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Linear Connections on the Quantum Plane

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Abstract. A general definition has been proposed recently of a linear connection and a metric in noncommutative geometry. It is shown that to within normalization there is a unique linear connection on the quantum plane and there is no metric.

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1. Linear Connections

There have been several models proposed for noncommutative geometries [2–4], some of which are based on quantum groups [8, 11–13]. A definition of a linear connection which uses only the left-module structure of the differential forms has been proposed by Chamseddine *et al.* [1]. An algebra of differential forms has, however, a natural structure of a bimodule. Recently, linear connections have been considered in the particular case of differential calculi based on derivations [5, 6] and, more generally, [10], which make essential use of this bimodule structure. We shall here apply the general definition to the particular case of the quantum plane. We shall show that to within normalization, there is a unique linear connection on the quantum plane. The connection is not metric compatible. There is, in fact, no metric in the sense we have defined it.

We first recall the definition of a linear connection in commutative geometry in a form [7] which allows for a noncommutative generalization. Let V be a differential manifold and let $(\Omega^*(V), \mathrm{d})$ be the ordinary differential calculus on V. Let H be a vector bundle over V associated to some principle bundle P. Let $\mathscr{C}(V)$ be the algebra of smooth functions on V and $\mathscr{H}$ the left $\mathscr{C}(V)$-module of smooth sections of H.

A connection on P is equivalent to a covariant derivative on H. This, in turn, can be characterized as a linear map

$$\mathscr{H} \xrightarrow{D} \Omega^1(V) \otimes_{\mathscr{C}(V)} \mathscr{H}, \tag{1.1}$$

which satisfies the condition

$$D(f\psi) = \mathrm{d}f \otimes \psi + f D\psi \tag{1.2}$$

for arbitrary $f \in \mathscr{C}(V)$ and $\psi \in \mathscr{H}$. There is an immediate extension of D to a map

$$\Omega^*(V) \otimes_{\mathscr{C}(V)} \mathscr{H} \to \Omega^*(V) \otimes_{\mathscr{C}(V)} \mathscr{H}$$

by requiring that it be an antiderivation of degree 1. From (1.2), it follows that the difference between two covariant derivatives is an algebra morphism of $\mathscr{H}$.

The definition of a connection as a covariant derivative has an immediate extension [2] to noncommutative geometry. Let $\mathscr{A}$ be an arbitrary algebra and $(\Omega^*(\mathscr{A}), \mathrm{d})$ be a differential calculus over $\mathscr{A}$. One defines a covariant derivative on a left $\mathscr{A}$-module $\mathscr{H}$ as a map

$$\mathscr{H} \xrightarrow{D} \Omega^1(\mathscr{A}) \otimes \mathscr{H}, \tag{1.3}$$

which satisfies the condition (1.2) but with $f \in \mathscr{A}$. There is again an extension of D to a map

$$\Omega^*(\mathscr{A}) \otimes_{\mathscr{A}} \mathscr{H} \to \Omega^*(\mathscr{A}) \otimes_{\mathscr{A}} \mathscr{H}$$

by requiring that it be an antiderivation of degree 1.

A linear connection on V can be defined as a connection on the cotangent bundle to V. It can be characterized as a linear map

$$\Omega^1(V) \xrightarrow{D} \Omega^1(V) \otimes_{\mathscr{C}(V)} \Omega^1(V) \tag{1.4}$$

which satisfies the condition

$$D(f\xi) = \mathrm{d}f \otimes \xi + fD\xi \tag{1.5}$$

for arbitrary $f \in \mathscr{C}(V)$ and $\xi \in \Omega^1(V)$.

