to $\mathscr{B}$. The kernel of this homomorphism is exactly $\mathscr{Z}_{\mathrm{Der}}(\mathscr{B})$.

Definition V.1: The subalgebra $\mathscr{B}$ of $\mathscr{A}$ is a quotient manifold algebra of $\mathscr{A}$ if the following three conditions are fulfilled

(i) $\mathscr{Z}(\mathscr{B}) = \mathscr{B} \cap \mathscr{Z}(\mathscr{A})$,
(ii) $\mathrm{Der}(\mathscr{B}) \simeq \mathscr{N}_{\mathrm{Der}}(\mathscr{B})/\mathscr{Z}_{\mathrm{Der}}(\mathscr{B})$,
(iii) $\mathscr{B} = \{a \in \mathscr{A}/Xa = 0 \forall X \in \mathscr{Z}_{\mathrm{Der}}(\mathscr{B})\}$.

Condition (i) gives to $\mathscr{N}_{\mathrm{Der}}(\mathscr{B})$ and $\mathscr{Z}_{\mathrm{Der}}(\mathscr{B})$ a structure of $\mathscr{Z}(\mathscr{B})$-module. Then $\mathscr{N}_{\mathrm{Der}}(\mathscr{B})/\mathscr{Z}_{\mathrm{Der}}(\mathscr{B})$ is naturally a $\mathscr{Z}(\mathscr{B})$-module, and condition (ii) is an isomorphism of $\mathscr{Z}(\mathscr{B})$-modules. One then has the short exact sequence of Lie algebras and $\mathscr{Z}(\mathscr{B})$-modules

$$0 \to \mathscr{Z}_{\mathrm{Der}}(\mathscr{B}) \to \mathscr{N}_{\mathrm{Der}}(\mathscr{B}) \overset{\rho}{\to} \mathrm{Der}(\mathscr{B}) \to 0. \tag{4}$$

Now, the Lie subalgebra $\mathscr{Z}_{\mathrm{Der}}(\mathscr{B})$ of $\mathrm{Der}(\mathscr{A})$ gives a Cartan operation on $\Omega_{\mathrm{Der}}(\mathscr{A})$. Condition (iii) says that $\mathscr{B}$ is exactly the basic algebra in $\mathscr{A}$ for this operation.

Let $\omega \in \Omega^n_{\mathrm{Der}}(\mathscr{A})$ be a basic element for the operation of $\mathscr{Z}_{\mathrm{Der}}(\mathscr{B})$, $i_X\omega = 0$, and $L_X\omega = 0$ for any $X \in \mathscr{Z}_{\mathrm{Der}}(\mathscr{B})$. Then $d\omega$ is also basic. One can then define $\widetilde{\omega} \in \Omega^n_{\mathrm{Der}}(\mathscr{B})$ by the relation

$$\widetilde{\omega}(\widetilde{X}_1,\ldots,\widetilde{X}_n) = \omega(X_1,\ldots,X_n)$$

for any $\widetilde{X}_1,\ldots,\widetilde{X}_n \in \mathrm{Der}(\mathscr{B})$ and any representatives $X_1,\ldots,X_n \in \mathscr{N}_{\mathrm{Der}}(\mathscr{B})$. By the Koszul formula and condition (iii), it is easy to show that $\omega(X_1,\ldots,X_n) \in \mathscr{B}$ for $X_i \in \mathscr{N}_{\mathrm{Der}}(\mathscr{B})$. Note that condition (ii) is essential to ensure the consistency of this definition. Condition (i) shows that $\widetilde{\omega}$ is $\mathscr{Z}(\mathscr{B})$-linear. The Koszul formula shows then that $d\widetilde{\omega} = \widetilde{d\omega}$.

So one has the following lemma:

Lemma V.1: One has a mapping of graded differential algebras

$$\Omega_{\mathrm{Der},B}(\mathscr{A}) \to \Omega_{\mathrm{Der}}(\mathscr{B}),$$

where $\Omega_{\mathrm{Der},B}(\mathscr{A})$ is the subalgebra of $\Omega_{\mathrm{Der}}(\mathscr{A})$ of basic elements for $\mathscr{Z}_{\mathrm{Der}}(\mathscr{B})$.

