On the noncommutative Riemannian geometry of $\mathrm{GL}_q(n)$

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A recently proposed definition of a linear connection in noncommutative geometry, based on a generalized permutation, is used to construct linear connections on $\mathrm{GL}_q(n)$. Restrictions on the generalized permutation arising from the stability of linear connections under involution are discussed. Candidates for generalized permutations on $\mathrm{GL}_q(n)$ are found. It is shown that, for a given generalized permutation, there exists one and only one associated linear connection. Properties of the linear connection are discussed, in particular its bicovariance, torsion, and commutative limit. © *1997 American Institute of Physics*. [S0022-2488(97)02506-1]

I. INTRODUCTION

Shortly after their discovery in the context of integrable models,[1–4] quantum groups were identified as interesting noncommutative generalizations of the algebra of functions on a Lie group manifold.[5–7] Noncommutative differential calculi[8] have been proposed where the main constraint is the bicovariance of the differential algebra.[9] In addition the R-matrix formulation[7] played a key role in further developments.[10–15]

The aim of this paper is to define linear connections and metrics on quantum groups. From the mathematical point of view, this is a step towards the understanding of which classical concepts can have a noncommutative generalization. From the physical point of view, it could be a first step towards the formulation of gravitational theories on quantum groups. Noncommutative manifolds in fact could represent a solution to the problem of short distance divergences of usual quantum field theories (see, e.g., Refs. 16 and 17) and could also offer a more satisfactory description of space–time. In this respect, quantum groups are an interesting toy model where qualitative differences between the noncommutative ($q\neq 1$) and the nondeformed ($q=1$) cases can be observed. In the context of the Dirac-operator-based differential calculus of Connes, an approach to the construction of such theories has been proposed using the Wodzicki residue of the Dirac operator.[18,19] However, many interesting differential calculi, such as those on quantum groups[9] and spaces,[20] are not defined by a Dirac operator. Here, as was proposed in Refs. 21–23, we follow the idea, which is suitable for all differential calculi, of a generalization to the noncommutative context of the usual commutative metrics and linear connections.

A general definition of linear connections, in the context of noncommutative geometry, has been recently proposed for the derivation-based differential calculus[24,25] and other differential calculi[24] in which case the construction relies on a generalized permutation. In Sec. II we fix our notation concerning quantum groups. In Sec. III we briefly review the construction of Ref. 24 and

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add some restrictions on the generalized permutations which arise from the requirement that the set of covariant derivatives be stable under complex conjugation. Section IV is devoted to the search for generalized permutations on $GL_q(n)$ which are restricted by the bicovariance condition. We find a two-parameter family of generalized permutations. In Sec. V we prove that for a given generalized permutation there exists only one linear connection. Properties of this linear connection are studied, in particular its bicovariance, torsion, and curvature. Finally, we examine the commutative limit of our linear connections. We show that the limit of one class of these when $q\to 1$ corresponds to left- and right-invariant linear connections on $GL(n)$. We collect our conclusions in Sec. VI.

II. QUANTUM GROUPS AND THEIR BICOVARIANT DIFFERENTIAL CALCULI

The quantum group $\mathrm{Fun}(GL_q(n))$ is a Hopf algebra $(\mathscr{A},\Delta,\epsilon,\kappa)$ generated, as an algebra, by the identity and T_i^j, $i,j=1,\ldots n$. An exchange of the order of the generators, while maintaining the classical Poincaré series, is obtained by the RTT relation[7]:

$$RT_1T_2=T_1T_2R. \tag{II.1}$$

Here, R is the R matrix, which is an element of $M_n(\mathbb{C})\otimes M_n(\mathbb{C})$ obeying the Yang–Baxter relation

$$(1\otimes R)(R\otimes 1)(1\otimes R)=(R\otimes 1)(1\otimes R)(R\otimes 1). \tag{II.2}$$

The R matrix of $GL_q(n)$ is given by[4]

$$R=q\sum_i E_{ii}\otimes E_{ii}+\sum_{i\neq j} E_{ij}\otimes E_{ji}+\lambda\sum_{i<j} E_{ii}\otimes E_{jj}, \tag{II.3}$$

where $\lambda=q-q^{-1}$. It satisfies the Hecke condition

$$(R-q)(R+1/q)=0. \tag{II.4}$$

The differential calculus on the quantum group is considerably restricted by the bicovariance condition.[9] This means that there exist a right and left coaction of $\mathscr{A}$ on Ω^1, the space of one-forms, such that

$$\Delta_L(adb)=\Delta(a)(1\otimes d)\Delta(b), \tag{II.5}$$

$$\Delta_R(adb)=\Delta(a)(d\otimes 1)\Delta(b), \tag{II.6}$$

$$(1\otimes\Delta_R)\Delta_L=(\Delta_L\otimes 1)\Delta_R. \tag{II.7}$$

Under some restrictions on q and the assumption that Ω^1 be generated as a left-module by $dT^i_{\ j}$, bicovariant differential calculi have been classified[26] and shown to be obtained by the constructive method of Jurco.[12].

For such differential calculi Ω^1 is generated as a left (or right) module by left-invariant one-forms ω^i_j $[\Delta_L(\omega^i_j)=1\otimes\omega^i_j]$:

$$\omega^i_j=\kappa(T^i_k)dT^k_j. \tag{II.8}$$

The differential algebra is entirely characterized by the commutation relations

$$\omega^i_ja=(1\otimes f^{ik}_{jl})\Delta(a)\omega^l_k, \tag{II.9}$$

where f^{ik}_{jl} are linear functionnals representing the algebra $\mathscr{A}$:

$$f^{ik}_{jl}(1)=\delta^i_l\delta^k_j, \quad f^{ik}_{jl}(ab)=f^{im}_{jn}(a)f^{nk}_{ml}(b).$$