If, for simplicity, we suppose V to be parallelizable, we can choose a globally defined moving frame θ^α on V. The connection form ω^α_β is defined then in terms of the covariant derivative of the moving frame

$$D\theta^\alpha = -\omega^\alpha{}_\beta \otimes \theta^\beta. \tag{1.6}$$

Because of (1.2), the covariant derivative $D\xi$ of an arbitrary element $\xi = \xi_\alpha \theta^\alpha \in \Omega^1(V)$ can be written as $D\xi = (D\xi_\alpha) \otimes \theta^\alpha$, where

$$D\xi_\alpha = \mathrm{d}\xi_\alpha - \omega^\beta{}_\alpha \xi_\beta. \tag{1.7}$$

Let π be the projection of $\Omega^1(V) \otimes_{\mathscr{C}(V)} \Omega^1(V)$ onto $\Omega^2(V)$ defined by the wedge product on the forms. The torsion form Θ^α can be defined as

$$\Theta^\alpha = (\mathrm{d} - \pi D)\theta^\alpha. \tag{1.8}$$

The derivative $D_X\xi$ along the vector field X,

$$D_X\xi = i_X D\xi, \tag{1.9}$$

is a linear map of $\Omega^1(V)$ into itself. In particular, $D_X\theta^\alpha = -\omega^\alpha{}_\beta(X)\theta^\beta$. Using D_X, an extension of D can be constructed to the tensor product $\Omega^1(V) \otimes_{\mathscr{C}(V)} \Omega^1(V)$.

We define

$$D_X(\theta^\alpha \otimes \theta^\beta) = D_X\theta^\alpha \otimes \theta^\beta + \theta^\alpha \otimes D_X\theta^\beta. \tag{1.10}$$

Now let σ be the action on $\Omega^1(V) \otimes_{\mathscr{C}(V)} \Omega^1(V)$ defined by the permutation of two derivations:

$$\sigma(\xi \otimes \eta)(X, Y) = \xi \otimes \eta(Y, X) \tag{1.11}$$

and define $\sigma_{12} = \sigma \otimes 1$ as acting on the tensor product over $\mathscr{C}(V)$ of three factors of $\Omega^1(V)$. Then (1.10) can be rewritten without explicitly using the vector field as

$$D(\theta^\alpha \otimes \theta^\beta) = D\theta^\alpha \otimes \theta^\beta + \sigma_{12}(\theta^\alpha \otimes D\theta^\beta). \tag{1.12}$$

Define $\pi_{12} = \pi \otimes 1$. If the torsion vanishes, one finds that

$$\pi_{12}D^2\theta^\alpha = -\Omega^\alpha{}_\beta \otimes \theta^\beta, \tag{1.13}$$

where Ω^α_β is the curvature 2-form. Notice that the equality

$$\pi_{12}D^2(f\theta^\alpha) = f\pi_{12}D^2\theta^\alpha \tag{1.14}$$

is a consequence of the identity

$$\pi(\sigma + 1) = 0. \tag{1.15}$$

The module $\Omega^1(V)$ has a natural structure as a right $\mathscr{C}(V)$-module and the corresponding condition equivalent to (1.5) is determined using the fact that $\mathscr{C}(V)$ is a commutative algebra

$$D(\xi f) = D(f\xi). \tag{1.16}$$

Using σ this can also be written in the form

$$D(\xi f) = \sigma(\xi \otimes \mathrm{d}f) + (D\xi)f. \tag{1.17}$$

By extension, a linear connection over a general noncommutative algebra $\mathscr{A}$ with a differential calculus $(\Omega^*(\mathscr{A}), \mathrm{d})$ can be defined as a linear map

$$\Omega^1 \xrightarrow{D} \Omega^1 \otimes_{\mathscr{A}} \Omega^1 \tag{1.18}$$

which satisfies the condition (1.5) for arbitrary $f \in \mathscr{A}$ and $\xi \in \Omega^1$. We have here set $\Omega^1(\mathscr{A}) = \Omega^1$. The module Ω^1 has again a natural structure as a right $\mathscr{A}$-module, but in the noncommutative case it is generally impossible to consistently impose the condition (1.16) and a substitute must be found. We must decide how it is appropriate to define $D(\xi f)$ in terms of $D(\xi)$ and $\mathrm{d}f$. We propose [10] to postulate the existence of a map

$$\Omega^1 \otimes_{\mathscr{A}} \Omega^1 \xrightarrow{\sigma} \Omega^1 \otimes_{\mathscr{A}} \Omega^1 \tag{1.19}$$