Remark 1: In degree 0, by the very definition, one has $\Omega^0_{\mathrm{Der},B}(\mathscr{A}) = \mathscr{B} = \Omega^0_{\mathrm{Der}}(\mathscr{B})$.

Remark 2: No canonical mapping can be constructed between the basic elements of $\Omega_{\mathrm{Der}}(\mathscr{A})$ and $\Omega_{\mathrm{Der}}(\mathscr{B})$ without more information on the algebras $\mathscr{A}$ and $\mathscr{B}$.

Remark 3: Condition (ii) can be relaxed if we define $\mathrm{Der}(\mathscr{B})$ to be the Lie algebra $\mathscr{N}_{\mathrm{Der}}^{\cdot}(\mathscr{B})/\mathscr{Z}_{\mathrm{Der}}(\mathscr{B})$, even if $\mathscr{B}$ admitted other derivations. In this situation, one has a kind of induced differential structure on $\mathscr{B}$ (see Example 1 below).

Proposition V.1: If the $\mathscr{Z}(\mathscr{A})$-module induced by $\mathscr{N}_{\mathrm{Der}}^{\cdot}(\mathscr{B})$ in $\mathrm{Der}(\mathscr{A})$ is $\mathrm{Der}(\mathscr{A})$ itself, then we have an isomorphism of graded differential algebras

$$\underline{\Omega}_{\mathrm{Der},B}(\mathscr{A}) \simeq \underline{\Omega}_{\mathrm{Der}}(\mathscr{B}).$$

Proof: First, let us prove that the mapping $\underline{\Omega}_{\mathrm{Der},B}(\mathscr{A}) \to \underline{\Omega}_{\mathrm{Der}}(\mathscr{B})$ constructed above is injective. If $\widetilde{\omega}$ is zero in $\Omega_{\mathrm{Der}}^n(\mathscr{B})$, then for any $X_1,\ldots,X_n \in \mathscr{N}_{\mathrm{Der}}^{\cdot}(\mathscr{B})$ we have $\omega(X_1,\ldots,X_n)=0$. Now, ω is $\mathscr{Z}(\mathscr{A})$-linear, so ω is zero on the $\mathscr{Z}(\mathscr{A})$-module induced by $\mathscr{N}_{\mathrm{Der}}^{\cdot}(\mathscr{B})$ in $\mathrm{Der}(\mathscr{A})$. This proves injectivity.

Let $\widetilde{\omega} \in \Omega_{\mathrm{Der}}^n(\mathscr{B})$ be any n-form. Define ω as an antisymmetric n-$\mathscr{Z}(\mathscr{B})$-linear mapping from $\mathscr{N}_{\mathrm{Der}}^{\cdot}(\mathscr{B}) \otimes_{\mathscr{Z}(\mathscr{B})} \cdots \otimes_{\mathscr{Z}(\mathscr{B})} \mathscr{N}_{\mathrm{Der}}^{\cdot}(\mathscr{B})$ to $\mathscr{B}$ by the relation

$$\omega(X_1,\ldots,X_n) = \widetilde{\omega}(\widetilde{X}_1,\ldots,\widetilde{X}_n) \in \mathscr{B} \subset \mathscr{A}$$

for any $X_1,\ldots,X_n \in \mathscr{N}_{\mathrm{Der}}^{\cdot}(\mathscr{B})$. Then we extend ω on the $\mathscr{Z}(\mathscr{A})$-module induced by $\mathscr{N}_{\mathrm{Der}}^{\cdot}(\mathscr{B})$, by $\mathscr{Z}(\mathscr{A})$ linearity. Notice that ω is already $\mathscr{Z}(\mathscr{B})$ linear. By hypothesis, ω is then an element of $\Omega_{\mathrm{Der}}^n(\mathscr{A})$. We have $i_X\omega=0$ for any $X \in \mathscr{Z}_{\mathrm{Der}}(\mathscr{B})$, so ω is horizontal for the action of $\mathscr{Z}_{\mathrm{Der}}(\mathscr{B})$ in $\underline{\Omega}_{\mathrm{Der}}(\mathscr{A})$. Now, notice that the $(n+1)$-form in $\underline{\Omega}_{\mathrm{Der}}(\mathscr{A})$ which comes from $d\widetilde{\omega}$ is exactly $d\omega$, because by the Koszul formula they coincide on $\mathscr{N}_{\mathrm{Der}}^{\cdot}(\mathscr{B})$. So $d\omega$ is also horizontal, and then ω is basic in $\underline{\Omega}_{\mathrm{Der}}(\mathscr{A})$. This proves surjectivity. $\square$

