LINEAR CONNECTIONS ON THE TWO-PARAMETER QUANTUM PLANE

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We apply a recently proposed definition of a linear connection in non-commutative geometry based on the natural bimodule structure of the algebra of differential forms to the case of the two-parameter quantum plane. We find that there exists a non-trivial family of linear connections only when the two parameters obey a specific relation.

1. Introduction

In the last few years, much attention has been attracted by non-commutative differential geometry. Many related attempts focused on generalization of differential forms [1, 2, 3], as well as covariant derivative [1]. This later generalization of covariant derivative only used a left or a right module structure. However, in order to extend the notion of linear connection to non-commutative geometry, one has to deal with one-forms. Then, the bimodule structure of the space of one-forms should be taken into account. This has been done in [4a, b] for the general derivation based differential calculus and in [5] for more general differential calculi. Other examples based on [5] have been worked out in [6] and [7].

In this letter, we adopt this last viewpoint to construct linear connections on the two-parameter quantum plane. In Sec. 2, we recall the general definition of a linear connection on a non-commutative algebra as well as some results already obtained in [6] on the one-parameter quantum plane [8]. In Sec. 3 we present our results for the two-parameter quantum plane.

2. Linear Connections Over a Non-commutative Algebra

2a. *Basic tools*

Let $\mathcal{A}$ and $(\Omega^*(\mathcal{A}), d)$ be respectively a non-commutative algebra and a differential calculus over it (d is the exterior derivative). Let $\Omega^k(\mathcal{A})$ ($k \geq 0$) be the

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algebra of differential forms of degree k on $\mathcal{A}$ and π: $\Omega^1(\mathcal{A})\otimes_{\mathcal{A}}\Omega^1(\mathcal{A}) \to \Omega^2(\mathcal{A})$ the projection defined by the product of forms ($\otimes_{\mathcal{A}}$ is the tensor product on $\mathcal{A}$).

We first recall the definition of a linear connection on $\mathcal{A}$ given in [4–6] that will be the main ingredient used in this letter.

Definition. A linear connection over $\mathcal{A}$ is determined from two maps σ and D, σ: $\Omega^1(\mathcal{A})\otimes_{\mathcal{A}}\Omega^1(\mathcal{A}) \to \Omega^1(\mathcal{A})\otimes_{\mathcal{A}}\Omega^1(\mathcal{A})$, D: $\Omega^1(\mathcal{A}) \to \Omega^1(\mathcal{A})\otimes_{\mathcal{A}}\Omega^1(\mathcal{A})$, satisfying the following properties: for any $f,g \in \Omega^0(\mathcal{A})$, $\alpha,\beta \in \Omega^1(\mathcal{A})$

$$\sigma \text{ is left and right-linear } \sigma(f\alpha\otimes\beta g) = f\sigma(\alpha\otimes\beta)g \tag{2.1}$$

$$\pi(\sigma+1) = 0 \tag{2.2}$$

$$D(f\alpha) = df\otimes\alpha + fD\alpha,\ D(\alpha f) = D\alpha f + \sigma(\alpha\otimes df)\,. \tag{2.3}$$

Concerning this definition, some comments are in order.

Basically, the map σ, which is a bimodule automorphism in $\Omega^1(\mathcal{A})\otimes_{\mathcal{A}}\Omega^1(\mathcal{A})$, can be viewed as a generalization of the permutation map that would appear in the case of a commutative algebra. The map D is a non-commutative extension of the covariant derivative (observe in particular the Leibnitz rules, property (2.3)). Furthermore, it can be easily seen that the right (resp. left) $\mathcal{A}$-linearity of σ, property (2.1), insures that $D((\xi f)g) = D(\xi(fg))$ (resp. $D(d((fg)h)) = D(d(f(gh)))$) for any $\xi \in \Omega^1(\mathcal{A})$, $f,g,h \in \Omega^0(\mathcal{A})$.

As far as the properties (2.2) and (2.3) are concerned, it must be pointed out that not each solution of (2.2) admits a covariant derivative, that is, a map D fulfilling the Leibnitz rules (2.3). In other words, a linear connection for each σ does not necessarily exist.

Finally, the above definition is nothing but a non-commutative extension of the definition of a linear connection in term of a covariant derivative introduced by Koszul [9] in the framework of commutative geometry. Indeed, for a commutative algebra $\mathcal{A}$, it follows from (2.3) that σ reduces to the usual permutation, so that (2.1) and (2.2) are satisfied. The Koszul definition then follows.