which satisfies (1.15) to replace the one defined by (1.11). We define then $D(\xi f)$ by Equation (1.17) but using (1.19) instead of (1.11). In the commutative case, there are many σ solutions to (1.15) but because of the relation (1.16), only the usual one has associated covariant derivatives. In general, there need not exist covariant derivatives for each solution to (1.15); the existence of a connection places restrictions on σ. From the identity

$$D((\xi f)g) = D(\xi(fg)),$$

we find that σ must be right $\mathscr{A}$-linear. From the identity

$$D(\mathrm{d}((fg)h)) = D(\mathrm{d}(f(gh))),$$

we find that σ must be also left $\mathscr{A}$-linear. In general,

$$\sigma^2 \neq 1. \tag{1.20}$$

The extension of D to $\Omega^1 \otimes \Omega^1$ is given by (1.12) but again using (1.19).

A covariant derivative is a map of the form (1.18) which satisfies the Leibniz rules (1.5) and (1.17). Because of (1.15) the image of the operator $\mathrm{d} - \pi D$ acting on $\Omega^1(\mathscr{A})$ is a bi-submodule Θ of $\Omega^2(\mathscr{A})$. It is the torsion submodule.

The extension of D to $\Omega^1 \otimes_{\mathscr{A}} \Omega^1$ is given by the analogue of (1.12):

$$D(\xi \otimes \eta) = D\xi \otimes \eta + \sigma_{12}(\xi \otimes D\eta). \tag{1.21}$$

We can then define the map

$$\Omega^1 \xrightarrow{\pi_{12}D^2} \Omega^2 \otimes_{\mathscr{A}} \Omega^1, \tag{1.22}$$

which can be extended by a projection

$$\Omega^2 \otimes_{\mathscr{A}} \Omega^1 \to (\Omega^2/\Theta) \otimes_{\mathscr{A}} \Omega^1. \tag{1.23}$$

After the projection, $\pi_{12}D^2$ is left $\mathscr{A}$-linear:

$$\pi_{12}D^2(f\xi) = f\pi_{12}D^2\xi. \tag{1.24}$$

It will not generally be right $\mathscr{A}$-linear. However, in the particular case which we shall consider in the next section, the right-module structure is completely determined by the left-module structure. There is a representation ρ of the algebra such that

$$\pi_{12}D^2(\xi f) = (\pi_{12}D^2\xi)\rho(f) \tag{1.25}$$

after the projection (1.23).

As a simple example, we mention the universal calculus which has the property that $\pi = 1$. Therefore, from (1.15) we see that $\sigma = -1$. If the torsion is to vanish the only possible covariant derivative then is the ordinary exterior derivative. Every torsion-free linear connection has vanishing curvature.

A metric g on V can be defined as a $\mathscr{C}(V)$-linear, symmetric map of $\Omega^1(V) \otimes_{\mathscr{C}(V)} \Omega^1(V)$ into $\mathscr{C}(V)$. This definition makes sense if one replaces $\mathscr{C}(V)$ by an algebra $\mathscr{A}$ and $\Omega^1(V)$ by $\Omega^1(\mathscr{A})$. By analogy with the commutative case, we shall say that the covariant derivative (1.17) is metric if the following diagram is commutative:

$$\begin{array}{ccc} \Omega^1 \otimes_{\mathscr{A}} \Omega^1 & \xrightarrow{D} & \Omega^1 \otimes_{\mathscr{A}} \Omega^1 \otimes_{\mathscr{A}} \Omega^1 \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle 1\otimes g} \\ \mathscr{A} & \xrightarrow{\mathrm{d}} & \Omega^1 \end{array} \tag{1.26}$$