B. Action

Let M be a manifold and G be a Lie group. An action of G on M gives a Lie algebra homomorphism $\mathfrak{g} \to \Gamma(M)$ from the Lie algebra $\mathfrak{g}$ of G to the Lie algebra of vector fields on M. Then one has a Cartan operation of the Lie algebra $\mathfrak{g}$ on the graded commutative differential algebra $\Omega(M)$ of de Rham differential forms on M.

In the noncommutative case, we will say we have an action of the Lie algebra $\mathfrak{g}$ on an associative algebra with unit $\mathscr{A}$ if there is a homomorphism of Lie algebras $\mathfrak{g} \to \mathrm{Der}(\mathscr{A})$. In this situation one has an operation of $\mathfrak{g}$ on the graded differential algebra $\underline{\Omega}_{\mathrm{Der}}(\mathscr{A})$.

Then one can take as subalgebra of $\mathscr{A}$ the basic algebra for this operation. In this situation, if $\mathscr{B}$ is a quotient manifold algebra, then one has the noncommutative version of the quotient manifold by the ''leaves'' defined by $\mathfrak{g}$.

If the homomorphism $\mathfrak{g} \to \mathrm{Der}(\mathscr{A})$ is injective (take the image of $\mathfrak{g}$ if not), one can identify $\mathfrak{g}$ with its image. Then, one has the inclusion $\mathfrak{g} \subset \mathscr{Z}_{\mathrm{Der}}(\mathscr{B})$, but the equality is not the generic case. Between these two Lie algebras, one has a third one, the $\mathscr{Z}(\mathscr{A})$-module induced by $\mathfrak{g}$ in $\mathrm{Der}(\mathscr{A})$, denoted by $\mathfrak{g}_{\mathscr{Z}(\mathscr{A})}$. If $\mathscr{B}$ is the basic algebra in $\mathscr{A}$ for the operation of $\mathfrak{g}$, the condition (iii) of Definition V.1 is fulfilled.

C. Examples

Example 1: The inner derivations.

Let $\mathscr{A}$ be an associative algebra with unit for which there are inner derivations. Suppose one has $H^1(\mathscr{A},\mathscr{Z}(\mathscr{A});\mathscr{A})=0$. Take the operation of $\mathfrak{g}=\mathrm{Int}(\mathscr{A})$ on $\mathscr{A}$. Then one has $\mathscr{B}=\mathscr{Z}(\mathscr{A})$, $\mathscr{Z}_{\mathrm{Der}}(\mathscr{B})=\mathfrak{g}$ because the first relative cohomology group vanishes, and $\mathscr{N}_{\mathrm{Der}}^{\cdot}(\mathscr{B})=\mathrm{Der}(\mathscr{A})$. Take then the induced differential structure on $\mathscr{B}$ by setting $\mathrm{Der}(\mathscr{B})=\mathrm{Out}(\mathscr{A})=\mathscr{N}_{\mathrm{Der}}^{\cdot}(\mathscr{B})/\mathfrak{g}$. The algebra of differential forms associated to $\mathscr{B}$ is then, by Proposition V.1, the algebra $\underline{\Omega}_{\mathrm{Out}}(\mathscr{A})$ introduced in Ref. 3.

Example 2: The noncommutative torus.