2b. *Application to the Manin quantum plane*

Let $\Omega^* = \Omega^0 \oplus \Omega^1 \oplus \Omega^2$ the algebra of forms of the Manin quantum plane [8] whose generators $x^i := (x,y)$, $\xi^i = dx^i := (\xi,\eta)$ obey the following relations

$$x^ix^j - q^{-1}\hat{R}^{ij}_{kl}x^kx^l = 0\,; \tag{2.4a}$$

$$x^i\xi^j - q\hat{R}^{ij}_{kl}\xi^kx^l = 0\,; \tag{2.4b}$$

$$\xi^i\xi^j + q\hat{R}^{ij}_{kl}\xi^k\xi^l = 0\,, \tag{2.4c}$$

where the tensor $\hat{R}^{ij}_{kl}$ is given in matrix form by

$$\hat{R} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}\,. \tag{2.5}$$

We recall that Ω^* is stable under the action of $SL_q(2,C)$ since the following identity $\hat{R}^{ij}_{kl}a^k_m a^l_n = a^i_k a^j_l \hat{R}^{kl}_{mn}$ holds (where a^i_j denote generically the four generators of $SL_q(2,C)$ written again in matrix form with $da^i_j = 0$ and d is the exterior derivative).

Now, the action of D on Eq. (2.4b) which is defined on Ω^1 determines the action of σ on $\xi^i \otimes \xi^j$. It is given [6] by

$$\sigma(\xi\otimes\xi) = q^{-2}(\xi\otimes\xi)\,; \tag{2.6a}$$

$$\sigma(\xi\otimes\eta) = q^{-1}(\eta\otimes\xi)\,, \tag{2.6b}$$

$$\sigma(\eta\otimes\xi) = q^{-1}(\xi\otimes\eta) + (q^{-2}-1)(\eta\otimes\xi)\,; \tag{2.6c}$$

$$\sigma(\eta\otimes\eta) = q^{-2}(\eta\otimes\eta)\,, \tag{2.6d}$$

so that $\sigma = q^{-1}\hat{R}^{-1}$ and can be proven to be left and right linear. Furthermore, σ obeys the property (2.2) of the general definition, namely $\pi(\sigma+1) = 0$, as it can be easily seen by computing the action of $\pi(\sigma+1)$ on $\xi^i\otimes\xi^j$ and using the relation (2.4c). Notice that $\sigma^2 \neq 1$; however, it satisfies a Hecke relation given by

$$(\sigma+1)(\sigma-q^{-2}) = 0\,, \tag{2.7}$$

where the simple (resp. triply degenerate) eigenvalue -1 (resp. q^{-2}) corresponds to the antisymmetric (resp. symmetric) eigenspaces with eigenvectors $\xi\otimes\eta - q\eta\otimes\xi$ (resp. $\xi\otimes\xi$, $\eta\otimes\eta$, $\eta\otimes\xi + q\xi\otimes\eta$).

Finally, the covariant derivative map D acting on the ξ^i's can be cast into the form

$$D\xi^i = \mu\, x^i\theta\otimes\theta \tag{2.8}$$

where μ is an arbitrary parameter (which has the dimension of a mass) and $\theta\in\Omega^1$ is the unique 1-form invariant under the coaction of $SL_q(2,C)$ which is given by $\theta = x\eta - qy\xi$ (up to an overall constant) with $\theta^2 = 0$.

3. The Two-Parameter Quantum Plane

The algebraic structure of the two-parameter quantum plane is now defined by

$$x^ix^j - q^{-1}\hat{R}^{ij}_{kl}(p,q)x^kx^l = 0\,; \tag{3.1a}$$

$$x^i\xi^j - p\hat{R}^{ij}_{kl}(p,q)\xi^kx^l = 0\,; \tag{3.1b}$$

$$\xi^i\xi^j + p\hat{R}^{ij}_{kl}(p,q)\xi^k\xi^l = 0 \tag{3.1c}$$

where again $x^i := (x,y)$, $\xi^i = dx^i := (\xi,\eta)$ and the tensor $\hat{R}^{ij}_{kl}(p,q)$ is given by

$$\hat{R} = \begin{pmatrix} q & 0 & 0 & 0\\ 0 & q-p^{-1} & qp^{-1} & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & q\end{pmatrix} \tag{3.2}$$

