ELSEVIER

Journal of Pure and Applied Algebra 114 (1996) 39–50

JOURNAL OF PURE AND APPLIED ALGEBRA

Basic cohomology of associative algebras

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Communicated by J.D. Stasheff; received 18 June 1994; revised 7 September 1994

Abstract

We define a new cohomology for associative algebras which we compute for algebras with units.

1. Introduction: Definition of the basic cohomology of an associative algebra

Let $\mathscr{A}$ be an associative algebra over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let $\mathscr{A}_{\text{Lie}}$ be the underlying Lie algebra (with the commutator as Lie bracket). For each integer $n \in \mathbb{N}$, let $C^n(\mathscr{A})$ be the vector space of n-linear forms on $\mathscr{A}$, i.e. $C^n(\mathscr{A}) = (\mathscr{A}^{\otimes^n})^*$. For $\omega \in C^n(\mathscr{A})$ and $\tau \in C^m(\mathscr{A})$ one defines $\omega.\tau \in C^{n+m}(\mathscr{A})$ by

$$\omega.\tau(A_1,\dots,A_{n+m}) = \omega(A_1,\dots,A_n)\tau(A_{n+1},\dots,A_{n+m}),\ \forall A_i \in \mathscr{A}.$$

Equipped with this product, $C(\mathscr{A}) = \bigoplus_n C^n(\mathscr{A})$ becomes an associative graded algebra with unit ($C^0(\mathscr{A}) = \mathbb{K}$). One defines a differential d on $C(\mathscr{A})$ by setting for $\omega \in C^n(\mathscr{A})$, $A_i \in \mathscr{A}$,

$$d\omega(A_1,\dots,A_{n+1}) = \sum_{k=1}^{n} (-1)^k \omega(A_1,\dots,A_{k-1},A_kA_{k+1},A_{k+2},\dots,A_{n+1}).$$

Indeed, d is the extension as antiderivation of $C(\mathscr{A})$ of minus the dual of the product of $\mathscr{A}$ and $d^2 = 0$ is then equivalent to the associativity of the product of $\mathscr{A}$. The graded differential algebra $C(\mathscr{A})$ is together with a bimodule $\mathscr{M}$ the basic building blocks of the Hochschild complex giving the Hochschild cohomology with value in $\mathscr{M}$. Here we

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SSDI 0022-4049(95)00166-2

do not want to introduce bimodules like $\mathscr{M}$. However it is well known, see below, that the cohomology of $C(\mathscr{A})$ is trivial whenever $\mathscr{A}$ has a unit. Nevertheless, there are two classical cohomologies which can be extracted from the differential algebra $C(\mathscr{A})$, namely the Lie algebra cohomology of $\mathscr{A}_{\mathrm{Lie}}$ and the cyclic cohomology of $\mathscr{A}$. In fact, let $\mathscr{S} : C(\mathscr{A}) \to C(\mathscr{A})$ and $\mathscr{C} : C(\mathscr{A}) \to C(\mathscr{A})$ be defined by

$$(\mathscr{S}\omega)(A_1,\dots,A_n) = \sum_{\pi\in\mathscr{S}_n} \varepsilon(\pi)\omega(A_{\pi(1)},\dots,A_{\pi(n)})$$

and

$$(\mathscr{C}\omega)(A_1,\dots,A_n) = \sum_{\gamma\in\mathscr{C}_n} \varepsilon(\gamma)\omega(A_{\gamma(1)},\dots,A_{\gamma(n)})$$

for $\omega \in C^n(\mathscr{A})$, $A_k \in \mathscr{A}$ and where $\mathscr{S}_n$ is the group of permutations of $\{1,\dots,n\}$ and $\mathscr{C}_n$ is the subgroup of cyclic permutations. One has $\mathscr{S} \circ d = \delta \circ \mathscr{S}$ where δ is the Chevalley–Eilenberg differential so $(\mathrm{Im}\,\mathscr{S}, \delta)$ is a differential algebra whose cohomology is the Lie algebra cohomology $H(\mathscr{A}_{\mathrm{Lie}})$ of the Lie algebra $\mathscr{A}_{\mathrm{Lie}}$ [3, 6, 5]. On the other hand, see Lemma 3 in [4, part II] one has $\mathscr{C} \circ d = b \circ \mathscr{C}$ where b is the Hochschild differential of $C(\mathscr{A}, \mathscr{A}^*)$ so $(\mathrm{Im}\,\mathscr{C}, b)$ is a complex whose cohomology is the cyclic cohomology $H_\lambda(\mathscr{A})$ of $\mathscr{A}$ up to a shift -1 in degree [4] (it is worth noticing, and this is not accidental, that the same shift occurs in the Loday–Quillen theorem [7]).

We want now to point out that there is another natural non-trivial cohomology which may be extracted from the differential algebra $C(\mathscr{A})$. This cohomology is connected with the existence of a canonical operation, in the sense of Cartan [2, 5], of the Lie algebra $\mathscr{A}_{\mathrm{Lie}}$ in the graded differential algebra $C(\mathscr{A})$. For $A \in \mathscr{A} = \mathscr{A}_{\mathrm{Lie}}$, define $i_A : C^n(\mathscr{A}) \to C^{n-1}(\mathscr{A})$ by

$$i_A\omega(A_1,\dots,A_{n-1}) = \sum_{k=0}^{n-1} (-1)^k \omega(A_1,\dots,A_k,A,A_{k+1},\dots,A_{n-1})$$

$\forall\omega \in C^n(\mathscr{A})$, $\forall A_i \in \mathscr{A}$, for $n \geq 1$ and $i_A C^0(\mathscr{A}) = 0$. For each $A \in \mathscr{A}$, i_A is an antiderivation of degree -1 of $C(\mathscr{A})$ and one has, with $L_A = i_A d + d i_A$, $i_A i_B + i_B i_A = 0$, $[L_A, i_B] = i_{[A,B]}$, $[L_A, L_B] = L_{[A,B]}$ which are the relations which characterize an operation of $\mathscr{A}_{\mathrm{Lie}}$ in $C(\mathscr{A})$. Notice that then, for $A \in \mathscr{A}$, the derivation L_A of degree 0 of $C(\mathscr{A})$ is given by

$$L_A\omega(A_1,\dots,A_n) = \sum_{k=1}^{n} \omega(A_1,\dots,[A_k,A],\dots,A_n)$$

