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Linear connections on matrix geometries

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Abstract. A general definition of a linear connection in non-commutative geometry has recently been proposed. Two examples are given of linear connections in non-commutative geometries which are based on matrix algebras. They both possess a unique metric connection.

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1. Introduction and motivation

The extension to non-commutative algebras of the notion of a differential calculus has been given both without (Connes 1986) and with (Dubois-Violette 1988) use of the derivations of the algebra. A definition has been given (Chamseddine *et al* 1993) of a possible non-commutative generalization of a linear connection which uses the left-module structure of the differential forms. Recently a different definition has been given (Mourad 1995, Dubois-Violette *et al* 1994) which makes essential use of the full bimodule structure of the differential forms. We shall use this definition here to consider linear connections on two examples of non-commutative geometries based on matrix algebras. Both have a unique linear connection, which is metric and torsion-free. In this respect they are similar to the quantum plane, which is not based on a finite-dimensional algebra.

The general definition of a linear connection is given in this section, and in section 2 some basic formulae from matrix geometry are recalled. In section 3 we consider an algebra of forms based on derivations and we show that there is a unique metric linear connections without torsion. This case is very similar to ordinary differential geometry and the calculations follow closely those of this section. In section 4 we consider a more abstract differential geometry whose differential calculus is not based on derivations. Here we find that there is a unique one-parameter family of connections, which is without torsion. The condition that the connection be metric fixes the value of the parameter.

We first recall the definition of a linear connection in commutative geometry, in a form (Koszul 1960) which allows for a non-commutative generalization. Let V be a differential manifold and let $(\Omega^*(V), \mathrm{d})$ be the ordinary differential calculus on V. Let H be a vector bundle over V associated with some principle bundle P. Let $\mathcal{C}(V)$ be the algebra of smooth functions on V and $\mathcal{H}$ the left $\mathcal{C}(V)$-module of smooth sections of H.

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A connection on P is equivalent to a covariant derivative on H, which in turn can be characterized as a linear map

$$\mathcal{H} \xrightarrow{\mathrm{D}} \Omega^1(V) \otimes_{\mathcal{C}(V)} \mathcal{H} \tag{1.1}$$

which satisfies the condition

$$\mathrm{D}(f\psi) = \mathrm{d}f \otimes \psi + f\mathrm{D}\psi \tag{1.2}$$

for arbitrary $f \in \mathcal{C}(V)$ and $\psi \in \mathcal{H}$.

The definition of a connection as a covariant derivative has an immediate extension to non-commutative geometry. Let $\mathcal{A}$ be an arbitrary algebra and $(\Omega^*(\mathcal{A}), \mathrm{d})$ a differential calculus over $\mathcal{A}$. We shall define in the next section a differential calculus $(\Omega^*(M_n), \mathrm{d})$ over the matrix algebras M_n. One defines a covariant derivative on a left $\mathcal{A}$-module $\mathcal{H}$ as a map

$$\mathcal{H} \xrightarrow{\mathrm{D}} \Omega^1(\mathcal{A}) \otimes \mathcal{H} \tag{1.3}$$

which satisfies the condition (1.2) but with $f \in \mathcal{A}$.

A linear connection on V can be defined as a connection on the cotangent bundle to V. It can be characterized as a linear map

$$\Omega^1(V) \xrightarrow{\mathrm{D}} \Omega^1(V) \otimes_{\mathcal{C}(V)} \Omega^1(V) \tag{1.4}$$

which satisfies the condition

$$\mathrm{D}(f\xi) = \mathrm{d}f \otimes \xi + f\mathrm{D}\xi \tag{1.5}$$

for arbitrary $f \in \mathcal{C}(V)$ and $\xi \in \Omega^1(V)$.

Suppose, for simplicity that V is parallelizable and choose θ^α to be a globally defined moving frame on V. The connection form $\omega^\alpha{}_\beta$ is defined in terms of the covariant derivative of the moving frame:

$$\mathrm{D}\theta^\alpha = -\omega^\alpha{}_\beta \otimes \theta^\beta . \tag{1.6}$$

Because of (1.5) the covariant derivative $\mathrm{D}\xi$ of an arbitrary element $\xi = \xi_\alpha\theta^\alpha \in \Omega^1(V)$ can be written as $\mathrm{D}\xi = (\mathrm{D}\xi_\alpha) \otimes \theta^\alpha$ where

$$\mathrm{D}\xi_\alpha = \mathrm{d}\xi_\alpha - \omega^\beta{}_\alpha\xi_\beta . \tag{1.7}$$