and reduces to (2.5) when $q = p$. Recall that $\hat{R}(p,q)$ satisfies the braid equation: $\hat{R}_{12}(p,q)\hat{R}_{23}(p,q)\hat{R}_{12}(p,q) = \hat{R}_{23}(p,q)\hat{R}_{12}(p,q)\hat{R}_{23}(p,q)$ with as usual $\hat{R}_{12}(p,q) := \hat{R}(p,q) \otimes 1$, $\hat{R}_{23}(p,q) := 1 \otimes \hat{R}(p,q)$. For the moment, p and q are independant arbitrary parameters. We will see in a while that a non-trivial family of linear connections can be obtained only when p and q obey a supplementary relation.

From the action of D on Eq. (3.1b), we determine the action of σ on $\xi^i \otimes \xi^j$. Namely, using the property (2.3) of the general definition, we obtain generically

$$\xi^i \otimes \xi^j = p\hat{R}^{ij}_{kl}(p,q)\sigma(\xi^k \otimes \xi^l)\,; \tag{3.3a}$$

$$x^i D\xi^j = p\hat{R}^{ij}_{kl}(p,q)(D\xi^k)x^l\,. \tag{3.3b}$$

The Eq. (3.3a) is verified when $\sigma = p^{-1}\hat{R}^{-1}(p,q)$ or equivalently

$$\sigma(\xi \otimes \xi) = p^{-1}q^{-1}(\xi \otimes \xi)\,; \tag{3.4a}$$

$$\sigma(\xi \otimes \eta) = p^{-1}(\eta \otimes \xi)\,, \tag{3.4b}$$

$$\sigma(\eta \otimes \xi) = q^{-1}(\xi \otimes \eta) + (p^{-1}q^{-1} - 1)(\eta \otimes \xi)\,; \tag{3.4c}$$

$$\sigma(\eta \otimes \eta) = p^{-1}q^{-1}(\eta \otimes \eta)\,. \tag{3.4d}$$

It is easy to verify the left and right-linearity of σ (property (2.1)). Notice that right-linearity stems from left-linearity, as a mere consequence of the braid equation for $\hat{R}(p,q)$. To see that, it is sufficient to combine $\sigma = p^{-1}\hat{R}^{-1}(p,q)$ with (3.3a) and (3.1b) together with the braid equation; then, the above statement follows. Notice also that σ fulfills a braid equation since $\hat{R}(p,q)$ does. Besides, combining (3.4) and (3.1c), we also verify that $\pi(\sigma + 1) = 0$ (property (2.2)) still holds.

Thus, we have determined a suitable σ map on the two parameter quantum plane. The remaining Eq. (3.3b) will fix a relation between p and q so that a non-trivial family of covariant derivative D can be associated to this σ. By non-trivial family, we mean that D can differ from the trivial case $D\xi = D\eta = 0$ which always verifies (3.3b). Namely, we find after some calculations that (3.3b) is verified provided $D\xi$ and $D\eta$ are given by

$$D\xi = xZ\theta \otimes \theta\,; \tag{3.5a}$$

$$D\eta = yZ\theta \otimes \theta\,; \tag{3.5b}$$

$$Z = \mu x^{n-1}y^{n-1} \tag{3.5c}$$

where μ is an overall complex parameter; (3.5) must be supplemented by

$$p = q^n \quad n \geq 1\,. \tag{3.6}$$

In (3.5), $\theta = x\eta - qy\xi$ and still verifies $\theta^2 = 0$ and

$$\sigma(\xi \otimes \theta) = q^{-1-2n}\theta \otimes \xi\,; \tag{3.7a}$$

$$\sigma(\theta \otimes \xi) = q^n(\xi \otimes \theta) - (1 - q^{-1-n})(\theta \otimes \xi) \tag{3.7b}$$

$$\sigma(\eta \otimes \theta) = q^{-2-n}\theta \otimes \eta\,; \tag{3.7c}$$

$$\sigma(\theta \otimes \eta) = q(\eta \otimes \theta) - (1 - q^{-1-n})(\theta \otimes \eta) \tag{3.7d}$$

$$\sigma(\theta \otimes \theta) = q^{-1-n}(\theta \otimes \theta) \tag{3.7e}$$

where we used (3.6).

It is interesting to observe that a non-trivial family of linear connection on the two-parameter quantum plane can be consistently found only when the two parameters are related to each other through (3.6).