for $\omega \in C^n(\mathscr{A})$, $A_i \in \mathscr{A}$. An element $\omega \in C(\mathscr{A})$ is called *horizontal* if $i_A\omega = 0$ for any $A \in \mathscr{A}$, it is called *invariant* if $L_A\omega = 0$ for any $A \in \mathscr{A}$ and it is called *basic* if it is horizontal and invariant, i.e. if $i_A\omega = 0$ and $L_A\omega = 0$ for any $A \in \mathscr{A}$. The set $C_{\mathrm{H}}(\mathscr{A})$ of horizontal elements of $\mathscr{A}$ is a graded subalgebra of $C(\mathscr{A})$ which is stable by the L_A, $A \in \mathscr{A}$. The set $C_{\mathrm{I}}(\mathscr{A})$ of invariant elements of $\mathscr{A}$ and the set $C_{\mathrm{B}}(\mathscr{A})$ of basic elements of $\mathscr{A}$ are two graded differential subalgebras of $C(\mathscr{A})$ $(C_{\mathrm{B}}(\mathscr{A}) \subset C_{\mathrm{I}}(\mathscr{A}))$; their cohomologies $H_{\mathrm{I}}(\mathscr{A})$ and $H_{\mathrm{B}}(\mathscr{A})$ are called the invariant cohomology and the basic cohomology of $\mathscr{A}$. As already claimed, if $\mathscr{A}$ has a unit then the cohomology $H(\mathscr{A})$ of $C(\mathscr{A})$ is trivial and it turns out that the same is true for the invariant cohomology; one has the following proposition.

Proposition 1. *If $\mathscr{A}$ has a unit, then one has $H^n(\mathscr{A}) = 0$, $H_{\mathrm{I}}^n(\mathscr{A}) = 0$ for $n \geq 1$ and $H^0(\mathscr{A}) = H_{\mathrm{I}}^0(\mathscr{A}) = \mathbb{K}$.*

Proof. Let $\mathbb{1}$ be the unit of $\mathscr{A}$ and let us define for $n \geq 1$, $h : C^n(\mathscr{A}) \to C^{n-1}(\mathscr{A})$ by $h\omega(A_1,\dots,A_{n-1}) = -\omega(\mathbb{1},A_1,\dots,A_{n-1})$, for $\omega \in C^n(\mathscr{A})$ and $A_i \in \mathscr{A}$. One has $(dh+hd)\omega = \omega$ and $(L_A h - hL_A)\omega = 0$ for $\omega \in C^n(\mathscr{A})$ and $A \in \mathscr{A}$. It follows that h is a contracting homotopy for $C^+(\mathscr{A}) = \bigoplus_{n\geq 1} C^n(\mathscr{A})$ and for $C_I^+(\mathscr{A}) = \bigoplus_{n\geq 1} C_I^n(\mathscr{A})$, which proves the result. □

The basic cohomology $H_{\mathrm{B}}(\mathscr{A})$ is however non-trivial. In fact, it is already non-trivial for $\mathscr{A} = \mathbb{K}$.

Proposition 2. *The basic cohomology $H_{\mathrm{B}}(\mathbb{K})$ of $\mathbb{K}$ is the free graded commutative algebra with unit generated by an element of degree two; $H_{\mathrm{B}}^{2k}(\mathbb{K}) = \mathbb{K}, H_{\mathrm{B}}^{2k+1}(\mathbb{K}) = 0$ and $H_{\mathrm{B}}(\mathbb{K})$ identifies to the algebra $\mathbb{K}[X^2]$ of polynomials in one indeterminate X^2 of degree two (X^2 being identified to a non-vanishing element of $H_{\mathrm{B}}^2(\mathbb{K})$).*

Proof. $C(\mathbb{K})$ can be identified to $\mathbb{K}[X]$ and coincides with $C_{\mathrm{I}}(\mathbb{K})$ since $L_1 = 0$. One has $i_1 = 0$ on the elements of even degrees and $i_1 \neq 0$ on the non-vanishing elements of odd degrees. Therefore, $C_{\mathrm{B}}(\mathbb{K}) = \bigoplus_k C^{2k}(\mathbb{K}) = \mathbb{K}[X^2] = H_{\mathrm{B}}(\mathbb{K})$. □

One sees that the basic cohomology of $\mathbb{K}$ coincides with its cyclic cohomology [4]. This is however a little accidental since, as we shall see, the basic cohomology of algebras is not Morita invariant. The basic cohomology of algebras is of course functorial, one has the following obvious result: *The basic coholomogy H_{B} is a contravariant functor from the category of associative algebras into the category of graded associative algebras.*

It is worth noticing here that one has $C_{\mathrm{B}}^1(\mathscr{A}) = 0$ and therefore $H_{\mathrm{B}}^1(\mathscr{A}) = 0$ for any associative $\mathbb{K}$-algebra $\mathscr{A}$.

In the next section we shall describe $H_{\mathrm{B}}(\mathscr{A})$ for an arbitrary associative $\mathbb{K}$-algebra $\mathscr{A}$ with unit.

2. The basic cohomology of unital algebras

In this section and the following one, $\mathscr{A}$ is an associative $\mathbb{K}$-algebra with a unit denoted by $\mathbb{1}$. Let $\mathscr{I}_{\mathrm{S}}^n(\mathscr{A}_{\mathrm{Lie}})$ denote the space of ad*-invariant homogeneous polynomials of degree n on the underlying Lie algebra $\mathscr{A}_{\mathrm{Lie}}$ of $\mathscr{A}$. We shall prove the following theorem which generalizes the Proposition 2 of Section 1.

Theorem 1. *The basic cohomology $H_{\mathrm{B}}(\mathscr{A})$ of $\mathscr{A}$ identifies with the algebra $\mathscr{I}_{\mathrm{S}}(\mathscr{A}_{\mathrm{Lie}})$ of invariant polynomials on the Lie algebra $\mathscr{A}_{\mathrm{Lie}}$ where the degree $2n$ is given to the* *homogeneous polynomials of degree n, i.e.* $H_{\mathrm{B}}^{2n}(\mathscr{A}) \simeq \mathscr{I}_{\mathrm{S}}^{n}(\mathscr{A}_{\mathrm{Lie}})$ *and* $H_{\mathrm{B}}^{2n+1}(\mathscr{A}) = 0$. *In particular,* $H_{\mathrm{B}}(\mathscr{A})$ *is commutative and graded commutative.*