They can be explicitly determined in terms of the R-matrix and some parameters.[26] Here we shall make the choice

$$f^{jk}_{il}(T^m_n)=(R^{-1})^{sk}_{in}(R^{-1})^{jm}_{sl} \tag{II.10}$$

for the differential calculus. In the limit $q\to 1$, this differential calculus reduces to the usual one on $GL(n)$. It has been considered in Refs. 11, 15, 14, 27, 28, and 13. In this case, the commutation relations are often written in the form

$$T_1 dT_2=R dT_1 T_2 R. \tag{II.11}$$

The space of two-forms is constructed as the image of $\Omega^1\otimes_{\mathscr{A}}\Omega^1$ under the ''multiplication'' map π:

$$\pi:\Omega^1\otimes_{\mathscr{A}}\Omega^1\to\Omega^1\otimes_{\mathscr{A}}\Omega^1, \tag{II.12}$$

$$\pi=1-\Lambda, \tag{II.13}$$

where Λ is a bimodule automorphism, obeying the Yang–Baxter equation, which generalizes the permutation map of the commutative case. Let η^i_j be right-invariant one-forms:

$$\eta^i_j=T^i_l\omega^l_k\kappa(T^k_j). \tag{II.14}$$

Then Λ is determined by[9]

$$\Lambda(\omega^i_j\otimes\eta^k_l)=\eta^k_l\otimes\omega^i_j. \tag{II.15}$$

When applied to $\omega^i_j\otimes\omega^k_l$, one can show that

$$\Lambda(\omega^i_j\otimes\omega^k_l)=\Lambda^{ik}_{jl}{}^{np}_{mq}\omega^m_n\otimes\omega^q_p, \tag{II.16}$$

$$\Lambda^{ik}_{jl}{}^{np}_{mq}=f^{ip}_{jq}(\kappa(T^k_m)T^n_l). \tag{II.17}$$

When applied to $dT_1\otimes dT_2$, the map Λ yields

$$\Lambda(dT_1\otimes dT_2)=R dT_1\otimes dT_2 R^{-1}. \tag{II.18}$$

The Hecke relation for the R matrix (II.4), combined with the previous equation, yields the following characteristic equation for Λ:

$$(\Lambda-1)(\Lambda+q^2)(\Lambda+q^{-2})=0. \tag{II.19}$$

Higher-order forms can be constructed in a similar way using the map Λ.[9] The bicovariant bimodule Ω is the direct sum of the space of n forms:

$$\Omega=\oplus_n\Omega^n. \tag{II.20}$$

It is equipped with an exterior derivative which is defined with the help of the right- and left-invariant one-form θ,

$$\theta=-\frac{q^{2n+1}}{\lambda}\sum_i q^{-2i}\omega^i_i, \tag{II.21}$$

by

$$d\omega = [\theta, \omega], \tag{II.22}$$

where [,] is the graded commutator and the product is in Ω.

For real values of q or for $|q|=1$, an involution may be defined on $GL_q(n)$ reducing it respectively to $U_q(n)$ or to $GL_q(n,\mathbb{R})$. Setting the q-determinant equal to one gives rise to $SU_q(n)$ and $SL_q(n,\mathbb{R})$.[7] The bicovariant differential calculus on these reductions is characterized either by a larger set of one-forms than the classical case[29] or by a modified Leibniz rule.[30]

III. LINEAR CONNECTIONS IN NONCOMMUTATIVE GEOMETRY

In this section we collect the main definitions and results concerning the general construction of linear connections as proposed in Ref. 24. We add some new restrictions on the generalized permutation by imposing the stability of the set of covariant derivatives under complex conjugation. In the following $\mathcal{A}$ is a unital associative algebra over $\mathbb{C}$ equipped with the differential calculus (Ω, d).

Definition 3.1: Let π be the multiplication in Ω. A generalized permutation, σ, is a bimodule automorphism of $\Omega^1 \otimes_{\mathcal{A}} \Omega^1$ satisfying

$$\pi \circ \sigma = -\pi. \tag{III.1}$$

A generalized flip, τ, is defined as a generalized permutation satisfying $\tau^2=1$.

Remarks:

(1) Note that $\sigma=-1$ is a generalized flip.

(2) When the algebra $\mathcal{A}$ is the algebra of C^∞ functions on a manifold M, the permutation

$$\tau(\omega \otimes \omega') = \omega' \otimes \omega \tag{III.2}$$

is a generalized flip.

(3) When Ω^2 is realized as a subspace of $\Omega^1 \otimes \Omega^1$ with an imbedding i verifying $\pi \circ i = l_{\Omega^2}$, then

$$1 - 2 i \circ \pi \tag{III.3}$$

is a generalized flip. The generalized flip of the derivation-based differential calculus proposed in Refs. 25 and 24 is of this form, as are the generalized flips of Refs. 31 and 23.

(4) If σ is a generalized permutation, then so is σ^{-1} as well as σ^{2n+1} for an arbitrary integer n.

(5) If σ and σ' are two generalized permutations, then so is $\mu(\sigma+1) + \mu'(\sigma'+1) - 1$ for all μ and μ' in $\mathbb{C}$. The $\sigma+1$ form a linear space.

Definitions 3.2: A linear connection associated to a generalized permutation σ, is a linear map, ∇^σ, from Ω^1 to $\Omega^1 \otimes_{\mathcal{A}} \Omega^1$ satisfying the two Leibniz rules

$$\nabla^\sigma(a\omega) = da \otimes \omega + a \nabla^\sigma \omega, \tag{III.4}$$

$$\nabla^\sigma(\omega a) = \sigma(\omega \otimes da) + \nabla^\sigma \omega a, \tag{III.5}$$

for any $a \in \mathcal{A}$ and any $\omega \in \Omega^1$.

Remarks:

(1) When the algebra $\mathcal{A}$ is the commutative algebra of smooth functions on a manifold the only possible linear connections are those associated to the permutation (III.2).

(2) If σ and σ' are two generalized permutations, then $\nabla^\sigma - \nabla^{\sigma'}$ is a left-module homomorphism.

(3) If ∇ and ∇' are two linear connections associated to the same generalized permutation, then their difference is a bimodule homomorphism.