Let π be the projection of $\Omega^1(V) \otimes_{\mathcal{C}(V)} \Omega^1(V)$ onto $\Omega^2(V)$. The torsion form Θ^α can be defined as

$$\Theta^\alpha = (\mathrm{d} - \pi\mathrm{D})\theta^\alpha . \tag{1.8}$$

The derivative $\mathrm{D}_X\xi$ along the vector field X,

$$\mathrm{D}_X\xi = i_X\mathrm{D}\xi \tag{1.9}$$

is a linear map of $\Omega^1(V)$ into itself. In particular, $\mathrm{D}_X\theta^\alpha = -\omega^\alpha{}_\beta(X)\theta^\beta$. Using D_X an extension of D can be constructed to the tensor product $\Omega^1(V)\otimes_{\mathcal{C}(V)}\Omega^1(V)$. We define

$$\mathrm{D}_X(\theta^\alpha\otimes\theta^\beta) = \mathrm{D}_X\theta^\alpha\otimes\theta^\beta + \theta^\alpha\otimes\mathrm{D}_X\theta^\beta\,. \tag{1.10}$$

Now let σ be the action on $\Omega^1(V)\otimes_{\mathcal{C}(V)}\Omega^1(V)$ defined by the permutation of two derivations:

$$\sigma(\xi\otimes\eta)(X,Y) = \xi\otimes\eta(Y,X) \tag{1.11}$$

and define $\sigma_{12} = \sigma\otimes 1$. Then (1.10) can be rewritten without explicitly using the vector field as

$$\mathrm{D}(\theta^\alpha\otimes\theta^\beta) = \mathrm{D}\theta^\alpha\otimes\theta^\beta + \sigma_{12}(\theta^\alpha\otimes\mathrm{D}\theta^\beta)\,. \tag{1.12}$$

Define $\pi_{12} = \pi\otimes 1$. If the torsion vanishes one finds that

$$\pi_{12}\mathrm{D}^2\theta^\alpha = -\Omega^\alpha{}_\beta\otimes\theta^\beta \tag{1.13}$$

where $\Omega^\alpha{}_\beta$ is the curvature 2-form. Notice that the equality

$$\pi_{12}\mathrm{D}^2(f\theta^\alpha) = f\pi_{12}\mathrm{D}^2\theta^\alpha \tag{1.14}$$

is a consequence of the identity

$$\pi(\sigma+1) = 0\,. \tag{1.15}$$

The module $\Omega^1(V)$ has a natural structure as a right $\mathcal{C}(V)$-module and the corresponding condition equivalent to (1.5) is determined using the fact that $\mathcal{C}(V)$ is a commutative algebra:

$$\mathrm{D}(\xi f) = \mathrm{D}(f\xi)\,. \tag{1.16}$$

Using σ this can also be written in the form

$$\mathrm{D}(\xi f) = \sigma(\xi\otimes\mathrm{d}f) + (\mathrm{D}\xi)f\,. \tag{1.17}$$

By extension, a linear connection over a general non-commutative algebra $\mathcal{A}$ with a differential calculus $(\Omega^*(\mathcal{A}),\mathrm{d})$ can be defined as a linear map

$$\Omega^1(\mathcal{A}) \xrightarrow{\mathrm{D}} \Omega^1(\mathcal{A})\otimes_{\mathcal{A}}\Omega^1(\mathcal{A}) \tag{1.18}$$

which satisfies the condition (1.5) for arbitrary $f\in\mathcal{A}$ and $\xi\in\Omega^1(\mathcal{A})$.

The module $\Omega^1(\mathcal{A})$ again has a natural structure as a right $\mathcal{A}$-module but in the non-commutative case it is impossible in general to consistently impose the condition (1.16) and a substitute must be found. We consider first the case where the differential calculus $(\Omega^*(\mathcal{A}),\mathrm{d})$ is defined using the derivations of $\mathcal{A}$ (Dubois-Violette 1988). Let X and Y be arbitrary derivations of $\mathcal{A}$ and suppose that the transposition σ in (1.11) maps $\Omega^1(\mathcal{A})\otimes_{\mathcal{A}}\Omega^1(\mathcal{A})$ into itself. Then we propose to define $\mathrm{D}(\xi f)$ by the equation (1.17) (Dubois-Violette and Michor 1994, 1995). A covariant derivative is a map of the form (1.18) which satisfies the Leibniz rules (1.5) and (1.17). The right Leibniz rule (1.18) can be made more transparent using the covariant derivative D_X with respect to the derivation X. The two Leibniz rules can be written as

$$\begin{aligned} D_X(f\xi) &= (Xf)\xi + fD_X\xi \\ D_X(\xi f) &= \xi Xf + (D_X\xi)f . \end{aligned} \tag{1.19}$$