The complete proof of this theorem will be given in the next section; here we just outline the main ideas of the proof in the case where $\mathscr{A}$ is finite dimensional (for notational simplicity). In this case, $C(\mathscr{A})$ is just the tensor algebra $T(\mathscr{A}^*)$ over the dual space $\mathscr{A}^*$ of $\mathscr{A}$. We use a familiar trick in equivariant cohomology [1, 2] to convert the operation i into a differential δ. Namely, consider $T(\mathscr{A}^*)$ as a subalgebra of $S(\mathscr{A}^*)\otimes T(\mathscr{A}^*)$ where $S(\mathscr{A}^*)$ is the symmetric algebra over $\mathscr{A}^*$ and define the endomorphism δ with $\delta(S^m(\mathscr{A}^*)\otimes T^n(\mathscr{A}^*)) \subset S^{m+1}(\mathscr{A}^*)\otimes T^{n-1}(\mathscr{A}^*)$ by $\delta = \sum_\alpha \mu(e^\alpha)\otimes i_{e_\alpha}$ where (e_α) is a basis of $\mathscr{A}$ with dual basis (e^α) and where $\mu(e^\alpha)$ denotes the multiplication by e^α in $S(\mathscr{A}^*)$. Then, since $i_{e_\alpha}i_{e_\beta} + i_{e_\beta}i_{e_\alpha} = 0$, one has $\delta^2 = 0$. One extends the differential d of $C(\mathscr{A}) = T(\mathscr{A}^*)$ to $S(\mathscr{A}^*)\otimes T(\mathscr{A}^*)$ by $id_{S(\mathscr{A}^*)}\otimes d$ which we again denote by d. The differentials d on δ do not anticommute; however $S^m(\mathscr{A}^*)\otimes T^n(\mathscr{A}^*)$ is canonically a subspace of $T^{m+n}(\mathscr{A}^*) = C^{m+n}(\mathscr{A})$ and if $\mathscr{I}^{m,n} = (S^m(\mathscr{A}^*)\otimes T^n(\mathscr{A}^*)) \cap C_{\mathrm{I}}^{m+n}(\mathscr{A})$ denotes the invariant elements then the algebra $\mathscr{I} = \oplus \mathscr{I}^{m,n}$ is stable by d and by δ and these differentials anticommute on $\mathscr{I}$. It is easy to show that the d cohomology of $\mathscr{I}$ satisfies $H^{m,n}(\mathscr{I}, d) = 0$ for $n \geq 1$ and $H^{m,0}(\mathscr{I}, d) = \mathscr{I}_{\mathrm{S}}^{m}(\mathscr{A}_{\mathrm{Lie}})$. Let α be a basic cocycle of $\mathscr{A}$ of degree $n \geq 2$. This means that $\alpha \in \mathscr{I}^{0,n}$ satisfies $d\alpha = 0$ and $\delta\alpha = 0$. From the triviality of the d-cohomology, it follows that there is a $\omega^{0,n-1} \in \mathscr{I}^{0,n-1}$ such that $\alpha = d\omega^{0,n-1}$. Now either $n = 2$ and then $\delta\omega^{0,1} \in \mathscr{I}_{\mathrm{S}}^{1}(\mathscr{I}_{\mathrm{Lie}})$ or if $n \geq 3$ one has $\delta\alpha = \delta d\omega^{0,n-1} = -d\delta\omega^{0,n-1} = 0$ which implies, in view of the triviality of the d-cohomology, that there is a $\omega^{1,n-3} \in \mathscr{I}^{1,n-3}$ such that $\delta\omega^{0,n-1} + d\omega^{1,n-3} = 0$ and thus one has a tower

$$
\begin{aligned}
\alpha &= d\omega^{0,n-1},\\
0 &= \delta\omega^{0,n-1} + d\omega^{1,n-3},\\
&\vdots\\
0 &= \delta\omega^{k,n-2k-1} + d\omega^{k+1,n-2k-3},\\
&\vdots
\end{aligned}
$$

which ends by $\delta\omega^{p-1,1} \in \mathscr{I}_{\mathrm{S}}^{p}(\mathscr{A}_{\mathrm{Lie}})$ if $n = 2p$ and by zero if $n = 2p+1$. By using again the triviality of the d-cohomology of $\mathscr{I}$ in appropriate degrees, one sees that $\alpha \to \omega^{0,n-1} \to \cdots \omega^{k,n-2k-1} \to \cdots$ are well-defined maps in cohomology, i.e. that by denoting by $H(\delta|d)$ the δ-cohomology modulo d of $\mathscr{I}$ one has chains of mappings:

$$
\begin{aligned}
&H_{\mathrm{B}}^{2p}(\mathscr{A}) \to H^{0,2p-1}(\delta|d) \to \cdots \to H^{k,2(p-k)-1}(\delta|d) \to \cdots\\
&\qquad \cdots \to H^{p-1,1}(\delta|d) \to \mathscr{I}_{\mathrm{S}}^{p}(\mathscr{A}_{\mathrm{Lie}}),\\
&H_{\mathrm{B}}^{2p+1}(\mathscr{A}) \to H^{0,2p}(\delta|d) \to \cdots \to H^{k,2(p-k)}(\delta|d) \to \cdots \to 0.
\end{aligned}
$$

We shall show in the next section (Proposition 4) that the δ-cohomology $H^{m,n}(\mathscr{I}, \delta)$ vanishes for $m \geq 1$. This implies that one can climb up the above tower and that, therefore the above chains of mappings are in fact chains of isomorphisms. Concerning the last result on the δ-cohomology of $\mathscr{I}$, we remark that it would be easy if instead of $T(\mathscr{A}^*)$ one has the exterior algebra $\Lambda(\mathscr{A}^*)$ and the idea of the proof is more or less to project on the latter situation. In any case see next section for the proof of the theorem.

By applying Theorem 1 it is easy to compute the basic cohomology for specific examples of algebras $\mathscr{A}$. For instance, if $\mathscr{A}$ is a commutative algebra with unit, then $H_{\mathrm{B}}^{2n}(\mathscr{A})$ is the space of homogeneous polynomials of degree n on $\mathscr{A}$ and $H_{\mathrm{B}}^{2n+1}(\mathscr{A}) = 0$, so with obvious identifications (for the degrees) $H_{\mathrm{B}}(\mathscr{A})$ is just the algebra of polynomials on $\mathscr{A}$. If $\mathscr{A}$ is the algebra $M_p(\mathbb{C})$ of complex $p \times p$-matrices, then $H_{\mathrm{B}}^{2n}(M_p(\mathbb{C}))$ is the space of homogeneous invariant polynomials of degree n on the Lie algebra $\mathfrak{gl}_p(\mathbb{C})$ of $GL_p(\mathbb{C})$ and $H_B^{2n+1}(M_p(\mathbb{C})) = 0$. Thus, $H_{\mathrm{B}}(M_p(\mathbb{C}))$ is the free graded-commutative (in fact commutative) algebra with unit generated by elements x_k for $k \in \{1, 2, \dots, p\}$ with x_k of degree $2k$ (x_k corresponds to an indecomposable homogeneous invariant polynomial of degree k on $\mathfrak{gl}_p(\mathbb{C})$). Since $H_{\mathrm{B}}(M_p(\mathbb{C}))$ depends on the integer p, one sees that *the basic cohomology is not Morita invariant.*