The preceding definition of the linear connection has the advantage of allowing an extension to the tensor product over $\mathscr{A}$ of several copies of Ω^1. This is formulated in the following

Proposition 3.3: A linear connection associated to a generalized permutation σ admits a unique extension as a linear map from $\underbrace{\Omega^1\otimes_{\mathscr{A}}\cdots\otimes_{\mathscr{A}}\Omega^1}_{s\ times}$ to $\underbrace{\Omega^1\otimes_{\mathscr{A}}\cdots\otimes_{\mathscr{A}}\Omega^1}_{s+1\ times}$ of the form

$$\nabla^\sigma(\omega\otimes\omega')=\nabla^\sigma(\omega)\otimes\omega'+\sigma_s(\omega\otimes\nabla^\sigma\omega'), \tag{III.6}$$

for any $\omega\in\Omega^1$ and any $\omega'\in\underbrace{\Omega^1\otimes_{\mathscr{A}}\cdots\otimes_{\mathscr{A}}\Omega^1}_{s-1\ times}$ with σ_s an automorphism of $\underbrace{\Omega^1\otimes_{\mathscr{A}}\cdots\otimes_{\mathscr{A}}\Omega^1}_{s+1\ times}$. The unique σ_s is given by

$$\sigma_s=\sigma\otimes\underbrace{1\otimes\cdots\otimes 1}_{s-1\ times}. \tag{III.7}$$

Proof: The proof can be carried out by induction. For $s=2$ an identification of $\nabla^\sigma(\omega f\otimes\omega')$ with $\nabla^\sigma(\omega\otimes f\omega')$, where f is an arbitrary element of $\mathscr{A}$ and ω and ω' are one-forms, gives $\sigma_2=\sigma\otimes 1$; so the proposition is true for $s=2$. Suppose it is true to order $s-1$ and let ω' be an element of the tensor product of $s-1$ copies of Ω^1. Then, by the induction hypothesis,

$$\nabla^\sigma f\omega'=df\otimes\omega'+f\nabla^\sigma\omega'. \tag{III.8}$$

The identification of $\nabla^\sigma(\omega\otimes f\omega')$ with $\nabla^\sigma(\omega f\otimes\omega')$ where ω is an element of Ω^1 completes the Proof. ●

Suppose that $\mathscr{A}$ is an algebra over $\mathbb{C}$ equipped with an involution $*$. Then Ω^1 carries a natural involution defined by $(bda)^*=(da^*)b^*$. The involution on $\Omega^1\otimes_{\mathscr{A}}\Omega^1$ is not *a priori* determined. In fact, if we define the antihomomorphism α by

$$\alpha(\omega\otimes\omega')=\omega'^*\otimes\omega^*, \tag{III.9}$$

and if ϕ is an automorphism of $\Omega^1\otimes_{\mathscr{A}}\Omega^1$ such that $(\phi\circ\alpha)^2=1$, then $\phi\circ\alpha$ defines an involution on $\Omega^1\otimes_{\mathscr{A}}\Omega^1$. We would like to define an involution on $\Omega^1\otimes_{\mathscr{A}}\Omega^1$ which in the commutative limit reduces to $(\omega\otimes\omega')^*=\tau(\omega'^*\otimes\omega^*)$, where τ is the usual permutation operator, and which allows us to define the complex conjugate of a linear connection, as in the commutative case, by

$$\bar{\nabla}^\sigma\omega=(\nabla^\sigma(\omega^*))^*. \tag{III.10}$$

The requirement that $\bar{\nabla}^\sigma$ be a linear connection imposes constraints on the involution on $\Omega^1\otimes_{\mathscr{A}}\Omega^1$ and on the generalized permutation, σ:

Proposition 3.4: Suppose that $\mathscr{A}$ is equipped with an involution $*$. Then the following assertions are equivalent:

(1) *The map $\bar{\nabla}^\sigma$ defined in (III.10) is a linear connection.*

(2) *The generalized permutation, σ, verifies*

$$(\sigma\circ\alpha)^2=1 \tag{III.11}$$

and the involution on $\Omega^1\otimes_{\mathscr{A}}\Omega^1$ *is given by*

$$(\omega\otimes\omega')^*=\sigma(\omega'^*\otimes\omega^*). \tag{III.12}$$

Proof: $2\Rightarrow 1$ is a direct calculation. We prove $1\Rightarrow 2$. Calculate, with the aid of Eq. (III.10), $\bar{\nabla}^\sigma(\omega a)$:

$$\bar{\nabla}^\sigma(\omega a)=(\nabla^\sigma(a^*\omega^*))^*=(da^*\otimes\omega^*)^*+(\nabla^\sigma\omega^*)^*a=(da^*\otimes\omega^*)^*+(\bar{\nabla}^\sigma\omega)a. \tag{III.13}$$

If the map $\bar{\nabla}^\sigma$ is a covariant derivative, then there exists a generalized permutation, ϕ, such that

$$\bar{\nabla}^\sigma(\omega a)=\phi(\omega\otimes da)+\bar{\nabla}\omega a. \tag{III.14}$$

Comparing the two equations (III.13) and (III.14) we obtain

$$(da^*\otimes\omega^*)^*=\phi(\omega\otimes da). \tag{III.15}$$

This equation is valid for arbitrary a and ω so the involution in $\Omega^1\otimes_{\mathscr{A}}\Omega^1$ verifies:

$$(\omega'\otimes\omega)^*=\phi(\omega^*\otimes\omega'^*). \tag{III.16}$$

The involution property, $**=1$, gives $(\phi\circ\alpha)^2=1$. It remains to prove that $\phi=\sigma$. In order to do this, calculate, using Eq. (III.10), $\bar{\nabla}^\sigma(a\omega)$:

$$\bar{\nabla}^\sigma(a\omega)=a(\nabla^\sigma(\omega^*))^*+(\sigma(\omega^*\otimes da^*))^*. \tag{III.17}$$

Since $\bar{\nabla}^\sigma$ is a linear connection we have

$$\bar{\nabla}^\sigma(a\omega)=a\bar{\nabla}^\sigma\omega+da\otimes\omega. \tag{III.18}$$

Comparing Eq. (III.17) and (III.8) we get

$$da\otimes\omega=(\sigma(\omega^*\otimes da^*))^*. \tag{III.19}$$

This equation is valid for arbitrary a and ω, so we have

$$\omega'\otimes\omega=(\sigma(\omega^*\otimes\omega'^*))^*. \tag{III.20}$$

Comparing Eqs. (III.16) and (III.20) leads to the quality of ϕ and σ. ●

Definition 3.5: For a given involution $*$ *on* $\Omega^1\otimes_{\mathscr{A}}\Omega^1$, *a generalized permutation* σ *is defined to be real if it satisfies the following property:*

$$\sigma\circ *=*\circ\sigma \tag{III.21}$$

on $\Omega^1\otimes_{\mathscr{A}}\Omega^1$.