Let T_q denote the complex associative algebra with unit of elements of the form

$$a = \sum_{k,l \in \mathbb{Z}} c_{kl} U^k V^l$$

with

$$\|a\|_m = \sup_{k,l \in \mathbb{Z}} |c_{kl}| (1 + |k| + |l|)^m < \infty$$

and the relation

$$UV = qVU$$

for $q \in \mathbb{C}$ such that $q^N = 1$ for $N \in \mathbb{N}$. We take N to be the minimal one for which this is true. The center of this algebra is the set of elements depending only on U^N and V^N.

The derivations of this algebra are the inner derivations and the derivations of the form

$$a(U^N, V^N) D_U + b(U^N, V^N) D_V,$$

where $D_U(U) = U$, $D_U(V) = 0$, $D_V(U) = 0$ and $D_V(V) = V$, and $a(U^N, V^N)$ and $b(U^N, V^N)$ belong to $\mathscr{Z}(T_q)$.

Take $\mathfrak{g} = \text{Int}(T_q)$. Then $\mathscr{B} = \mathscr{Z}(T_q)$, $\mathscr{Z}_{\text{Der}}(\mathscr{B}) = \mathfrak{g}$, and $\mathscr{N}_{\text{Der}}^{\cdot}(\mathscr{B}) = \text{Der}(T_q)$. We are then in the situation of the previous example. Then the differential algebra of forms on $\mathscr{Z}(T_q)$ is the basic algebra of the differential algebra of forms on T_q. But now, remark that the center $\mathscr{Z}(T_q)$ is isomorphic to the algebra $C^\infty(S^1 \times S^1)$ of smooth functions on the (ordinary) torus. This isomorphism is $U^N \mapsto e^{2\pi i t}$ and $V^N \mapsto e^{2\pi i s}$. Then an element $a \in \mathscr{Z}(T_q)$ is mapped on the Fourier expansion of an element of $C^\infty(S^1 \times S^1)$. Thus, the algebra of forms on $\mathscr{Z}(T_q)$ is the de Rham algebra of forms on the torus.

D. Connections

Let $\mathscr{B}$ be a quotient manifold algebra of $\mathscr{A}$. Then $\mathscr{A}$ is a central bimodule over the algebra $\mathscr{B}$. Let $\psi{:}\text{Der}(\mathscr{B}) \to \mathscr{N}_{\text{Der}}^{\cdot}(\mathscr{B})$ be a splitting of the short exact sequence (4), considered as a short exact sequence of $\mathscr{Z}(\mathscr{B})$-modules (forgetting the Lie algebra structures).

Proposition V.2: For any $X \in \text{Der}(\mathscr{B})$, the mapping

$$\nabla_X{:}\mathscr{A} \to \mathscr{A}, \quad a \mapsto \psi(X)a,$$

is a connection on the central bimodule $\mathscr{A}$ over $\mathscr{B}$.

The curvature of this connection is the obstruction on ψ to be a splitting of the short exact sequence (4) of Lie algebras.

Proof: This is an immediate consequence of the fact that ψ is a $\mathscr{Z}(\mathscr{B})$-modules homomorphism, such that $\psi(X)b = Xb$ for any $b \in \mathscr{B} \subset \mathscr{A}$. The curvature of this connection is

$$R(X,Y) = [\psi(X), \psi(Y)] - \psi([X, Y]),$$

which proves the Proposition. □

Such a connection gives a projection $P{:}\mathscr{N}_{\text{Der}}^{\cdot}(\mathscr{B}) \to \mathscr{Z}_{\text{Der}}(\mathscr{B}) \subset \mathscr{N}_{\text{Der}}^{\cdot}(\mathscr{B})$ of $\mathscr{Z}(\mathscr{B})$-modules defined by $P(X) = X - \psi \circ \rho(X)$. Then one has $\mathscr{N}_{\text{Der}}^{\cdot}(\mathscr{B}) = \text{Ker } P \oplus \mathscr{Z}_{\text{Der}}(\mathscr{B})$.

Conversely, a projection $P{:}\mathscr{N}_{\text{Der}}^{\cdot}(\mathscr{B}) \to \mathscr{Z}_{\text{Der}}(\mathscr{B}) \subset \mathscr{N}_{\text{Der}}^{\cdot}(\mathscr{B})$ of $\mathscr{Z}(\mathscr{B})$-modules defines a split, and so a connection on $\mathscr{A}$.