3. Proof of the theorem

Let $\mathscr{P}^{m,n}$ denote the space of homogeneous polynomial mappings of degree m of $\mathscr{A}$ in $C^n(\mathscr{A})$. The direct sum $\mathscr{P} = \bigoplus_{m,n} \mathscr{P}^{m,n}$ is an associative bigraded algebra in a natural way. One defines the total degree of an element of $\mathscr{P}^{m,n}$ to be $2m + n$; $\mathscr{P}$ is a graded algebra for the total degree and $C(\mathscr{A}) = \bigoplus_n \mathscr{P}^{0,n}$ is a graded subalgebra of $\mathscr{P}$. The composition with the differential d of $C(\mathscr{A})$ is a differential, again denoted by d, of the graded algebra $\mathscr{P}$ which extends the differential d of $C(\mathscr{A})$. One has $d\mathscr{P}^{m,n} \subset \mathscr{P}^{m,n+1}$. By using the operation $A \mapsto i_A$, one can define another differential, δ, on $\mathscr{P}$. Namely, if $\omega \in \mathscr{P}$ is the polynomial mapping $A \mapsto \omega_A$ of $\mathscr{A}$ in $C(\mathscr{A})$, then $\delta\omega$ is the polynomial mapping $A \mapsto (\delta\omega)_A = i_A\omega_A$ of $\mathscr{A}$ in $C(\mathscr{A})$. One has $\delta\mathscr{P}^{m,n} \subset \mathscr{P}^{m+1,n-1}$ so δ is of total degree $2-1 = 1$ and the fact that δ is an antiderivation satisfying $\delta^2 = 0$ follows from the fact that, for any $A \in \mathscr{A}$, i_A is an antiderivation of $C(\mathscr{A})$ satisfying $i_A^2 = 0$. Notice that $C_{\mathrm{H}}^n(\mathscr{A})$ is the kernel of $\delta \restriction C^n(\mathscr{A}) = \mathscr{P}^{0,n}$ $(: \mathscr{P}^{0,n} \to \mathscr{P}^{1,n-1})$.

As a vector space, $\mathscr{P}^{m,n}$ can be identified to the subspace of elements of $C^{m+n}(\mathscr{A})$ which are symmetric in their m first arguments: For $\omega \in \mathscr{P}^{m,n}$, $A \mapsto \omega_A$, there is a unique $\xi_\omega \in C^{m+n}(\mathscr{A})$ symmetric in the m first arguments such that

$$\omega_A(A_1, \dots, A_n) = \xi_\omega(\underbrace{A, \dots, A}_{m}, A_1, \dots, A_n), \quad \forall A, A_i \in \mathscr{A}.$$

Let $\mathscr{I}^{m,n}$ denote the subspace of $\mathscr{P}^{m,n}$ consisting of the $\omega \in \mathscr{P}^{m,n}$ such that $\xi_\omega \in C_{\mathrm{I}}^{m+n}(\mathscr{A})$, (i.e. such that ξ_ω is invariant). $\mathscr{I} = \oplus \mathscr{I}^{m,n}$ is a graded subalgebra (also a bigraded subalgebra in the obvious sense) of $\mathscr{P}$ which is stable by d and δ and, furthermore, d and δ anticommute on $\mathscr{I}$.

Notice that one has $\mathscr{I}^{m,0} = \mathscr{I}^m_{\mathrm{S}}(\mathscr{A}_{\mathrm{Lie}})$ and $\mathscr{I}^{0,n} = C^n_{\mathrm{I}}(\mathscr{A})$ and that $C^n_{\mathrm{B}}(\mathscr{A})$ is the kernel of $\delta \restriction C^n_{\mathrm{I}}(\mathscr{A}) = \mathscr{I}^{0,n}$ $(: \mathscr{I}^{0,n} \to \mathscr{I}^{1,n-1})$. The algebras $\mathscr{P}$ and $\mathscr{I}$ are bigraded and d and δ are bihomogeneous, therefore the d and the δ cohomologies of $\mathscr{P}$ and $\mathscr{I}$ are also bigraded algebras. By using composition with the homotopy h of the proof of Proposition 1 and by noticing that $\mathscr{I}$ is stable by this composition, one obtains the following generalization of Proposition 1.

Proposition 3. *One has* $H^{m,n}(\mathscr{P},d) = 0, H^{m,n}(\mathscr{I},d) = 0$ *for* $n \geq 1$ *and* $H^{m,0}(\mathscr{P},d) = \mathscr{P}^{m,0}$, $H^{m,0}(\mathscr{I},d) = \mathscr{I}^{m,0} = \mathscr{I}^m_{\mathrm{S}}(\mathscr{A}_{\mathrm{Lie}})$.

Concerning the cohomology of δ one has the following result.

Proposition 4. *One has* $H^{m,n}(\mathscr{P},\delta) = 0, H^{m,n}(\mathscr{I},\delta) = 0$ *for* $m \geq 1$ *and* $H^{0,n}(\mathscr{P},\delta) = C^n_{\mathrm{H}}(\mathscr{A})$, $H^{0,n}(\mathscr{I},\delta) = C^n_{\mathrm{B}}(\mathscr{A})$.

Proof. The last part of the proposition $(m=0)$ is obvious since one has $H^{0,n}(\mathscr{P},\delta)= \ker(\delta \restriction C^n(\mathscr{A}))$ and $H^{0,n}(\mathscr{I},\delta) = \ker(\delta \restriction C^n_{\mathrm{I}}(\mathscr{A}))$. Therefore, from now on, assume that one has $m \geq 1$. Define a linear mapping ℓ of $\mathscr{P}$ in itself with $\ell(\mathscr{P}^{m,n}) \subset \mathscr{P}^{m-1,n+1}$ by

$$(\ell\omega)_A(A_1,\ldots,A_{n+1}) = \frac{\mathrm{d}}{\mathrm{d}t}\omega_{A+tA_1}(A_2,\ldots,A_{n+1})|_{t=0}$$

for $\omega \in \mathscr{P}^{m,n}$. One has $(\delta\ell + \ell\delta)\omega = m\omega + \mathscr{H}\omega$ where $\mathscr{H}\omega$ is given by

$$(\mathscr{H}\omega)_A(A_1,\ldots,A_n) = \sum_{p=2}^{n+1}(-1)^p\omega_A(A_2,\ldots,A_{p-1},A_1,A_p,\ldots,A_n)$$

$(\omega \in \mathscr{P}^{m,n})$. Notice that if ω is such that $\omega_A(A_1,\ldots,A_n)$ is antisymmetric in $A_1,\ldots,A_n$, then $\mathscr{H}\omega = n\omega$ and therefore ℓ gives an homotopy for such ω. □

The following lemma, which is a combinatorial statement in the algebra of the permutation group, will lead to an homotopy for the general case. The proof of this lemma (which is probably known) will be given in the appendix.

Lemma 1. *One has on* $\mathscr{P}^{m,n}$, $\prod_{p=0}^{n-2}(\mathscr{H} - p\ id) = \prod_{p=0}^{n-1}(\mathscr{H} - p\ id) = \mathscr{S}$, *where* $\mathscr{S}\omega$ *is given as before (antisymmetrisation) by* $(\mathscr{S}\omega)_A(A_1,\ldots,A_n) = \sum_{\pi\in\mathscr{S}_n}\varepsilon(\pi)\omega_A(A_{\pi(1)},\ldots, A_{\pi(n)})$, *i.e.* $(\mathscr{S}\omega)_A = \mathscr{S}\omega_A$.