A metric g on V can be defined as a $\mathcal{C}(V)$-bilinear, symmetric map of $\Omega^1(V) \otimes_{\mathcal{C}(V)} \Omega^1(V)$ into $\mathcal{C}(V)$. This definition makes sense if one replaces $\mathcal{C}(V)$ by an algebra $\mathcal{A}$ and $\Omega^1(V)$ by a differential calculus $\Omega^1(\mathcal{A})$ over $\mathcal{A}$. By analogy with the commutative case we shall say that the covariant derivative (1.17) is metric if the following diagram is commutative:

$$\begin{array}{ccc} \Omega^1 \otimes_{\mathcal{A}} \Omega^1 & \xrightarrow{D} & \Omega^1 \otimes_{\mathcal{A}} \Omega^1 \otimes_{\mathcal{A}} \Omega^1 \\ g \downarrow & & \downarrow 1 \otimes g \\ \mathcal{A} & \xrightarrow{d} & \Omega^1 . \end{array} \tag{1.20}$$

Here we have set $\Omega^1(\mathcal{A}) = \Omega^1$. In general symmetry must be defined with respect to the map σ. By a symmetric metric then we mean one which satisfies the condition

$$g\sigma = g . \tag{1.21}$$

2. Matrix geometries

Non-commutative geometry is based on the fact that one can formulate (Koszul 1960) much of the ordinary differential geometry of a manifold in terms of the algebra of smooth functions defined on it. It is possible to define a finite non-commutative geometry based on derivations by replacing this algebra by the algebra M_n of $n \times n$ complex matrices (Dubois-Violette *et al* 1989, 1990). Since M_n is of finite dimension as a vector space, all calculations reduce to pure algebra. Matrix geometry is interesting in being similar is certain aspects to the ordinary geometry of compact Lie groups; it constitutes a transition to the more abstract formalism of general non-commutative geometry (Connes 1986, 1990). Our notation is that of Dubois-Violette *et al* (1989). See also Madore (1995). In this section we recall some important formulae.

Let λ_r, for $1 \leqslant r \leqslant n^2 - 1$, be an anti-Hermitian basis of the Lie algebra of the special unitary group SU_n in n dimensions. The λ_r generate M_n and the derivations

$$e_r = \mathrm{ad}\,\lambda_r \tag{2.1}$$

form a basis for the Lie algebra of derivations $\mathrm{Der}(M_n)$ of M_n.

We define df for $f \in M_n$ by

$$df(e_r) = e_r(f) . \tag{2.2}$$

In particular,

$$d\lambda^r(e_s) = -C^r{}_{st}\lambda^t .$$

We raise and lower indices with the Killing metric g_{rs} of SU_n.

We define the set of 1-forms $\Omega^1(M_n)$ to be the set of all elements of the form $f\mathrm{d}g$ with f and g in M_n. The set of all differential forms is a differential algebra $\Omega^*(M_n)$. The couple $(\Omega^*(M_n), \mathrm{d})$ is a differential calculus over M_n.

There is a convenient system of generators of $\Omega^1(M_n)$ as a left- or right-module completely characterized by the equations

$$\theta^r(e_s) = \delta^r_s \,. \tag{2.3}$$

The θ^r are related to the $\mathrm{d}\lambda^r$ by the equations

$$\mathrm{d}\lambda^r = C^r{}_{st}\lambda^s\theta^t \qquad \theta^r = \lambda_s\lambda^r\mathrm{d}\lambda^s \,. \tag{2.4}$$

The θ^r satisfy the same structure equations as the components of the Maurer–Cartan form on the special unitary group SU_n:

$$\mathrm{d}\theta^r = -\tfrac{1}{2}C^r{}_{st}\,\theta^s\theta^t \,. \tag{2.5}$$

The product on the right-hand side of this formula is the product in $\Omega^*(M_n)$. We shall refer to the θ^r as a frame or Stehbein. If we define $\theta = -\lambda_r\theta^r$ we can write the differential $\mathrm{d}f$ of an element $f \in \Omega^0(M_n)$ as a commutator:

$$\mathrm{d}f = -[\theta, f] \,. \tag{2.6}$$

3. A differential calculus with derivations

From equation (2.5) we see that the linear connection defined by

$$\mathrm{D}\theta^r = -\omega^r{}_s \otimes \theta^s \qquad \omega^r{}_s = -\tfrac{1}{2}C^r{}_{st}\,\theta^t \tag{3.1}$$

has vanishing torsion. With this connection the geometry of M_n looks like the invariant geometry of the group SU_n. Since the elements of the algebra commute with the frame θ^r, we can define D on all of $\Omega^*(M_n)$ using (1.5). The map σ is given by

$$\sigma(\theta^r \otimes \theta^s) = \theta^s \otimes \theta^r \,. \tag{3.2}$$

It follows that D also satisfies (1.17).