In Ref. 11, one can find another approach to group actions, bundles, and connections in noncommutative differential geometry.

Let $P(M,G)$ be a principal bundle, where M is the base manifold and G is the structure group, and let $\mathfrak{g}$ be its Lie algebra. Denote by $\mathscr{A}$ the (commutative) algebra of smooth functions

on P, and $\mathscr{B}$ the algebra of smooth functions on M. Then one can consider that $\mathscr{B} \subset \mathscr{A}$ because of the projection $P \rightarrow M$. The Lie algebra $\mathfrak{g}$ can be injectively mapped into $\Gamma(P)$, the vector fields on P, and, more precisely, into the vertical vector fields. Thus $\mathfrak{g}$ operates on $\mathscr{A}$. The algebra $\mathscr{B}$ is obviously the basic algebra for this operation and $\mathscr{Z}_{\text{Der}}(\mathscr{B})$ is exactly the Lie algebra of vertical vector fields on P.

It is well known that a connection on P can be given as a $\mathscr{B}$-linear mapping $\Gamma(M) \rightarrow \Gamma(P), X \mapsto X^h$, the horizontal lift, with its usual properties, one of them being $[\mathfrak{g}, X^h] = 0$ for all $X \in \Gamma(M)$. In fact this mapping is a splitting of (4) [remember here that $\mathscr{B} = \mathscr{Z}(\mathscr{B})$].

Then one could think that the connections introduced by the construction of Proposition V.2 are generalizations of connections on principal bundles. However, this is not completely true, because a principal bundle has many more properties than a couple $(\mathscr{A}, \mathscr{B})$ of an algebra and a quotient manifold algebra. For example, one can introduce associated bundles, on which connections can be transported.

In order to obtain a similar situation, one must introduce a more restrictive definition. Given a couple $(\mathscr{A}, \mathscr{B})$ of an algebra and a quotient manifold algebra, suppose there exists a Lie algebra $\mathfrak{g}$ and an injective homomorphism of Lie algebras $\mathfrak{g} \rightarrow \text{Der}(\mathscr{A})$ such that $\mathscr{B}$ is the basic algebra for the operation of $\mathfrak{g}$ on $\mathscr{A}$ [then $\mathfrak{g} \subset \mathscr{Z}_{\text{Der}}(\mathscr{B})$]. A connection on this triple $(\mathscr{A}, \mathscr{B}, \mathfrak{g})$ is a splitting $\psi : \text{Der}(\mathscr{B}) \rightarrow \mathscr{N}_{\text{Der}}(\mathscr{B})$ of $\mathscr{Z}(\mathscr{B})$-modules compatible with the operation of $\mathfrak{g}$ in the sense $[\mathfrak{g}, \psi(X)] = 0$ for all $X \in \text{Der}(\mathscr{B})$. Such a connection is also given by a covariant projection $P : \mathscr{N}_{\text{Der}}(\mathscr{B}) \rightarrow \mathscr{Z}_{\text{Der}}(\mathscr{B})$ where the covariance means $[Y, P(X)] = P([Y, X])$ for all $Y \in \mathfrak{g}$ and $X \in \mathscr{N}_{\text{Der}}(\mathscr{B})$.

In this situation, if V is a linear space on which $\mathfrak{g}$ is represented by $\eta : \mathfrak{g} \rightarrow \text{End}(V)$, then one has an associated central bimodule over $\mathscr{B}$ defined by

$$\mathscr{M}_V = \{ a_i \otimes v^i \in \mathscr{A} \otimes V / (Y a_i) \otimes v^i + a_i \otimes \eta(Y) v^i = 0 \, \forall Y \in \mathfrak{g} \},$$

where the structure of bimodule over $\mathscr{B}$ is localized on $\mathscr{A}$.