Let $\omega \in \mathscr{P}^{m,n}$ with $m \geq 1$ be such that $\delta\omega = 0$. Then $\delta\ell\omega = m\omega + \mathscr{H}\omega$, so one also has $\delta\mathscr{H}\omega = 0$ and, by induction, $\delta\mathscr{H}^p\omega = 0$ for any integer p, i.e. one has $\delta P(\mathscr{H})\omega = 0$ for any polynomial P. Define $\omega_r \in \mathscr{P}^{m,n}$, for $r = 1,2,\ldots,n$, by $\omega_1 = \omega$, $\omega_2 = \mathscr{H}\omega - (n-2)\omega,\ldots$, $\omega_r = \prod_{p=2}^{r}(\mathscr{H} - (n-p)id)\omega,\ldots$, $\omega_n = \mathscr{H}(\mathscr{H} - id)\ldots(\mathscr{H} - (n-2)id)\omega$. One has $\delta\ell\omega_r = m\omega_r + \mathscr{H}\omega_r = (m+n-r-1)\omega_r + \omega_{r+1}$, i.e.

$$\omega_r = \delta\ell\left(\frac{\omega_r}{m+n-(r+1)}\right) - \frac{\omega_{r+1}}{m+n-(r+1)}$$

for $r \leq n-1$. This implies that

$$\omega = \delta\ell\left(\sum_{r=1}^{n-1} \frac{(-1)^{r+1}}{\prod_{p=2}^{r+1}(m+n-p)}\omega_r\right) - \frac{(-1)^n}{\prod_{p=2}^{n}(m+n-p)}\omega_n.$$

On the other hand, it follows from the lemma and the previous discussion (antisymmetry) that $\omega_n = \delta\ell((1/(m+n))\omega_n)$ and therefore one has an homotopy formula, for $\omega \in \mathscr{P}^{m,n}$ with $m \geq 1$ satisfying $\delta\omega = 0$, of the form $\omega = \delta\delta'\omega$ where $\delta' = \ell \circ Q^{m,n}(\mathscr{H})$ and where the polynomial $Q^{m,n}$ is easily computed from the previous formulae. Since ℓ and $\mathscr{H}$ preserve $\mathscr{I}$ this achieves the proof of Proposition 4. □

The proof of the Theorem 1 will now follow from $H^{m,n}(\mathscr{I},d) = 0$ for $n \geq 1$, $H^{m,n}(\mathscr{I},\delta) = 0$ for $m \geq 1$,

$$H^{m,0}(\mathscr{I},d) = \mathscr{I}^m_{\mathrm{S}}(\mathscr{A}_{\mathrm{Lie}}) \quad \text{and} \quad H^{0,n}(\mathscr{I},\delta) = C^n_{\mathrm{B}}(\mathscr{A})$$

by a standard spectral sequence argument in the bicomplex $(\mathscr{I},d,\delta)$.

Let $H(\delta|d)$ denote the δ-cohomology modulo d of $\mathscr{I}$, i.e.

$$H^{m,n}(\delta|d) = Z^{m,n}(\delta|d)/B^{m,n}(\delta|d),$$

where $Z^{m,n}(\delta|d)$ is the space of the $\alpha^{m,n} \in \mathscr{I}^{m,n}$ for which there is an $\alpha^{m+1,n-2} \in \mathscr{I}^{m+1,n-2}$ such that $\delta\alpha^{m,n} + d\alpha^{m+1,n-2} = 0$ and where $B^{m,n}(\delta|d) = \delta\mathscr{I}^{m-1,n+1} + d\mathscr{I}^{m,n-1}$ $(\subset \mathscr{I}^{m,n})$. With these notations, one has the following result.

Proposition 5. *One has the following isomorphisms*:

- $H^{2p}_{\mathrm{B}}(\mathscr{A}) \simeq H^{k,2(p-k)-1}(\delta|d) \simeq \mathscr{I}^p_{\mathrm{S}}(\mathscr{A}_{\mathrm{Lie}})$ *for* $1 \leq k \leq p-2$,
- $H^{2p+1}_{\mathrm{B}}(\mathscr{A}) \simeq H^{k,2(p-k)}(\delta|d) \simeq 0$ *for* $1 \leq k \leq p-1$, $H^4_{\mathrm{B}}(\mathscr{A}) \simeq \mathscr{I}^2_{\mathrm{S}}(\mathscr{A}_{\mathrm{Lie}})$,
- $H^3_{\mathrm{B}}(\mathscr{A}) \simeq 0$ *and* $H^2_{\mathrm{B}}(\mathscr{A}) \simeq \mathscr{I}^1_{\mathrm{S}}(\mathscr{A}_{\mathrm{Lie}})$.

Proof. Let $\alpha^{m,n} \in \mathscr{I}^{m,n}$ be a δ-cocycle modulo d, i.e. there is a $\alpha^{m+1,n-2} \in \mathscr{I}^{m+1,n-2}$ such that $\delta\alpha^{m,n} + d\alpha^{m+1,n-2} = 0$. By applying δ, one obtains $\delta d\alpha^{m+1,n-2} = -d\delta\alpha^{m+1,n-2} = 0$; therefore, if $n \geq 4$, there is in view of Proposition 3 a $\alpha^{m+2,n-4} \in \mathscr{I}^{m+2,n-4}$ such that $\delta\alpha^{m+1,n-2} + d\alpha^{m+2,n-4} = 0$, which means that $\alpha^{m+1,n-2}$ is also a δ-cocycle modulo d. If $\alpha^{m,n}$ is exact, i.e. if there are $\beta^{m-1,n+1} \in \mathscr{I}^{m-1,n+1}$ and $\beta^{m,n-1} \in \mathscr{I}^{m,n-1}$ such that $\alpha^{m,n} = \delta\beta^{m-1,n+1} + d\beta^{m,n-1}$, then $d(\alpha^{m+1,n-2} - \delta\beta^{m,n-1}) = 0$ which implies, again by Proposition 3 (since $n-2 \geq 2 > 0$), that there is a $\beta^{m+1,n-3}$ such that $\alpha^{m+1,n-2} = \delta\beta^{m,n-1} + d\beta^{m+1,n-3}$, i.e. $\alpha^{m+1,n-2}$ is also exact. Therefore, there is a well-defined linear mapping $\partial : H^{m,n}(\delta|d) \to H^{m+1,n-2}(\delta|d)$ for $n \geq 4$ such that $\partial[\alpha^{m,n}] = [\alpha^{m+1,n-2}]$. Let now $\alpha^{m+1,n-2} \in \mathscr{I}^{m+1,n-2}$ be a δ-cocycle modulo d, i.e. there is $\alpha^{m+2,n-4} \in \mathscr{I}^{m+2,n-4}$ such that $\delta\alpha^{m+1,n-2} + d\alpha^{m+2,n-4} = 0$. By applying d, one obtains $\delta d\alpha^{m+1,n-2} = 0$ which implies, in view of Proposition 4, that there is a $\alpha^{m,n} \in \mathscr{I}^{m,n}$ such that $\delta\alpha^{m,n} + d\alpha^{m+1,n-2} = 0$. This means that ∂ is surjective. Assume that $[\alpha^{m+1,n-2}] = 0$, i.e. $\alpha^{m+1,n-2} = \delta\beta^{m,n-1} + d\beta^{m+1,n-3}$ $(\beta \in \mathscr{I})$; then one has $\delta(\alpha^{m,n} - d\beta^{m,n-1}) = 0$ which implies that $[\alpha^{m,n}] = 0$ if $m \geq 1$ or that