Consider a general covariant derivative. We can suppose it to be of the form

$$\mathrm{D}\theta^r = -\omega^r{}_{st}\,\theta^s \otimes \theta^t \tag{3.3}$$

with $\omega^r{}_{st}$ an arbitrary element of M_n for each value of (r, s, t). Then from (1.5) and (1.17) we find that

$$0 = \mathrm{D}([f, \theta^r]) = [f, \mathrm{D}\theta^r] \tag{3.4}$$

and so the $\omega^r{}_{st}$ must be all in the centre of M_n. They are complex numbers. If we define the torsion as in (1.8) and require that it vanish then we have

$$\omega^r{}_{[st]} = C^r{}_{st} \,. \tag{3.5}$$

Define a metric on M_n by the equation $g(\theta^r \otimes \theta^s) = g^{rs}$. It satisfies the symmetry condition (1.21). The commutativity of the diagram (1.20) is the formal analogue of the condition that a connection be metric. If we impose it we see that

$$\omega^r{}_{(st)} = 0 \,. \tag{3.6}$$

The linear connection (3.1) is the unique torsion-free metric connection on $\Omega^1(M_n)$. From the formula analogous to (1.13) we find that the curvature 2-form is given by

$$\Omega^r{}_s = \tfrac{1}{8}C^r{}_{st}C^t{}_{uv}\theta^u\theta^v .$$

The connection (3.1) has been used (Dubois-Violette *et al* 1989, Madore 1990, Madore and Mourad 1993, Madore 1995) in the construction of non-commutative generalizations of Kaluza–Klein theories. In particular, the Dirac operator has a natural coupling to it, determined by a correspondence principle.

4. A differential calculus without derivations

Equation (1.17) can be extended, in principle, to the case of a differential calculus which is not based on derivations if we postulate (Mourad 1995) the existence of a map

$$\Omega^1 \otimes_{\mathcal{A}} \Omega^1 \xrightarrow{\sigma} \Omega^1 \otimes_{\mathcal{A}} \Omega^1 \tag{4.1}$$

to replace the one defined by (1.11). We then define $\mathrm{D}(\xi f)$ by equation (1.17), but using (4.1) instead of (1.11). From the identity

$$\mathrm{D}((\xi f)g) = \mathrm{D}(\xi(fg))$$

we find that σ must be right $\mathcal{A}$-linear. From the identity

$$\mathrm{D}(\mathrm{d}((fg)h)) = \mathrm{D}(\mathrm{d}(f(gh)))$$

we find that σ must be also left $\mathcal{A}$-linear (Dubois-Violette *et al* 1994). In general,

$$\sigma^2 \neq 1\,. \tag{4.2}$$

The extension of D to $\Omega^1 \otimes \Omega^1$ is given by (1.12) but with σ defined by (4.1).

As an example we shall consider a differential calculus over an algebra of matrices with a differential defined by a graded commutator (Connes and Lott 1990). Consider the matrix algebra M_n with a $\mathbb{Z}_2$ grading. One can define on M_n a graded derivation $\hat{\mathrm{d}}$ by the formula

$$\hat{\mathrm{d}} f = -[\theta, f] \tag{4.3}$$

where θ is an arbitrary anti-Hermitian odd element and the commutator is taken as a graded commutator. We find that $\hat{\mathrm{d}}\theta = -2\theta^2$ and for any $\alpha \in M_n$,

$$\hat{\mathrm{d}}^2\alpha = [\theta^2, \alpha]\,. \tag{4.4}$$

The $\mathbb{Z}_2$ grading of M_n can be expressed as the direct sum $M_n = M_n^+ \oplus M_n^-$ where M_n^+ (M_n^-) are the even (odd) elements of M_n. It can be induced from a decomposition $\mathbb{C}^n = \mathbb{C}^l \oplus \mathbb{C}^{n-l}$ for some integer l. The elements of M_n^+ are diagonal with respect to the decomposition; the elements of M_n^- are off-diagonal.