Now, if one wants to find an involution $*$ on $\Omega^1\otimes_{\mathscr{A}}\Omega^1$ such that $\bar{\nabla}^\sigma$ is a linear connection, then one should take, according to Proposition 3.4, (III.12) as a definition of $*$. The condition for this to be possible is $(\sigma\circ\alpha)^2=1$. If one further demands that σ be real, then one has to use the following:

Proposition 3.6: Suppose that the generalized permutation, σ, *verifies Eq. (III.11) and that the involution on* $\Omega^1\otimes_{\mathscr{A}}\Omega^1$ *is given by* $\sigma\circ\alpha$. *Then* σ *is real iff it is a generalized flip.*

Proof: The reality condition reads

$$\sigma\circ\sigma\circ\alpha=\sigma\circ\alpha\circ\sigma. \tag{III.22}$$

Since σ is an automorphism, this equation leads to

$$\sigma\circ\alpha=\alpha\circ\sigma. \tag{III.23}$$

The relation in $(\sigma\circ\alpha)^2=1$ gives $\sigma^2=1$. ●

The definition of the complex conjugate of a linear connection allows the following:

Definition 3.7: A real linear connection associated to a generalized permutation σ is defined by $\bar{\nabla}^\sigma=\nabla^\sigma$.

Remark: The involution on $\Omega^1\otimes_{\mathscr{A}}\Omega^1$ defined above induces an involution on Ω^2 by $(\omega\wedge\omega')^*=\pi((\omega\otimes\omega')^*)=-\omega'^*\wedge\omega^*$. This is due to the property (III.1).

Definition 3.8: The torsion T of a linear connection ∇^σ is defined as the linear map from Ω^1 to Ω^2 given by

$$T=d-\pi\circ\nabla^\sigma. \tag{III.24}$$

Proposition 3.9: The torsion map is a bimodule homomorphism.

Proof: It is an immediate consequence of the condition (III.1). ●

Definition 3.10: The curvature $\mathscr{R}$ of a linear connection ∇^σ is defined as the linear map from Ω^1 to $\Omega^2\otimes_{\mathscr{A}}\Omega^1$ given by

$$\mathscr{R}=((T\otimes 1)+(\pi\otimes 1)\nabla^\sigma)\nabla^\sigma. \tag{III.25}$$

Proposition 3.11: The curvature is a left-module homomorphism.

Proof: A straightforward calculation. ●

Definition 3.12: A metric g is defined as an element of $\Omega^1\otimes_{\mathscr{A}}\Omega^1$ satisfying

$$\pi(g)=0. \tag{III.26}$$

If $\Omega^1\otimes_{\mathscr{A}}\Omega^1$ is equipped with an involution, a real metric is defined by $g^*=g$.

The definition of a nondegenerate metric requires some more structure on the algebra $\mathscr{A}$. This structure must guarantee that the dimension of Ω^1 as a left module be well defined. For example, if $\mathscr{A}$ is a Hopf algebra, then it is well known that this is so (see, e.g., Ref. 9). If it exists, let ω^a, $a=1,\ldots,N$, be a free basis of Ω^1 as a left module. Then a metric can be written uniquely in the form

$$g=g_{ab}\omega^a\otimes\omega^b, \tag{III.27}$$

with $g_{ab}\in\mathscr{A}$. We will call a metric nondegenerate if the matrix whose elements are g_{ab} is invertible.

Definition 3.13: A metric g and a linear connection ∇^σ are said to be compatible if the condition $\nabla^\sigma g=0$ is satisfied.

IV. DETERMINATION OF σ ON $GL_Q(N)$

In addition to the previous requirements on σ, it is natural, in the context of quantum groups, to add the requirement of bicovariance:

Definition 4.1: A generalized permutation, σ, is called bicovariant iff

$$(1\otimes\sigma)\Delta_L=\Delta_L\sigma,\quad (\sigma\otimes 1)\Delta_R=\Delta_R\sigma. \tag{IV.1}$$

Following the R-matrix technique, that is the determination of all unknown maps from the R-matrix and q, we will determine the candidates for the map σ in terms of R. We recall from (II.13) and (III.1) that the generalized permutation σ is an automorphism of $\Omega^1\otimes_{\mathscr{A}}\Omega^1$ verifying

$$(1-\Lambda)\circ(\sigma+1)=0, \tag{IV.2}$$

the bicovariance requirements (IV.1), and when $\mathscr{A}$ is equipped with an involution, that is for real q and for $|q|=1$, the involution property (III.11).

In order to find candidates for σ, we shall prove the following Proposition, which, in its first part, is a generalization of Proposition 3.1 of Ref. 9:

Proposition 4.2: Let α_{ij}, $i,j=0,1$, *be complex numbers.*

(1) *There exists a unique bimodule homomorphism*, Φ, *of* $\Omega^1\otimes_{\mathscr{A}}\Omega^1$ *such that*

$$\Phi(dT_1\otimes dT_2)=\sum_{i,j}\alpha_{ij}R^i dT_1\otimes dT_2 R^j. \tag{IV.3}$$

Moreover,

(2) *The map* Φ *is bicovariant.*

(3) *The map* Φ *is a generalized permutation iff*

$$\alpha_{01}-\alpha_{10}=0,\quad \alpha_{00}+\lambda\alpha_{10}-\alpha_{11}=-1, \tag{IV.4}$$

where, we recall, $\lambda=q-q^{-1}$.