$\alpha^{0,n} - d\beta^{0,n-1} \in C^n_{\mathrm{B}}(\mathscr{A})$ if $m = 0$, again by Proposition 4. *Thus,* $\partial : H^{m,n}(\delta|d) \to H^{m+1,n-2}(\delta|d)$ *are isomorphisms for* $n \geq 4$ *and* $m \geq 1$ *and, for* $m = 0$ $(n \geq 4)$, $\partial : H^{0,n}(\delta|d) \to H^{1,n-2}(\delta|d)$ *is surjective and its kernel is the image of* $C^n_{\mathrm{B}}(\mathscr{A}) = H^{0,n}(\mathscr{I},\delta)$ *in* $H^{0,n}(\delta|d)$.

On the other hand, if $\alpha^{0,n} \in \mathscr{I}^{0,n}$ is a δ-cocycle modulo d, i.e. $\delta\alpha^{0,n} + d\alpha^{1,n-2} = 0$, then $d\alpha^{0,n} \in C^{n+1}_{\mathrm{I}}(\mathscr{A})$ is a basic cocycle of $\mathscr{A}$ i.e. $d\alpha^{0,n} \in Z^{n+1}_{\mathrm{B}}(\mathscr{A})$ and if $\alpha^{0,n}$ is exact, i.e. $\alpha^{0,n} = d\beta^{0,n}$ with $\beta^{0,n} \in \mathscr{I}^{0,n}$, then $d\alpha^{0,n} = 0$. Therefore, with obvious notations, one has a linear mapping $d^\sharp : H^{0,n}(\delta|d) \to H^{n+1}_{\mathrm{B}}(\mathscr{A})$, $d^\sharp[\alpha^{0,n}] = [d\alpha^{0,n}]$. If $z^{n+1} \in C^{n+1}_{\mathrm{B}}(\mathscr{A})$ is closed, i.e. $z^{n+1} \in Z^{n+1}_{\mathrm{B}}(\mathscr{A})$ then, in view of Proposition 1, there is a $\alpha^{0,n} \in C^n_{\mathrm{I}}(\mathscr{A}) = \mathscr{I}^{0,n}$ such that $z^{n+1} = d\alpha^{0,n}$; one has $d\delta\alpha^{0,n} = 0$, which implies that $\alpha^{0,n}$ is a δ-cocycle modulo d if $n \geq 2$ (by Proposition 3). Thus, $d^\sharp$ is surjective for $n \geq 2$ and one obviously has $\mathit{ker}(d^\sharp) =$ image of $C^n_{\mathrm{B}}(\mathscr{A})$ in $H^{0,n}(\delta|d)$. Applying this for $n \geq 4$ and the previous results, one obtains isomorphisms:

$$H^{2p}_{\mathrm{B}}(\mathscr{A}) \simeq H^{k,2(p-k)-1}(\delta|d) \quad \text{for } 1 \leq k \leq p-2$$

and

$$H^{2p+1}_{\mathrm{B}}(\mathscr{A}) \simeq H^{k,2(p-k)}(\delta|d) \quad \text{for } 1 \leq k \leq p-1.$$

Thus, to achieve the proof, it remains to show that one has:

(i) $H^{m,2}(\delta|d) = 0$ for $m \geq 1$ and $H^{0,2}(\delta|d) =$ image of $C^2_{\mathrm{B}}(\mathscr{A})$,

(ii) $H^{m,3}(\delta|d) \simeq \mathscr{I}^{m+2}_{\mathrm{S}}(\mathscr{A}_{\mathrm{Lie}})$ for $m \geq 1$ and $H^{0,3}(\delta|d)$/image of $C^3_{\mathrm{B}}(\mathscr{A}) \simeq \mathscr{I}^2_{\mathrm{S}}(\mathscr{A}_{\mathrm{Lie}})$,

(iii) $H^2_{\mathrm{B}}(\mathscr{A}) \simeq \mathscr{I}^1_{\mathrm{S}}(\mathscr{A}_{\mathrm{Lie}})$ (remembering that $C^1_{\mathrm{B}}(\mathscr{A}) = 0$).

Let $\alpha^{m,2} \in \mathscr{I}^{m,2}$ be a δ-cocyle modulo d; then (since $d\alpha^{m+1,0} \equiv 0$) $\alpha^{m,2}$ is a δ-cocycle, i.e. $\delta\alpha^{m,2} = 0$, which implies, by Proposition 4, that $\alpha^{m,2} \in \delta\mathscr{I}^{m-1,1}$ for $m \geq 1$ and, for $m = 0$, $\alpha^{0,2} \in C^2_{\mathrm{B}}(\mathscr{A}) = H^{0,2}(\mathscr{I},\delta)$. This proves (i).