It is possible to construct over M_n^+ a differential algebra $\Omega^* = \Omega^*(M_n^+)$ (Connes and Lott 1991). Let $\Omega^0 = M_n^+$ and let $\Omega^1 \equiv \overline{\mathrm{d}\Omega^0} \subset M_n^-$ be the M_n^+-bimodule generated by the image of Ω^0 in M_n^- under $\hat{\mathrm{d}}$. Define

$$\Omega^0 \xrightarrow{\mathrm{d}} \Omega^1 \tag{4.5}$$

using directly (4.3): $\mathrm{d} = \hat{\mathrm{d}}$. Let $\overline{\mathrm{d}\Omega^1}$ be the M_n^+-module generated by the image of Ω^1 in M_n^+ under $\hat{\mathrm{d}}$. It would be natural to try to set $\Omega^2 = \overline{\mathrm{d}\Omega^1}$ and define

$$\Omega^1 \xrightarrow{\mathrm{d}} \Omega^2 \tag{4.6}$$

once again using (4.3). Every element of Ω^1 can be written as a sum of elements of the form $f_0\hat{\mathrm{d}}f_1$. If we attempt to define an application (4.6) using again directly (4.3),

$$\mathrm{d}(f_0\hat{\mathrm{d}}f_1) = \hat{\mathrm{d}}f_0\hat{\mathrm{d}}f_1 + f_0\hat{\mathrm{d}}^2 f_1 \tag{4.7}$$

then we see that, in general, d^2 does not vanish. To remedy this problem we eliminate simply the unwanted terms. Let $\mathrm{Im}\,\hat{d}^2$ *be the submodule of* $\overline{d\Omega^1}$ *consisting of those elements which* contain a factor which is the image of $\hat{d}^2$ and define Ω^2 by

$$\Omega^2 = \overline{d\Omega^1}/\mathrm{Im}\,\hat{d}^2 . \tag{4.8}$$

Then by construction the second term on the right-hand side of (4.7) vanishes as an element of Ω^2 and we have a well defined map (4.6) with $d^2 = 0$. This procedure can be continued to arbitrary order by iteration. For each $p \geqslant 2$ we let $\mathrm{Im}\,\hat{d}^2$ be the submodule of $\overline{d\Omega^{p-1}}$ defined as above and we define Ω^p by

$$\Omega^p = \overline{d\Omega^{p-1}}/\mathrm{Im}\,\hat{d}^2 . \tag{4.9}$$

Since $\Omega^p\Omega^q \subset \Omega^{p+q}$ the complex Ω^* is a differential algebra. The Ω^p need not vanish for large values of p. In fact if $\theta^2 \propto 1$ we see that $\hat{d}^2 = 0$ and the sequence defined by (4.9) never stops. However, $\Omega^p \subseteq M_n^+(M_n^-)$ for p even (odd) and so it stabilizes for large p.

We shall consider in some detail the case $n = 3$ with the grading defined by the decomposition $\mathbb{C}^3 = \mathbb{C}^2 \oplus \mathbb{C}$. The most general possible form for θ is

$$\theta = \eta_1 - \eta_1^* \tag{4.10}$$

where

$$\eta_1 = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} . \tag{4.11}$$

Without loss of generality we can choose the euclidean 2-vector η_{1i} of unit length. The general construction yields $\Omega^0 = M_3^+ = M_2 \times M_1$ and $\Omega^1 = M_3^-$ but after that the quotient by elements of the form $\mathrm{Im}\,\hat{d}^2$ reduces the dimension. One finds $\Omega^2 = M_1$ and $\Omega^p = 0$ for $p \geqslant 3$. Let e be the unit of M_1. It generates Ω^2 and can also be considered as an element of Ω^0.

To form a basis for Ω^1 we must introduce a second matrix η_2. It is convenient to choose it of the same form as η_1. We have then in Ω^2 the identity

$$\eta_i \eta_j^* = 0 .$$

We shall further impose that

$$\eta_i^* \eta_j = \delta_{ij} e . \tag{4.12}$$

It follows that

$$d\eta_1 = e \qquad d\eta_2 = 0 .$$

We can uniquely fix η_2 by requiring that there be a unitary element $u \in M_2 \subset M_3^+$ which exchanges η_1 and η_2:

$$\eta_2 = u\eta_1 \qquad \eta_1 = -u\eta_2 . \tag{4.13}$$

We have also

$$\eta_2 u = 0 \qquad \eta_1 u = 0\,. \tag{4.14}$$

The vector space of 1-forms is of dimension 4 over the complex numbers. The dimension of $\Omega^1 \otimes_{\mathbb{C}} \Omega^1$ is equal to 16 but the dimension of the tensor product $\Omega^1 \otimes_{M_3^+} \Omega^1$ is equal to 5. One finds in fact over M_3^+ the relations