In this case, Φ *obeys the characteristic equation*

$$(\Phi+1)(\Phi-\lambda_1)(\Phi-\lambda_2)=0, \tag{IV.5}$$

$$\lambda_1=-1+\alpha_{10}(q+q^{-1})+\alpha_{11}(1+q^2), \tag{IV.6}$$

$$\lambda_2=-1-\alpha_{10}(q+q^{-1})+\alpha_{11}(1+q^{-2}). \tag{IV.7}$$

Proof: An element ν of $\Omega^1\otimes_{\mathscr{A}}\Omega^1$ can be written in a unique way as

$$\nu=\sum a^{ij}_{kl}dT^k_i\otimes dT^l_j=\mathrm{Tr}(a(dT_1\otimes dT_2)), \tag{IV.8}$$

where $a\in M_n(\mathscr{A})\otimes M_n(\mathscr{A})$. This is a consequence of the fact that the dT generate Ω^1 as a left module. The action of Φ on ν is defined by

$$\Phi(\nu)=\mathrm{Tr}(a\alpha_{ij}R^i dT_1\otimes dT_2R^j). \tag{IV.9}$$

It clearly satisfies (IV.3). It remains to check that Φ defined in this way is a bimodule homomorphism. The left-module homomorphism property is assured by construction. To check the right-module homomorphism property it suffices to verify that

$$\Phi(dT_1\otimes dT_2T_3)=\Phi(dT_1\otimes dT_2)T_3. \tag{IV.10}$$

This is so because the T generate the algebra. The left-hand side of Eq. (IV.10) can be written, after successive use of Eq. (II.11), as

$$\Phi(dT_1\otimes R_{23}^{-1}T_2dT_3R_{23}^{-1})=R_{23}^{-1}R_{12}^{-1}\Phi(T_1dT_2\otimes dT_3)R_{12}^{-1}R_{23}^{-1}. \tag{IV.11}$$

Here the subscripts of the R-matrix denote the two spaces on which it acts. Next, we use the left-module property to write the right-hand side of Eq. (IV.11) as

$$R_{23}^{-1}R_{12}^{-1}T_1\Phi(dT_2\otimes dT_3)R_{12}^{-1}R_{23}^{-1}=\alpha_{ij}R_{23}^{-1}R_{12}^{-1}R_{23}^{i}T_1dT_2\otimes dT_3R_{23}^{j}R_{12}^{-1}R_{23}^{-1}. \quad \text{(IV.12)}$$

The right-hand side of Eq. (IV.10) can be written as

$$\alpha_{ij}R_{12}^{i}dT_1\otimes dT_2R_{12}^{j}T_3=\alpha_{ij}R_{12}^{i}dT_1\otimes dT_2T_3R_{12}^{j}. \quad \text{(IV.13)}$$

The commutation relations (II.11) allow us to write this term as

$$\alpha_{ij}R_{12}^{i}dT_1\otimes R_{23}^{-1}T_2dT_3R_{23}^{-1}R_{12}^{j}=\alpha_{ij}R_{12}^{i}R_{23}^{-1}R_{12}^{-1}T_1dT_2\otimes dT_3R_{12}^{-1}R_{23}^{-1}R_{12}^{j}. \quad \text{(IV.14)}$$

As a consequence of the Yang–Baxter equation we have

$$\begin{aligned} R_{12}^{i}R_{23}^{-1}R_{12}^{-1}&=R_{23}^{-1}R_{12}^{-1}R_{23}^{i},\\ R_{23}^{i}R_{12}^{-1}R_{23}^{-1}&=R_{12}^{-1}R_{23}^{-1}R_{12}^{i}. \end{aligned} \quad \text{(IV.15)}$$

The right hand sides of Eqs. (IV.12) and (IV.14) are thus equal. This proves the first point of the Proposition.

In order to prove the bicovariance of Φ, it suffices to prove that

$$\Delta_L\Phi(dT_1\otimes dT_2)=(1\otimes\Phi)\Delta_L(dT_1\otimes dT_2), \quad \text{(IV.16)}$$

$$\Delta_R\Phi(dT_1\otimes dT_2)=(\Phi\otimes 1)\Delta_R(dT_1\otimes dT_2). \quad \text{(IV.17)}$$

This is due to the fact that $dT_1\otimes dT_2$ generate $\Omega^1\otimes_{\mathscr{A}}\Omega^1$ as a left module. Using Eq. (IV.3) and

$$\Delta_L(dT_1\otimes dT_2)=T_1T_2\otimes dT_1\otimes dT_2, \quad \text{(IV.18)}$$

$$\Delta_R(dT_1\otimes dT_2)=dT_1\otimes dT_2\otimes T_1T_2, \quad \text{(IV.19)}$$

Eqs. (IV.16) and (IV.17) can be written as

$$\alpha_{ij}R^{i}T_1T_2\otimes dT_1\otimes dT_2R^{j}=\alpha_{ij}T_1T_2\otimes R^{i}dT_1\otimes dT_2R^{j}, \quad \text{(IV.20)}$$

$$\alpha_{ij}R^{i}dT_1\otimes dT_2\otimes T_1T_2R^{j}=\alpha_{ij}R^{i}dT_1\otimes dT_2R^{j}\otimes T_1T_2. \quad \text{(IV.21)}$$

These equations are true due to the commutation relations (II.1). This proves point 2 of the Proposition.