Let $\alpha^{m,3} \in \mathscr{I}^{m,3}$ be a δ-cocycle modulo d, i.e. there is a $\alpha^{m+1,1} \in \mathscr{I}^{m+1,1}$ such that $\delta\alpha^{m,3} + d\alpha^{m+1,1} = 0$. Then one has $\delta\alpha^{m+1,1} = P^{m+2} \in \mathscr{I}^{m+2}_{\mathrm{S}}(\mathscr{A}_{\mathrm{Lie}}) = \mathscr{I}^{m+2,0}$. If $\alpha^{m,3} = \delta\beta^{m-1,4} + d\beta^{m,2}$ for $\beta^{m-1,4} \in \mathscr{I}^{m-1,4}$ and $\beta^{m,2} \in \mathscr{I}^{m,2}$ (i.e. if $\alpha^{m,3}$ is exact), one has $d(\alpha^{m+1,1} - \delta\beta^{m,2}) = 0$ which implies, by Proposition 3 and by $d\mathscr{I}^{m+1,0} = 0$, that $\alpha^{m+1,1} = \delta\beta^{m,2}$ and therefore $\delta\alpha^{m+1,1} = P^{m+2} = 0$. Thus, there is a well-defined linear mapping $j : H^{m,3}(\delta|d) \to \mathscr{I}^{m+2}_{S}(\mathscr{A}_{\mathrm{Lie}})$, $(j([\alpha^{m,3}]) = P^{m+2})$. Let P^{m+2} be an arbitrary element of $\mathscr{I}^{m+2}_{\mathrm{S}}(\mathscr{A}_{\mathrm{Lie}})$; then, by Proposition 4, there is a $\alpha^{m+1,1} \in \mathscr{I}^{m+1,1}$ such that $\delta\alpha^{m+1,1} = P^{m+2}$ and, since $dP^{m+2} = 0$, one has $\delta d\alpha^{m+1,1} = 0$ which implies again by Proposition 4 that there is a $\alpha^{m,3}$ such that $\delta\alpha^{m,3} + d\alpha^{m+1,1} = 0$. This shows that j is surjective. If $\delta\alpha^{m+1,1} = 0$, then, by Proposition 4, $\alpha^{m+1,1} = \delta\beta^{m,2}$ and therefore $\delta(\alpha^{m,3} - d\beta^{m,2}) = 0$ which implies again by Proposition 4 that $\alpha^{m,3} = \delta\beta^{m-1,4} + d\beta^{m,2}$ if $m \geq 1$ and, for $m = 0$, $\alpha^{0,3} - d\beta^{0,2} \in C^3_{\mathrm{B}}(\mathscr{A})$. This proves (ii).

Finally, let $z^2 \in C^2_{\mathrm{I}}(\mathscr{A})$ be a basic cocycle, i.e. $dz^2 = 0$ and $\delta z^2 = 0$; then $z^2 = d\alpha^1$ for a unique $\alpha^1 \in C^1_{\mathrm{I}}(\mathscr{A})$ (since $dC^0(\mathscr{A}) = 0$ and by Proposition 1). Conversely, if $\alpha^1 \in C^1_{\mathrm{I}}(\mathscr{A})$ then $d\alpha^1$ is basic; therefore $H^2_{\mathrm{B}}(\mathscr{A}) \simeq C^1_{\mathrm{I}}(\mathscr{A})$ since $C^1_{\mathrm{B}}(\mathscr{A}) = 0$. But one has canonically $C^1_{\mathrm{I}}(\mathscr{A}) = \mathscr{I}^1_{\mathrm{S}}(\mathscr{A}_{\mathrm{Lie}})$. $\square$

This proves of course Theorem 1, but it is worth noticing that in the above proof there is also a computation of the δ-cohomology modulo d of $\mathscr{I}$.

4. Sketch of another approach: Connection with the Lie algebra cohomology

There is another way to study the basic cohomology of $\mathscr{A}$ which connects it with the Lie algebra cohomology of $\mathscr{A}_{\mathrm{Lie}}$: It is to study the spectral sequence corresponding to the filtration of the differential algebra $C(\mathscr{A})$ associated to the operation i of the Lie algebra $\mathscr{A}_{\mathrm{Lie}}$ in the differential algebra $C(\mathscr{A})$ [5]. This filtration $\mathscr{F}$ is defined by

$$\mathscr{F}^p(C^n(\mathscr{A})) = \{\omega \in C^n(\mathscr{A}) | i_{A_1} \cdots i_{A_{n-p+1}}(\omega) = 0,\ \forall A_i \in \mathscr{A}\}$$

for $0 \le p \le n$ and $\mathscr{F}^p(C(\mathscr{A})) = \bigoplus_{n \ge p} \mathscr{F}^p(C^n(\mathscr{A}))$.

One has

$$\mathscr{F}^0(C(\mathscr{A})) = C(\mathscr{A}),\ \mathscr{F}^p(C(\mathscr{A})) \cdot \mathscr{F}^q(C(\mathscr{A})) \subset \mathscr{F}^{p+q}(C(\mathscr{A}))$$

and

$$d\mathscr{F}^p(C(\mathscr{A})) \subset \mathscr{F}^p(C(\mathscr{A})),$$

i.e. $\mathscr{F}$ is a (decreasing) filtration of graded differential algebra. To such a filtration corresponds a convergent spectral sequence $(E_r, d_r)_{r \in \mathbb{N}}$, where $E_r = \bigoplus_{p,q \in \mathbb{N}} E_r^{p,q}$ is a bigraded algebra and d_r is a homogeneous differential on E_r of bidegree $(r, 1-r)$. The triviality of the cohomology of $C(\mathscr{A})$ (i.e. Proposition 1) implies that $E_\infty^{p,q} = 0$ for $(p,q) \neq (0,0)$ and $E_\infty^{0,0} = \mathbb{K}$. The spectral sequence starts with the graded space E_0 associated to the filtration, i.e. $E_0^{p,q} = \mathscr{F}^p(C^{p+q}(\mathscr{A}))/\mathscr{F}^{p+1}(C^{p+q}(\mathscr{A}))$ and d_0 is induced by the differential d of $C(\mathscr{A})$. If $\omega \in \mathscr{F}^p(C^{p+q}(\mathscr{A}))$ then $i_{A_1} \cdots i_{A_q}\omega$ is in $C_{\mathrm{H}}^p(\mathscr{A})$ and is antisymmetric in $A_1, \ldots, A_q$. Therefore $(A_1, \ldots, A_q) \mapsto i_{A_1} \cdots i_{A_q}\omega$ is a q-cochain of the Lie algebra $\mathscr{A}_{\mathrm{Lie}}$ with values in $C_{\mathrm{H}}^p(\mathscr{A})$ for the representation $A \mapsto L_A$ of the Lie algebra $\mathscr{A}_{\mathrm{Lie}}$ in $C_{\mathrm{H}}^p(\mathscr{A})$. This defines a linear map of $\mathscr{F}^p(C^{p+q}(\mathscr{A}))$ in the space of q-cochains of $\mathscr{A}_{\mathrm{Lie}}$ with values in $C_{\mathrm{H}}^p(\mathscr{A})$. The kernel of this map is, by definition, $\mathscr{F}^{p+1}(C^{p+q}(\mathscr{A}))$. In our case, it is straightforward to show that this map is surjective, i.e. that $E_0^{p,q}$ identifies with the space of q-cochains of the Lie algebra $\mathscr{A}_{\mathrm{Lie}}$ with values in the space $C_{\mathrm{H}}^p(\mathscr{A})$ of horizontal elements of $C^p(\mathscr{A})$ and that then, d_0 coincides with the Chevalley–Eilenberg differential. Thus, $E_1 = H(E_0, d_0)$ is the Lie algebra cohomology of $\mathscr{A}_{\mathrm{Lie}}$ with value in $C_{\mathrm{H}}(\mathscr{A})$, $E_1^{p,q} = H^q(\mathscr{A}_{\mathrm{Lie}}, C_{\mathrm{H}}^p(\mathscr{A}))$. In particular, $E_1^{0,*}$ is the ordinary cohomology of $\mathscr{A}_{\mathrm{Lie}}$ (i.e. with value in the trivial representation in $\mathbb{K}$) and $E_1^{*,0}$ is the space of invariant elements of $C_{\mathrm{H}}(\mathscr{A})$, i.e. the space $C_{\mathrm{B}}(\mathscr{A})$ of basic elements of $C(\mathscr{A})$, $E_1^{n,0} = C_{\mathrm{B}}^n(\mathscr{A})$. Furthermore, on $E_1^{*,0} = C_{\mathrm{B}}(\mathscr{A})$, d_1 is just the differential d of $C(\mathscr{A})$ restricted to $C_{\mathrm{B}}(\mathscr{A})$. Therefore, $E_2^{*,0}$ is the basic cohomology $H_{\mathrm{B}}(\mathscr{A})$ of $\mathscr{A}$, $E_2^{n,0} = H_{\mathrm{B}}^n(\mathscr{A})$. This shows that the spectral sequence connects the basic cohomology of $\mathscr{A}$ to the Lie algebra cohomology of its underlying Lie algebra $\mathscr{A}_{\mathrm{Lie}}$. The connection between the Lie algebra cohomology of