$$\begin{aligned} &\eta_i \otimes \eta_j = 0 \qquad \eta_i^* \otimes \eta_j^* = 0 \\ &\eta_2^* \otimes \eta_1 = 0 \qquad \eta_1^* \otimes \eta_2 = 0 \qquad \eta_2^* \otimes \eta_2 = \eta_1^* \otimes \eta_1\,. \end{aligned} \tag{4.15}$$

which leave

$$\eta_{ij} = \eta_i \otimes \eta_j^* \qquad \zeta = \eta_1^* \otimes \eta_1 \tag{4.16}$$

as independent basis elements. We can therefore make the identification

$$\Omega^1 \otimes_{M_3^+} \Omega^1 = M_3^+ = \Omega^0\,. \tag{4.17}$$

To define a covariant derivative we must first introduce the map σ of (4.1). Because of the left and right M_3^+-linearity the map σ is entirely determined by its action on ζ and, for example, η_{11}:

$$\sigma(\eta_{11}) = \sum_{ij} a_{ij}\eta_{ij} + a\zeta \qquad \sigma(\zeta) = \sum_{ij} b_{ij}\eta_{ij} + b\zeta\,.$$

If we multiply both sides of the second equation by u we find that $b_{ij} = 0$; if we multiply both sides of the first equation by u^2 we find that $a = 0$. Let v be a matrix such that $v\eta_1 = \eta_1$ and $v\eta_2 \neq \eta_2$. From the conditions of left and right linearity we have the equations

$$\sigma(\eta_{11}) = v\sigma(\eta_{11}) = \sum_{ij} a_{ij} v\eta_{ij} \qquad \sigma(\eta_{11}) = \sigma(\eta_{11})v^* = \sum_{ij} a_{ij}\eta_{ij}v^*$$

from which we conclude that

$$a_{11} = \mu \qquad a_{12} = a_{21} = a_{22} = 0$$

where μ is an arbitrary complex number. If we impose the condition (1.15) we find that $1 + b = 0$. So σ is given by

$$\sigma(\eta_{11}) = \mu\eta_{11} \qquad \sigma(\zeta) = -\zeta\,. \tag{4.18}$$

The Hecke relation

$$(\sigma + 1)(\sigma - \mu) = 0$$

is satisfied. Suppose that $\mu \neq -1$ and define Λ^* (S^*) to be the quotient of the tensor algebra by the ideal generated by the eigenvectors of μ (-1). As a complex vector space Λ^* is of dimension 10. The map $\zeta \mapsto e$ induces an isomorphism of Λ^* with Ω^*. As a complex vector space S^* is of dimension 13. It is an unusual fact that it is of finite dimension. If

$\mu = -1$ then $\sigma = -1$ also and (1.15) is trivially satisfied. In this case it is natural to define Λ^* to be the entire tensor algebra. On the universal differential calculus the projection π of (1.15) is the identity and σ must be equal to -1.

The covariant derivative of η_i must be of the form

$$\mathrm{D}\eta_i = \sum_{jk} c_{ijk}\eta_{jk} + c_i\zeta\,.$$

The exterior derivative of u is given by

$$\mathrm{d}u = \eta_2 - \eta_2^*\,. \tag{4.19}$$

From equation (1.5) we find then that D must satisfy the constraints

$$\mathrm{D}\eta_1 = \zeta - u\mathrm{D}\eta_2 \qquad \mathrm{D}\eta_2 = u\mathrm{D}\eta_1 \tag{4.20}$$

and therefore that $c_1 = 1$, $c_2 = 0$ and c_{2ij} is determined in terms of c_{1ij}:

$$c_{211} = -c_{121} \qquad c_{212} = -c_{122} \qquad c_{221} = c_{111} \qquad c_{222} = c_{112}\,.$$

From the condition (1.17) one finds the additional constraints

$$(\mathrm{D}\eta_1)u - \sigma(\eta_{12}) = 0 \qquad (\mathrm{D}\eta_2)u - \sigma(\eta_{22}) = 0$$

which both imply that

$$c_{111} = -\mu \qquad c_{112} = 0 \qquad c_{121} = 0 \qquad c_{122} = 0\,. \tag{4.21}$$

The covariant derivative is uniquely defined then in terms of σ:

$$\mathrm{D}\eta_1 = -\mu\eta_{11} + \zeta \qquad \mathrm{D}\eta_2 = -\mu\eta_{21}\,. \tag{4.22}$$