Point 3 is a straightforward calculation using Eqs. (II.18) and the Hecke condition (II.4). ●

Proposition 4.2 gives us a two-parameter family of bicovariant generalized permutations. We turn to examine some of their properties. First, note that the maps Φ have the same eigenspaces even though their eigenvalues might be different. In fact, if we introduce the projectors

$$\begin{aligned} \Pi_1(dT_1\otimes dT_2)&=\hat{P}_q dT_1\otimes dT_2\hat{P}_q,\\ \Pi_2(dT_1\otimes dT_2)&=\hat{P}_{-q^{-1}}dT_1\otimes dT_2\hat{P}_{-q^{-1}},\\ \Pi_3(dT_1\otimes dT_2)&=\hat{P}_{-q^{-1}}dT_1\otimes dT_2\hat{P}_q,\\ \Pi_4(dT_1\otimes dT_2)&=\hat{P}_q dT_1\otimes dT_2\hat{P}_{-q^{-1}}, \end{aligned} \quad \text{(IV.22)}$$

with

$$\hat{P}_q = \frac{R+q^{-1}}{q+q^{-1}}, \quad \hat{P}_{-q^{-1}} = \frac{q-R}{q+q^{-1}}, \tag{IV.23}$$

then the generalized permutation Φ can be written as

$$\sigma_{\lambda_1,\lambda_2} = \lambda_1 \Pi_1 + \lambda_2 \Pi_2 - \Pi_3 - \Pi_4, \tag{IV.24}$$

and the expression for Λ is

$$\Lambda = \Pi_1 + \Pi_2 - q^2 \Pi_3 - q^{-2} \Pi_4. \tag{IV.25}$$

In the commutative limit $\Pi_1 + \Pi_2$ tends to the projector onto symmetric elements of $\Omega^1 \otimes_{\mathscr{A}} \Omega^1$ and $\Pi_3 + \Pi_4$ to the projector onto antisymmetric elements. The multiplication map π may be expressed in terms of these projections as

$$\pi = (1+q^2)\Pi_3 + (1+q^{-2})\Pi_4. \tag{IV.26}$$

So Ω^2 can be identified with the projection of $\Omega^1 \otimes_{\mathscr{A}} \Omega^1$:

$$\Omega^2 = (\Pi_3 + \Pi_4)\Omega^1 \otimes_{\mathscr{A}} \Omega^1. \tag{IV.27}$$

An imbedding i of Ω^2 in $\Omega^1 \otimes_{\mathscr{A}} \Omega^1$ verifying $\pi \circ i = 1_{\Omega^2}$, exists and is given by

$$i = \frac{1}{1+q^2}\Pi_3 + \frac{1}{1+q^{-2}}\Pi_4. \tag{IV.28}$$

With the aid of this imbedding we obtain the expression (III.3) for σ:

$$\sigma_\Lambda = 1 - 2 i \circ \pi = -1 + 2(\Pi_1 + \Pi_2). \tag{IV.29}$$

Note that this σ verifies $\sigma^2 = 1$; it is equal to -1 on Ω^2 and to $+1$ on $(\Pi_1 + \Pi_2)(\Omega^1 \otimes_{\mathscr{A}} \Omega^1)$. It corresponds to $\lambda_1 = \lambda_2 = 1$ in Eq. (IV.24).

Another simple solution to Eqs. (IV.4), which in addition obeys the Yang–Baxter equation, is given by σ_{q^{-2},q^2},

$$\sigma_R(dT_1 \otimes dT_2) = R^{-1} dT_1 \otimes dT_2 R^{-1}. \tag{IV.30}$$

This σ is to be compared with the σ found in Ref. 32 for the quantum plane. Indeed, it could be obtained in the same way from the differential calculus (see Lemma 5.13).

We turn now to consider involutions for $|q| = 1$. Then one can consider the involution $(T^i_j)^* = T^i_j$ on $GL_q(n)$. This involution is compatible with the relations on the algebra because, for $|q| = 1$ and R given by (II.3), one has

$$\bar{R}^{ij}_{kl} = (R^{-1})^{ji}_{lk} \tag{IV.31}$$

where $\bar{R}$ is the complex conjugate of R. In this case, the quantum group is $GL_q(n, \mathbb{R})$.

Proposition 4.3: Let $|q| = 1$. *A generalized permutation*, $\sigma_{\lambda_1,\lambda_2}$, *defines an involution iff*

$$|\lambda_1| = |\lambda_2| = 1. \tag{IV.32}$$

Proof: For $|q| = 1$ relations (IV.31) imply that

$$\alpha \circ \Pi_i \circ \alpha = \Pi_i, \quad i = 1,2,3,4. \tag{IV.33}$$

So, the condition (III.11) reads

$$|\lambda_1|^2\Pi_1+|\lambda_2|^2\Pi_2+\Pi_3+\Pi_4=1, \tag{IV.34}$$

which completes the Proof. ●

Remark: The previously defined σ_Λ and σ_R satisfy Eqs. (III.11).

V. LINEAR CONNECTIONS ON $\mathrm{GL}_Q(N)$

In this section we determine linear connections on the quantum group $\mathrm{GL}_q(n)$ and study their properties.

Proposition 5.1: Let σ be any generalized permutation. The map ∇_0^σ defined by

$$\begin{aligned}&\nabla_0^\sigma:\Omega^1\rightarrow\Omega^1\otimes_{\mathscr{A}}\Omega^1,\\ &\nabla_0^\sigma(\omega)=\theta\otimes\omega-\sigma(\omega\otimes\theta).\end{aligned} \tag{V.1}$$

is a linear connection associated to σ.

Proof: Calculate first

$$\nabla_0^\sigma(a\omega)=([\theta,a]+a\theta)\otimes\omega-\sigma(a\omega\otimes\theta) \tag{V.2}$$

and then use the expression of the exterior derivative and the bimodule property to obtain

$$\nabla_0^\sigma(a\omega)=da\otimes\omega+a\nabla_0^\sigma\omega. \tag{V.3}$$

Similarly, calculate

$$\begin{aligned}\nabla_0^\sigma(\omega a)&=\theta\otimes\omega a+\sigma(\omega\otimes([\theta,\, a]-\theta a)),\\ &=\sigma(\omega\otimes da)+(\nabla_0^\sigma\omega)a.\end{aligned} \tag{V.4}$$

This completes the Proof. ●

Remarks:

(1) The linear connection ∇_0^σ can be defined on any differential calculus where the exterior derivative is a graded commutator. See Ref. 31 for another example.