Proposition V.3: Let $\psi : \text{Der}(\mathscr{B}) \rightarrow \mathscr{N}_{\text{Der}}(\mathscr{B})$ be a connection on $(\mathscr{A}, \mathscr{B}, \mathfrak{g})$. Then, the mapping

$$\nabla_X^V : \mathscr{M}_V \rightarrow \mathscr{M}_V, \quad a_i \otimes v^i \mapsto (\psi(X) a_i) \otimes v^i,$$

is well defined and is a connection on $\mathscr{M}_V$. This is the associated connection to ψ on $\mathscr{M}_V$.

Proof: $\nabla_X^V \mathscr{M}_V \subset \mathscr{M}_V$ because $[\mathfrak{g}, \psi(X)] = 0$. Other properties of ∇_X^V are immediate consequences of the definition of ψ as in Proposition V.2. $\qquad \square$

In the case of a principal bundle, $\mathscr{M}_V$ is the module over $\mathscr{B}$ of sections of the associated vector bundle for (V, η). This module of sections is considered here as the module of equivariant mapping $P \rightarrow V$.

Let us now turn to a different problem. From the point of view of characteristic classes (even if there is not yet such a theory for the definition of connection used here), what is important in a connection is its curvature. Given a couple $(\mathscr{A}, \mathscr{B})$ of an algebra and a quotient manifold algebra, suppose one has a central bimodule $\mathscr{M}$ over $\mathscr{A}$ and a connection ∇ on $\mathscr{M}$, such that its curvature is zero if one of its argument is in $\mathscr{Z}_{\text{Der}}(\mathscr{B})$. Then one can transport the connection on a central bimodule over $\mathscr{B}$. Define the reduced central bimodule over $\mathscr{B}$

$$\mathscr{M}^{\mathscr{Z}_{\text{Der}}(\mathscr{B})} = \{ m \in \mathscr{M} / \nabla_X m = 0 \, \forall X \in \mathscr{Z}_{\text{Der}}(\mathscr{B}) \}.$$

For any $\widetilde{X} \in \text{Der}(\mathscr{B})$, take any $X \in \mathscr{N}_{\text{Der}}(\mathscr{B})$ such that $\rho(X) = \widetilde{X}$. Then define, for any $m \in \mathscr{M}^{\mathscr{Z}_{\text{Der}}(\mathscr{B})}$,

$$\widetilde{\nabla}_{\widetilde{X}} m = \nabla_X m.$$

Then, because the curvature of ∇ is zero on $\mathscr{Z}_{\text{Der}}(\mathscr{B})$, this is a well-defined mapping from $\mathscr{M}^{\mathscr{Z}_{\text{Der}}(\mathscr{B})}$ into itself. It is easy to verify that $\widetilde{\nabla}$ is a connection, the curvature of which is

$$\widetilde{R}(\widetilde{X},\widetilde{Y})m = R(X,Y)m$$

for any $m \in \mathscr{M}^{\mathscr{Z}\mathrm{Der}(\mathscr{B})}$.

In the case where $\mathrm{Der}(\mathscr{A}) = \mathrm{Int}(\mathscr{A})$, it has been shown in Ref. 3 that any central bimodule $\mathscr{M}$ over $\mathscr{A}$ admits the canonical connection $\nabla_{ad(a)}m = am - ma$. The curvature of this connection is zero.

Now, in the general case $[\mathrm{Der}(\mathscr{A}) \neq \mathrm{Int}(\mathscr{A})]$, if one can take this connection on $\mathrm{Int}(\mathscr{A})$ and a prolongement on $\mathrm{Der}(\mathscr{A})$, then the curvature is zero on $\mathrm{Int}(\mathscr{A})$. So one can hope to transport the connection on a reduced module over $\mathscr{B} = \mathscr{Z}(\mathscr{A})$ while keeping the same information on the curvature.

VI. CONCLUSION

In this paper we have proposed definitions of the noncommutative generalization of a submanifold and of a noncommutative quotient manifold. Various examples and related constructions seem to give them an importance for the study of derivations-based differential structures of algebras. What must be notice is the different use of the two generalizations of differential forms: $\Omega_{\mathrm{Der}}(\mathscr{A})$ and $\Omega_{\mathrm{Der}}(\mathscr{A})$. This shows the importance of introducing various generalizations of a commutative concept adapted to different situations.

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