The lack of symmetry is due to the fact that the form θ which determines the exterior derivative is defined in terms of η_1. The torsion vanishes. Recently (Dubois-Violette *et al* 1994) the quantum plane has been shown to possess a one-parameter family of covariant derivatives, which also are torsion-free. If one takes the covariant derivative of the identity $\theta e = \eta_1$ and its adjoint one finds that

$$\mathrm{D}\eta_1^* = -\eta_{11} - \zeta \qquad \mathrm{D}\eta_2^* = -\eta_{12} \tag{4.23}$$

and therefore that

$$\mathrm{D}\theta = (\sigma - 1)\,\theta \otimes \theta\,. \tag{4.24}$$

In fact one finds that for a general element $\eta \in \Omega^1$

$$\mathrm{D}\eta = \sigma(\eta \otimes \theta) - \theta \otimes \eta\,. \tag{4.25}$$

Using the identification (4.17) one sees that

$$\begin{aligned} \hat{\mathrm{d}}\eta_1 &= -\theta \otimes \eta_1 - \eta_1 \otimes \theta = \eta_{11} + \zeta \\ \hat{\mathrm{d}}\eta_1^* &= -\theta \otimes \eta_1^* - \eta_1^* \otimes \theta = -\eta_{11} - \zeta\,. \end{aligned} \tag{4.26}$$

Therefore when $\mu = -1$ one can identify D with $\hat{\mathrm{d}}$.

Let g be a metric and set

$$h_{ij} = g(\eta_{ij}) \qquad h = g(\zeta)\,. \tag{4.27}$$

If we suppose that the metric is bilinear then h_{ij} is given in terms of h_{11}. For example,

$$h_{21} = u h_{11}\,. \tag{4.28}$$

The condition that the connection be metric compatible is expressed by the equations

$$\mathrm{d}h_{11} = -\mu\eta_1 h + \eta_1^* h_{11} \qquad \mathrm{d}h = -\eta_1 h + \mu\eta_1^* h_{11}\,. \tag{4.29}$$

This equation has no solutions unless $\mu^2 = 1$. If $\mu = 1$ then to within an overall scale the unique bilinear metric is given by

$$h_{ij} = \eta_i\eta_j^* \qquad h = -e \tag{4.30}$$

where the right-hand sides are considered as elements of M_3^+. Therefore h_{ij} (h) takes its values in the M_2 (M_1) factor of M_3^+. From (4.18) we see that the metric (4.30) is not symmetric. With the normalization we have chosen the frames η_i have unit norm with respect to the metric:

$$\mathrm{Tr}(h_{ij}) = \delta_{ij}\,. \tag{4.31}$$

If $\mu = 1$ then it is possible to extend the definition of the complex conjugation to the tensor product by the formula

$$(\xi \otimes \eta)^* = \sigma(\eta^* \otimes \xi^*)\,.$$

With this definition the covariant derivative (4.25) is real and the metric (4.30) is real on η_{ij} and imaginary on ζ.

The curvature can be defined by a formula analogous to (1.13). Using (1.12) we find

$$\begin{aligned} &\mathrm{D}\eta_{11} = \zeta \otimes \eta_1^* - \mu\eta_{11} \otimes \eta_1 && \mathrm{D}\eta_{12} = \zeta \otimes \eta_2^* \\ &\mathrm{D}\eta_{21} = -\mu\eta_{21} \otimes \eta_1 && \mathrm{D}\zeta = \mu\zeta \otimes \eta_1^* - \eta_{11} \otimes \eta_1 \end{aligned} \tag{4.32}$$

from which we conclude that

$$\begin{aligned} &\mathrm{D}^2\eta_1 = (\mu^2 - 1)\eta_{11} \otimes \eta_1 && \mathrm{D}^2\eta_2 = \mu^2\eta_{21} \otimes \eta_1 \\ &\mathrm{D}^2\eta_1^* = (\mu + 1)(\eta_{11} \otimes \eta_1 - \zeta \otimes \eta_1^*) && \mathrm{D}^2\eta_2^* = -\zeta \otimes \eta_2^*\,. \end{aligned} \tag{4.33}$$

The curvature is given by the projection of $D^2\eta_1$ onto $\Omega^2 \otimes_{M_3^+} \Omega^1$:

$$\begin{aligned} &\pi_{12}\mathrm{D}^2\eta_1 = 0 && \pi_{12}\mathrm{D}^2\eta_2 = 0 \\ &\pi_{12}\mathrm{D}^2\eta_1^* = -(\mu + 1)e \otimes \eta_1^* && \pi_{12}\mathrm{D}^2\eta_2^* = -e \otimes \eta_2^*\,. \end{aligned} \tag{4.34}$$

Although by construction the operator $\pi_{12}\mathrm{D}^2$ is left linear it is not right linear.