(2) For $\sigma=-1$ the resulting covariant derivative ∇_0^σ is $i\circ d$, where i is the embedding of Ω^2 into $\Omega^1\otimes_{\mathscr{A}}\Omega^1$, by Eq. (II.21).

Proposition 5.2: The extension of ∇_0^σ to the tensor product of s copies of Ω^1 is given by

$$\nabla_0^\sigma\nu=\theta\otimes\nu+\sigma_s(\nu\otimes\theta),\quad \forall\,\nu\in\Omega^1\otimes_{\mathscr{A}}\cdots\Omega^1. \tag{V.5}$$

Proof: A direct application of Proposition 3.3. ●

Proposition 5.3: There are no nonvanishing bimodule homomorphisms from Ω^1 to $\Omega^1\otimes_{\mathscr{A}}\Omega^1$.

Proof: We will use the following Lemma proved in Refs. 7, 13, and 27.

Lemma 5.4: Let c be the q-determinant of T,

$$c=\det_q T=\sum_p\,(-q)^{l(p)}T^1_{p(1)}T^2_{p(2)}\cdots T^n_{p(n)}, \tag{V.6}$$

where the sum is over all permutations on n elements and $l(p)$ is the number of transpositions in the permutation p. Then c is in the center of $\mathscr{A}$ and verifies $\omega c=q^{-2}c\omega$ for all ω in Ω^1.

An immediate consequence of the preceding Lemma is the following.

Corollary 5.5: $\nu c = q^{-4} c\nu$, $\forall \nu \in \Omega^1 \otimes_{\mathscr{A}} \Omega^1$.

We are now in position to prove the Proposition. Let ϕ be a bimodule homomorphism from Ω^1 to $\Omega^1 \otimes_{\mathscr{A}} \Omega^1$ and let $\omega \in \Omega^1$. By the homomorphism property and Lemma 5.4, we get

$$\phi(\omega)c = q^{-2} c\phi(\omega). \tag{V.7}$$

On the other hand, since $\phi(\omega) \in \Omega^1 \otimes_{\mathscr{A}} \Omega^1$, by Corollary 5.5 we obtain

$$\phi(\omega)c = q^{-4} c\phi(\omega). \tag{V.8}$$

Comparing these two equations we prove the Proposition. ●

As a direct consequence of the preceding and of the third remark following Definition 3.2 we obtain the following

Theorem 5.6: *For any generalized permutation σ on $\mathrm{GL}_q(n)$, there exists one and only one associated linear connection, given by (V.1).*

We now turn to the study of some of the properties of the linear connection ∇_0^σ.

Proposition 5.7: For any generalized permutation σ, the linear connection ∇_0^σ has vanishing torsion.

Proof: Calculate $\pi \circ \nabla_0^\sigma$

$$\pi \circ \nabla_0^\sigma \omega = \theta \wedge \omega + \omega \wedge \theta = d\omega, \tag{V.9}$$

where we have used the property (III.1). The proof of the Proposition follows from (III.24). ●

Proposition 5.8: For any generalized permutation σ, the linear connection ∇_0^σ has the expression

$$\nabla_0^\sigma \omega^a = (\Lambda - \sigma)\omega^a \otimes \theta = ((\lambda_1 - 1)\Pi_1 + (\lambda_2 - 1)\Pi_2 + (q^2 - 1)\Pi_3 + (q^{-2} - 1)\Pi_4)\omega^a \otimes \theta \tag{V.10}$$

on the left invariant one-forms ω^a, and

$$\nabla_0^\sigma \eta^a = (\Lambda^{-1} - \sigma)\eta^a \otimes \theta = ((\lambda_1 - 1)\Pi_1 + (\lambda_2 - 1)\Pi_2 + (q^{-2} - 1)\Pi_3 + (q^2 - 1)\Pi_4)\eta^a \otimes \theta \tag{V.11}$$

on the right-invariant one-forms η^a

Proof: This is an immediate consequence of the definition of Λ, the right invariance of θ, and Eq. (IV.24). •

Definition 5.9: A bicovariant linear connection, ∇, *is defined by the properties*

$$(1 \otimes \nabla)\Delta_L = \Delta_L \nabla \quad (\textit{left covariance}), \tag{V.12}$$

$$(\nabla \otimes 1)\Delta_R = \Delta_R \nabla \quad (\textit{right covariance}). \tag{V.13}$$

Proposition 5.10: The linear connections associated to the generalized permutations $\sigma_{\lambda_1,\lambda_2}$ of formula (IV.24) are bicovariant.

Proof: First, one sees that Λ and $\sigma_{\lambda_1,\lambda_2}$ are bicovariant. Then, using formula (V.10) and the left invariance of ω^a one sees that formula (V.12) is true when applied to ω^a. Now, the one-forms ω^a form a basis of the left module Ω^1. Then, formula (III.4) and the previous result show that the associated linear connection is left invariant.

For the right invariance, one has to consider the right-invariant one-forms η^a, which constitute a basis of Ω^1 as a right module and formulas (V.11) and (III.5). •

The following Proposition allows one to calculate explicitly the covariant derivative associated to a generalized permutation given by Eq. (IV.24):

Proposition 5.11: Define ν, γ, *and* β *by*

$$\nu = q + q^{-1}, \quad \gamma = \frac{\lambda_1 - \lambda_2}{q^{-2} - q^2}, \quad \beta = \frac{\lambda_1 q^2 - \lambda_2 q^{-2}}{q^{-2} - q^2}; \tag{V.14}$$

the linear connection associated to the generalized permutation, $\sigma_{\lambda_1,\lambda_2}$, *acts on left-invariant one-forms as follows*:

$$\nabla_0^{\sigma_{\lambda_1,\lambda_2}} \omega^i_j = -\frac{1}{\nu^2}(1-\gamma-\beta)\omega^i_k \wedge \omega^k_j - \gamma \omega^i_k \otimes \omega^k_j + \frac{1}{2}(1-\gamma+\beta)(\omega^i_j \otimes \theta + \theta \otimes \omega^i_j)$$
$$+ \frac{\lambda^2}{2\nu^2}(1-\gamma-\beta)(\omega^i_j \otimes \theta - \theta \otimes \omega^i_j). \tag{V.15}$$

Proof: First we note that $\sigma_{\lambda_1,\lambda_2}$ of (IV.24) can be written as

$$\sigma_{\lambda_1,\lambda_2} = (\lambda_1 + 1)\Pi_1 + (\lambda_2 + 1)\Pi_2 - 1, \tag{V.16}$$

so that $\nabla \omega^i_j$ can be expressed as

$$\nabla \omega^i_j = \theta \otimes \omega^i_j + \omega^i_j \otimes \theta - [(\lambda_1 + 1)\Pi_1 + (\lambda_2 + 1)\Pi_2]\omega^i_j \otimes \theta. \tag{V.17}$$

It remains to calculate the term in the brackets of (V.17). We will do so by calculating it for two different values of the couple (λ_1, λ_2) with the aid of the following two Lemmata.