$\mathscr{A}_{\mathrm{Lie}}$ and the ad*-invariant polynomials, i.e. $H_{\mathrm{B}}(\mathscr{A})$ in our case, is well known but an interest of the last approach could be to catch the primitive parts.

Appendix: Proof of Lemma 1

Let $\mathscr{S}_n$ be the group of permutations of $\{1,\dots,n\}$. In the algebra of this group, let us define the antisymmetrisation operator

$$\mathscr{S} = \sum_{\pi\in\mathscr{S}_n} \varepsilon(\pi)\pi$$

and the operators

$$\mathscr{H}_{(k)} = \sum_{\substack{\pi\in\mathscr{S}_n \\ \pi^{-1}(k+1)<\cdots<\pi^{-1}(n)}} \varepsilon(\pi)\pi$$

for any $1 \le k \le n$, where $\varepsilon(\pi)$ denotes the signature of the permutation π.
Notice that

$$\mathscr{H}_{(n)} = \mathscr{H}_{(n-1)} = \mathscr{S}$$

and one easily shows that

$$\mathscr{H} \equiv \mathscr{H}_{(1)} = \sum_{p=1}^{n} (-1)^{p+1}\gamma_p,$$

where γ_p is the permutation $(1,\dots,p,\dots,n) \mapsto (2,\dots,p,1,p+1,\dots,n)$.
With these definitions, one has the following result.

Lemma. *For any* $1 \le k \le n-1$,

$$\mathscr{H}\mathscr{H}_{(k)} = k\mathscr{H}_{(k)} + \mathscr{H}_{(k+1)}.$$

Proof.

$$\begin{aligned}\mathscr{H}\mathscr{H}_{(k)} &= \left(\sum_{p=1}^{n}(-1)^{p+1}\gamma_p\right)\left(\sum_{\substack{\pi\in\mathscr{S}_n \\ \pi^{-1}(k+1)<\cdots<\pi^{-1}(n)}} \varepsilon(\pi)\pi\right)\\ &= \sum_{p=1}^{k}\sum_{\substack{\pi\in\mathscr{S}_n \\ \pi^{-1}(k+1)<\cdots<\pi^{-1}(n)}} (-1)^{p+1}\varepsilon(\pi)\gamma_p\pi\\ &\quad + \sum_{p=k+1}^{n}\sum_{\substack{\pi\in\mathscr{S}_n \\ \pi^{-1}(k+1)<\cdots<\pi^{-1}(n)}} (-1)^{p+1}\varepsilon(\pi)\gamma_p\pi.\end{aligned}$$

Now, define $\pi' = \gamma_p\pi \in \mathscr{S}_n$; one has $\varepsilon(\pi') = (-1)^{p+1}\varepsilon(\pi)$. For $p \le k$, one has $\pi'^{-1}(q) = \pi^{-1}(q)$ for any $k+1 \le q \le n$.

So, in the first summation, for a fixed p, the sum over the $\pi \in \mathscr{S}_n$ such that $\pi^{-1}(k+1) < \cdots < \pi^{-1}(n)$ can be replaced by the sum over the $\pi' \in \mathscr{S}_n$ such that $\pi'^{-1}(k+1) < \cdots < \pi'^{-1}(n)$. Thus,

$$\sum_{p=1}^{k} \sum_{\substack{\pi\in\mathscr{S}_n \\ \pi^{-1}(k+1)<\cdots<\pi^{-1}(n)}} (-1)^{p+1}\varepsilon(\pi)\gamma_p\pi = \sum_{p=1}^{k} \sum_{\substack{\pi'\in\mathscr{S}_n \\ \pi'^{-1}(k+1)<\cdots<\pi'^{-1}(n)}} \varepsilon(\pi')\pi' = k\mathscr{H}_{(k)}.$$

Now, for $p \geq k+1$, one has $\pi'^{-1}(q) = \pi^{-1}(q-1)$ for any $k+2 \leq q \leq p$ and $\pi'^{-1}(q) = \pi^{-1}(q)$ for any $p+1 \leq q \leq n$. So one has only

$$\pi'^{-1}(k+2) < \cdots < \pi'^{-1}(n),$$

and in the second summation the sum over p and π can be replaced by the sum over the $\pi' \in \mathscr{S}_n$ such that $\pi'^{-1}(k+2) < \cdots < \pi'^{-1}(n)$. Thus,

$$\sum_{p=k+1}^{n} \sum_{\substack{\pi\in\mathscr{S}_n \\ \pi^{-1}(k+1)<\cdots<\pi^{-1}(n)}} (-1)^{p+1}\varepsilon(\pi)\gamma_p\pi = \mathscr{H}_{(k+1)}. \qquad \square$$

By induction, this lemma shows that for any $1 \leq k \leq n$

$$\mathscr{H}_{(k)} = \prod_{p=0}^{k-1}(\mathscr{H} - p\ id),$$

where we recall $\mathscr{H} \equiv \mathscr{H}_{(1)}$.

So for $k = n$ and $k = n-1$, one has

$$\mathscr{H}_{(n)} = \prod_{p=0}^{n-1}(\mathscr{H} - p\ id) = \mathscr{S},$$

$$\mathscr{H}_{(n-1)} = \prod_{p=0}^{n-2}(\mathscr{H} - p\ id) = \mathscr{S}.$$

Now, notice that the operators $\mathscr{H}$ and $\mathscr{S}$ of Lemma 1 are representations of the operators $\mathscr{H}$ and $\mathscr{S}$ above in the linear space $\mathscr{P}^{m,n}$ (in fact only in $C^n(\mathscr{A})$). This proves Lemma 1.

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