For no value of μ does the curvature vanish. However, the analogue of the square of the curvature tensor does vanishes. In fact, the tensor product of the curvature tensor with itself vanishes identically. There are four different frames corresponding to the four different ways of choosing η_i and η_i^* as generators of Ω^1. The action of the matrix u which takes one into the other is a change of frame. Since h_{ij} is not proportional to the identity matrix, the frames cannot be considered as the analogues of orthonormal frames and since $h_{22} \neq h_{11}$ the change of frame u cannot be considered as 'orthonormal'. If we define

$$\pi_{12}D^2\eta_i = -R_{(i)}e \otimes \eta_i \qquad \pi_{12}D^2\eta_i^* = -\bar{R}_{(i)}e \otimes \eta_i^* \tag{4.35}$$

we see that R vanishes in all frames and that

$$\bar{R}_{(1)} = \mu + 1 \qquad \bar{R}_{(2)} = 1\,. \tag{4.36}$$

When we take η_1^* into η_2^* by the right action of u we change the value of $\bar{R}$ from $\mu + 1$ to 1.

Since $\eta_1 = \theta e$ and $\eta_1^* = -e\theta$ we could also choose θ as frame. If we rewrite (4.35) is this frame,

$$\pi_{12}D^2\theta = -R_{(\theta)}e \otimes \theta \tag{4.37}$$

we see that the component of the curvature can be defined by one matrix, proportional to the identity matrix, given by

$$R_{(\theta)} = \mu + 1\,. \tag{4.38}$$

The analogue of the Ricci tensor would be obtained by using the metric to 'contract two indices' of the curvature tensor. We can do this here if we identify Ω^2 with the vector space Λ^2. We can define a left-linear map Ric of Ω^1 into itself by

$$\mathrm{Ric}(\xi) = -(1 \otimes g)\pi_{12}D^2\xi\,. \tag{4.39}$$

From, for example, the identity

$$(1 \otimes g)(\zeta \otimes \eta_1^*) = \eta_1^* h_{11} = \eta_1^* \tag{4.40}$$

ones sees that Ric is given by the equations

$$\mathrm{Ric}(\eta_i) = R_{(i)}\eta_i \qquad \mathrm{Ric}(\eta_i^*) = \bar{R}_{(i)}\eta_i^*\,. \tag{4.41}$$

The geometry is therefore not 'Ricci-flat'. There is no analogue of the Ricci scalar.

There does not seem to be any way to construct a frame-independent quantity so the best we can do is declare θ to be a preferred frame and consider the component (4.38) of the curvature in this frame as the curvature of the geometry of M_3^+. If we require that it be metric the connection is unique and any action would yield it as extremal. We could on the other hand consider μ as an unknown parameter and chose as the action

$$\mathrm{Tr}(R_{(\theta)}^2) = 3(\mu + 1)^2\,. \tag{4.42}$$

The action then has a minimal which corresponds to a connection which is not metric and whose curvature component vanishes in the frame θ.

Additional structure could be put on the algebra M_3^+. For example, one could replace the M_2 component with the algebra of quaternions or require that the matrices be Hermitian. In the latter case the two possible frames are $\eta_1 + \eta_1^*$ and $\eta_2 + \eta_2^*$. They yield each one curvature component whose values are given by (4.36).

We mentioned that the Dirac operator has a natural coupling to the geometry of the previous section. There is also a generalized correspondence principle which can be used as a guide in introducing the Dirac operator coupled to the geometry of the quantum plane. The coupling of the Dirac operator to the geometry considered here is however more problematic. There is no possible correspondence principle since the geometry is not a deformation of a commutative geometry. It is natural to require that a spinor be an element of a left M_3-module and that the Dirac operator be an Hermitian element of M_3^- but otherwise there is no restriction. In ordinary geometry the exterior derivative can be identified with the commutator of the Dirac operator (Connes 1986) and this has been used as motivation for proposing $\mathrm{i}\theta$ as the Dirac operator in the present case, without any consideration of curvature (Connes and Lott 1990). This or any other element of M_3^- could be considered as automatically coupled to the curvature since there is a unique metric connection.

Since we have a differential calculus we have an associated cohomology. By definition $H^0 = M_3^+$ and $H^p = 0$ for $p \geqslant 3$. The unique 2-cocyle e is the coboundary of η_1 and so $H^2 = 0$. The vector space Z^1 of 1-cocycles is of complex dimension 2, generated by η_2 and η_2^*. From equation (4.19) we see that $\eta_2 - \eta_2^*$ is a coboundary; it is easy to verify that so also is $\eta_2 + \eta_2^*$. Therefore $H^1 = 0$ and the cohomology is trivial.

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