Lemma 5.12: The covariant derivative associated to σ_Λ *acts on left-invariant one-forms as follows*:

$$\nabla_0^{\sigma_\Lambda} \omega^i_j = -\frac{2}{\nu^2}\omega^i_k \wedge \omega^k_j - \frac{\lambda^2}{\nu^2}(\theta \otimes \omega^i_j - \omega^i_j \otimes \theta). \tag{V.18}$$

Proof: The Proof is a straightforward calculation exploiting the fact that σ_Λ can be expressed in terms of Λ as

$$\sigma_\Lambda = -1 + 2\frac{(\Lambda + q^2)(\Lambda + q^{-2})}{\nu^2}, \tag{V.19}$$

as well as the equation

$$d\omega^i_j = (1-\Lambda)\theta \otimes \omega^i_j = -\omega^i_k \wedge \omega^k_j, \tag{V.20}$$

which allows us to eliminate $\Lambda(\theta \otimes \omega^i_j)$ in $\nabla^{\sigma_\Lambda} \omega^i_j$. ●

Lemma 5.13: The covariant derivative associated to σ_R *is determined by*

$$\nabla^{\sigma_R} dT^i_j = 0. \tag{V.21}$$

Proof: Calculate the covariant derivative associated to σ_R of the two sides of Eq. (II.11). ●

The Proof of the Proposition is completed after expressing $\nabla^{\sigma_{\lambda_1,\lambda_2}} \omega^i_j$ in terms of $\nabla^{\sigma_\Lambda} \omega^i_j$ and $\nabla^{\sigma_R} \omega^i_j$ as

$$\nabla^{\sigma_{\lambda_1,\lambda_2}} \omega^i_j = \tfrac{1}{2}(1-\gamma+\beta)(\theta \otimes \omega^i_j + \omega^i_j \otimes \theta) + \tfrac{1}{2}(1-\gamma-\beta)\nabla^{\sigma_\Lambda}\omega^i_j + \gamma \nabla^{\sigma_R}\omega^i_j. \tag{V.22}$$

This equation is obtained after the evaluation of $\Pi_k \omega^i_j \otimes \theta$, $k=1,2$, in terms of $\nabla^{\sigma_\Lambda} \omega^i_j$ and $\nabla^{\sigma_R} \omega^i_j$. ●

Finally, we consider the limit of the linear connections determined above when $q \to 1$. In this limit the differential calculus tends to the usual commutative differential calculus. The one-form θ has a singular limit but $\lambda\theta$ tends to the right- and left-invariant one-form α on $GL(n)$. First of all, a necessary condition for the limit to be nonsingular is that the generalized permutation tend to the flip operator that is $\lambda_1 \to 1$ and $\lambda_2 \to 1$. A more precise statement, giving a necessary and sufficient condition for the limit to be nonsingular, is the following:

Proposition 5.14: Let

$$\mu_i = \frac{\lambda_i - 1}{\lambda}, \quad i = 1,2, \tag{V.23}$$

the linear connection associated to $\sigma_{\lambda_1,\lambda_2}$ admits a nonsingular limit iff μ_1 and μ_2 have finite limits $\mu_i|_{q=1}$ when q tends to 1. The linear connection, in the limit, is determined by

$$\nabla \omega^i_j = -\frac{1}{2}(1-\gamma_0)\omega^i_k \wedge \omega^k_j - \gamma_0 \omega^i_k \otimes \omega^k_j - \frac{\mu_0}{2}(\alpha \otimes \omega^i_j + \omega^i_j \otimes \alpha), \tag{V.24}$$

where

$$\gamma_0 = \frac{\mu_2|_{q=1} - \mu_1|_{q=1}}{2}, \quad \mu_0 = \frac{\mu_2|_{q=1} + \mu_1|_{q=1}}{2}. \tag{V.25}$$

Proof: A direct application of Proposition 5.11. ●

Remark: When μ_1 and μ_2 tend to 0, which is the case of $\sigma_\wedge$, γ_0 and μ_0 vanish and the limiting linear connection is given by

$$\nabla \omega^i_j = -\tfrac{1}{2}\omega^i_k \wedge \omega^k_j. \tag{V.26}$$

VI. CONCLUSION

The main result of this paper is the existence and uniqueness, for generic q, of the linear connection associated to a given generalized permutation. This connection is bicovariant and torsion-free. This is in contrast to the commutative case ($q=1$) where there are an infinite number of linear connections not necessarily bicovariant and torsion-free and where the generalized permutation is constrained to be the flip operator. It is also in contrast to the cases with q a root of unity where Proposition 5.3 is not in general valid. The arbitrariness in the deformed case lies merely in the generalized permutation for which we have found a two-parameter family [Eq. (IV.24)]. These parameters may be arbitrary functions of q and are constrained by the involution property (Proposition 4.3). The commutative limit is nonsingular for a class of such functions which tend to the identity when $q \to 1$. The commutative limit of the linear connection is a subset of right- and left-invariant linear connections on $GL(n)$.

We have used the differential calculus (II.10) to obtain our results. Had we used another differential calculus with the usual commutative limit the qualitative aspects of our conclusions, in particular the uniqueness of the linear connection associated to a given generalized permutation, are expected to remain the same.

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