In general, symmetry must be defined with respect to the map σ. We impose then on g the condition

$$g\sigma = g. \tag{1.27}$$

2. The Quantum Plane

In this section, we apply our general prescription to the quantum plane [9], which possesses a natural map σ. The algebra of forms $\Omega^* = \Omega^0 \oplus \Omega^1 \oplus \Omega^2$ has four generators $x^i = (x, y)$ and $\xi^i = \mathrm{d}x^i = (\xi, \eta)$ which satisfy the relations

$$xy = qyx, \tag{2.1a}$$

$$x\xi = q^2\xi x, \qquad x\eta = q\eta x + (q^2 - 1)\xi y, \qquad y\xi = q\xi y, \qquad y\eta = q^2\eta y, \tag{2.1b}$$

$$\xi^2 = 0, \qquad \eta^2 = 0, \qquad \eta\xi + q\xi\eta = 0. \tag{2.1c}$$

Here q is a complex number, which we shall suppose in general not to be a root of unity. If we apply d to both sides of (2.1a), we obtain a relation which is a consequence of (2.1b). If we apply d to (2.1b), we obtain (2.1c). The conditions (2.1) can be written in the form

$$x^i x^j - q^{-1}\hat{R}^{ij}{}_{kl} x^k x^l = 0, \tag{2.2a}$$

$$x^i \xi^j - q\hat{R}^{ij}{}_{kl} \xi^k x^l = 0, \tag{2.2b}$$

$$\xi^i \xi^j + q\hat{R}^{ij}{}_{kl} \xi^k \xi^l = 0. \tag{2.2c}$$

By grouping the indices the 4-index tensor $\hat{R}^{ij}{}_{kl}$ can be written as a 4×4 matrix:

$$\hat{R} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}. \tag{2.3}$$

If the generators (a, b, c, d) of $SL_q(2, \mathbb{C})$ are written in the form of a matrix

$$a^i_j = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{2.4}$$

then the invariance of Ω^* under the action of $SL_q(2, \mathbb{C})$ follows from the identity

$$\hat{R}^{ij}{}_{kl} a^k_m a^l_n = a^i_k a^j_l \hat{R}^{kl}{}_{mn}. \tag{2.5}$$

Introduce the trivial differential calculus on $SL_q(2, \mathbb{C})$ with the differential d given by

$$da^i_j = 0. \tag{2.6}$$

The result of the coaction of $SL_q(2, \mathbb{C})$ on x^i and ξ^i is then

$$x^{i\prime} = a^i_j \otimes x^j, \qquad \xi^{i\prime} = a^i_j \otimes \xi^j \tag{2.7}$$

and from (2.5) it follows that $x^{i\prime}$ and $\xi^{i\prime}$ satisfy the same relations as x^i and ξ^j.

We introduce the 1-form

$$\theta = x\eta - qy\xi. \tag{2.8}$$

It is easily seen that

$$\theta^2 = 0 \tag{2.9}$$

and that θ is invariant under the coaction of $SL_q(2, \mathbb{C})$:

$$\theta' = 1 \otimes \theta. \tag{2.10}$$

It is, in fact, to within multiplication by a complex number, the only invariant element of Ω^1. From (2.1), we deduce the commutation relations

$$x^i\theta = q\theta x^i, \qquad \xi^i\theta = -q^{-3}\theta\xi^i. \tag{2.11}$$

To fix the definition of a covariant derivative, we must first introduce the operator σ of Equation (1.19). If we take the covariant derivative of both sides of Equation (2.2b), we find that σ must be equal to the inverse of the matrix $q\hat{R}$. Written out in detail, it becomes

$$\begin{aligned} &\sigma(\xi \otimes \xi) = q^{-2}\xi \otimes \xi, \qquad \sigma(\xi \otimes \eta) = q^{-1}\eta \otimes \xi, \\ &\sigma(\eta \otimes \xi) = q^{-1}\xi \otimes \eta - (1 - q^{-2})\eta \otimes \xi, \qquad \sigma(\eta \otimes \eta) = q^{-2}\eta \otimes \eta. \end{aligned} \tag{2.12}$$