Therefore, the corresponding map σ is given in the tensor form by

$$\sigma = \begin{pmatrix} q^{-1-n} & 0 & 0 & 0 \\ 0 & 0 & q^{-n} & 0 \\ 0 & q^{-1} & q^{-1-n} - 1 & 0 \\ 0 & 0 & 0 & q^{-1-n} \end{pmatrix} \tag{3.8}$$

and satisfies the Hecke relation $(\sigma + 1)(\sigma - q^{-1-n}) = 0$ where the simple (resp. triply degenerate) eigenvalue -1 (resp. q^{-1-n}) corresponds to the antisymmetric (resp. symmetric) eigenspace with eigenvectors $\xi \otimes \eta - q^n \eta \otimes \xi$ (resp. $\xi \otimes \xi$, $\eta \otimes \eta$, $\eta \otimes \xi + q\xi \otimes \eta$).

Some remarks are in order. Firstly, it happens that σ is actually the only generalized permutation for which there exits a covariant derivative.

Next, one recovers the results of Sec. 2b when $n = 1$. We point out that the case $p = q$ is very similar, as far as the structure of the set of linear connections is concerned, to the cases $p = q^n$.

Finally, in (3.6) we have restricted $n \geq 1$ since working on the quantum plane forces us to consider only positive powers in x and y. However, the case $pq = 1$ is interesting. In this last situation, the algebra is formally the non-commutative torus (for which negative powers of x and y are allowed). The differential calculus is then based on derivations in the sense of [4b]. An easy calculation using σ given in (3.4) for $pq = 1$ leads to an eight complex parameter family of linear connections. This family can also be obtained using the definition of linear connection based on derivations given in [2, 4a].

Let $\pi_{12} := \pi \otimes 1$ and $\sigma_{12} := \sigma \otimes 1$. The covariant derivative map defined above can be extended to $\Omega^1 \otimes_{\mathcal{A}} \Omega^1$ by

$$D(\alpha \otimes \beta) = D\alpha \otimes \beta + \sigma_{12}(\alpha \otimes D\beta)\,, \tag{3.9}$$

for any $\alpha, \beta \in \Omega^1$. Consider now the following map

$$\pi_{12} D^2 : \Omega^1 \to (\Omega^2/\Theta) \otimes_{\mathcal{A}} \Omega^1\,, \tag{3.10}$$

where Θ is a submodule of Ω^2, called the torsion module, given by the image of Ω^1 by $d - \pi D$. This map is left-linear, namely $\pi_{12} D^2(f\alpha) = f\pi_{12} D^2(\alpha)$ for any $f \in \Omega^0$ $(\Omega^0 = \mathcal{A})$, $\alpha \in \Omega^1$ since σ verifies $\pi(\sigma + 1) = 0$.

By noticing that $(d - \pi D)\xi^i = 0$ holds, thanks to $\theta^2 = 0$ and combining (3.1), (3.9) and $\pi(\sigma + 1) = 0$, we obtain

$$\pi_{12} D^2 \theta = 0 . \tag{3.11}$$

Now, using again (3.1), (3.5), (3.6), and (3.8) we find that

$$\pi_{12} D^2 \xi^i = \Omega^i \otimes \theta , \tag{3.12}$$

with

$$\Omega^i = f(q) x^i Z \xi \eta , \tag{3.13a}$$

$$f(q) = \frac{1}{1-q^{n+1}} \left(1 - q^{3(n+1)} - q^{n+1} + q^{(2-n)(n+1)} - q^{(1-n^2)} + q^{(n+1)(1-2n)}\right) . \tag{3.13b}$$

Then, from (3.12) and (3.13), we obtain the 2-form curvature Ω^i_j given by

$$\pi_{12} D^2 \xi^i = -\Omega^i_j \otimes \xi^j , \tag{3.14}$$

$$\Omega^i_j = f(q) \begin{pmatrix} q^{-n-1} xy & -q^{-1-2n} x^2 \\ q^{-1-n} y^2 & -q^{-1-2n} yx \end{pmatrix} Z \xi \eta . \tag{3.15}$$

4. Conclusion

In this letter, working on the two-parameter quantum plane, we have shown that there exists a non-trivial family of linear connections only when the parameters are related through $p = q^n$, $n \geq 1$. In this respect, the usual one-parameter quantum plane ($n = 1$) is fully representative of the $p = q^n$ situation. It remains to see whether this specific relation between p and q is purely accidental or reflects a deeper property.

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