The extension to $\Omega^1 \otimes_{\Omega^0} \Omega^1$ is given by Ω^0-linearity. One verifies immediately that the condition (1.15) is satisfied. As a result of the linearity, one finds

$$\begin{aligned} &\sigma(\xi \otimes \theta) = q^{-3}\theta \otimes \xi, \qquad \sigma(\theta \otimes \xi) = q\xi \otimes \theta - (1 - q^{-2})\theta \otimes \xi, \\ &\sigma(\eta \otimes \theta) = q^{-3}\theta \otimes \eta, \qquad \sigma(\theta \otimes \eta) = q\eta \otimes \theta - (1 - q^{-2})\theta \otimes \eta, \end{aligned}$$

as well as

$$\sigma(\theta \otimes \theta) = q^{-2}\theta \otimes \theta.$$

Although $\sigma^2 \neq 1$, one finds that σ satisfies the Hecke relation

$$(\sigma + 1)(\sigma - q^{-2}) = 0.$$

The vector space Ω^1 of 1-forms can be considered as defined by the relations (2.1a,b). Suppose that $q^2 \neq -1$. The exterior algebra is obtained by dividing the tensor algebra over Ω^1 by the ideal generated by the three eigenvectors $\xi \otimes \xi$, $\eta \otimes \eta$ and $\eta \otimes \xi + q\xi + \eta$ corresponding to the eigenvalue q^{-2}. The relations (2.1c) are then satisfied and the exterior algebra can be identified with the algebra of forms Ω^*. The symmetric algebra is obtained by dividing the tensor algebra by the ideal generated by the eigenvector $\xi \otimes \eta - q\eta \otimes \xi$ corresponding to the eigenvalue -1. If $q^2 = -1$, then $\sigma = -1$. We shall see later that the curvature must vanish when $q^2 = -1$. The braid relation

$$\sigma_{12}\sigma_{23}\sigma_{12} = \sigma_{23}\sigma_{12}\sigma_{23}$$

assures us that on the tensor product of three copies of Ω^1 the two ways of taking the product yield the same answer. We shall not, however, explicitly use this fact. One can show that there exist Ω^0-linear maps of the form (1.19) which satisfy (1.15) but which have no associated covariant derivatives.

There is a unique one-parameter family of covariant derivatives compatible with the algebraic structure (2.1) of Ω^*. It is given by

$$D\xi^i = \mu^4 x^i \theta \otimes \theta. \tag{2.13}$$

The parameter μ must have the dimensions of inverse length. From the invariance of θ it follows that D is invariant under the coaction of $SL_q(2, \mathbb{C})$. From Equation (2.9), one sees that the torsion vanishes.

Using (1.21), one finds the equality

$$\pi_{12} D^2 \xi^i = \Omega^i \otimes \theta, \tag{2.14}$$

where the 2-form Ω^i is given by

$$\Omega^i = \mu^4 q^{-2}(q^2 + 1)(q^4 + 1)x^i \xi\eta. \tag{2.15}$$

It vanishes for $q = \pm i$ and $q^2 = \pm i$ but it does not vanish when $q = 1$. There is a preferred family of nontrivial linear connections on the ordinary complex 2-plane which are stable under the quantum deformation. Equation (2.14) can also be written in the usual form analogous to (1.13)

$$\pi_{12} D^2 \xi^i = -\Omega^i{}_j \otimes \xi^j, \tag{2.16}$$

with the curvature 2-form given by

$$\Omega^i{}_j = \mu^4(1 + q^{-2})(1 + q^{-4})\begin{pmatrix} q^2 xy & -qx^2 \\ q^2 y^2 & -xy \end{pmatrix}\xi\eta. \tag{2.17}$$

The operator $\pi_{12} D^2$ is a left-module morphism by construction. One finds that the representation ρ of (1.25) is given by $\rho(f)(x^i) = f(q^2 x^i)$. Since Ω^3 vanishes, the Bianchi identities are trivially satisfied. The form θ satisfies the equation

$$\pi_{12} D^2 \theta = 0. \tag{2.18}$$

The metric is a Ω^0-linear map from $\Omega^1 \otimes_{\Omega^0} \Omega^1$ into Ω^0 which satisfies (1.27). It is straightforward to see that there can be no metric. One can consistently impose the condition $g\sigma = -g$ but this map resembles rather a symplectic form. In the limit $q = 1$, there is a metric, the ordinary Euclidean metric, but the connection (2.13) is not a metric connection. This can be seen from the absence of any symmetry in the matrix on the right-hand side of (2.17).

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